ON A FUNCTIONAL EQUATION ON GROUPS

JUKANG K. CHUNG, SOON-MO JUNG, AND PRASANNA K. SAHOO

ABSTRACT. We present the general solution of the functional equation
\[ f(x_1y_1, x_2y_2) + f(x_1y_1^{-1}, x_2) + f(x_1, x_2y_2^{-1}) = f(x_1y_1^{-1}, x_2y_2^{-1}) + f(x_1y_1, x_2) + f(x_1, x_2y_2). \]
Furthermore, we also prove the Hyers-Ulam stability of the above functional equation.

1. Introduction

In a paper of Kannappan and Sahoo [8], it was shown that if the quadratic polynomial \( f(x, y) = ax^2 + bx + cy^2 + dy + exy + \alpha \) with \( a \neq 0 \) and \( c \neq 0 \), is a solution of the functional differential equation
\[
\begin{align*}
&f(x + h, y + k) - f(x, y) \\
&= hf_x(x + \theta h, y + \theta k) + k f_y(x + \theta h, y + \theta k), \quad 0 < \theta < 1,
\end{align*}
\]
for all \( x, y, h, k \in \mathbb{R} \) (the set of reals) with \( h^2 + k^2 \neq 0 \), then \( \theta = 1/2 \). Conversely, if a function \( f \) satisfies the above equation with \( \theta = 1/2 \), then the only solution is a quadratic polynomial. Here \( f_x \) and \( f_y \) represent the first partial derivatives of \( f \) with respect to \( x \) and \( y \), respectively.

To establish this result, Kannappan and Sahoo [8] used the following functional equation
\[ f(x_1 + x_2 + y_2) + f(x_1 - y_1, x_2) + f(x_1, x_2 - y_2) = f(x_1 - y_1, x_2 - y_2) + f(x_1 + y_1, x_2) + f(x_1, x_2 + y_2) \]
for all \( x_1, x_2, y_1, y_2 \in \mathbb{R} \). In the present paper we solve the above functional equation on groups without any regularity assumption about \( f \). We shall not assume the group to be abelian, so we write the above equation as
\[
\begin{align*}
f(x_1y_1, x_2y_2) + f(x_1y_1^{-1}, x_2) + f(x_1, x_2y_2^{-1}) = f(x_1y_1^{-1}, x_2y_2^{-1}) + f(x_1y_1, x_2) + f(x_1, x_2y_2)
\end{align*}
\]
(1)
for all $x_1, x_2, y_1, y_2 \in G$. Throughout this paper, $\mathbb{C}$ will denote the set of complex numbers.

This paper is organized as follows. In Section 2, we give some preliminary results that will be used in solving the functional equation (1). In Section 3, we present the general solution of (1). In Section 4, we prove the Hyers-Ulam stability of the functional equations (2) and (3), and then apply these results to prove the stability of the functional equation (1).

2. Some preliminary results

The following lemma was given by Aczel, Chung and Ng [1] and can also be obtained from Chung, Ebanks, Ng and Sahoo [2].

**Lemma 1.** Let $G$ be a group in which $x^2 = y$ has a solution for all $x, y \in G$. The general solution $f : G \to \mathbb{C}$ of the functional equation

$$f(xy) + f(xy^{-1}) = 2f(x) \quad (x, y \in G)$$

satisfying also

$$f(xy) = f(yx) \quad (x, y \in G)$$

is given by

$$f(x) = \phi(x) + a,$$

where $a \in \mathbb{C}$ is an arbitrary constant and $\phi$ an arbitrary homomorphism of $G$ into the additive group $(\mathbb{C}, +)$ of $\mathbb{C}$.

Next, we generalize the above lemma. In fact, this lemma can also be deduced from Chung, Ebanks, Ng and Sahoo [2]. However, we give a short proof for the sake of completeness.

**Lemma 2.** Let $G$ be a group in which $x^2 = y$ has a solution for all $x, y \in G$. The general solution, $f_1, f_2, f_3 : G \to \mathbb{C}$, of

$$f_1(xy) + f_2(xy^{-1}) = f_3(x) \quad (x, y \in G)$$

satisfying also

$$f_1(xy) = f_1(yx), \quad f_2(xy) = f_2(yx) \quad (x, y \in G)$$

are given by

$$f_1(x) = \phi(x) + a, \quad f_2(x) = \phi(x) + b, \quad f_3(x) = 2\phi(x) + a + b,$$

where $a, b \in \mathbb{C}$ are arbitrary constants and $\phi : G \to \mathbb{C}$ is a solution of

$$\phi(xy) = \phi(x) + \phi(y) \quad (x, y \in G).$$
Proof. It is obvious that (5) with (6) satisfies (3) and (4). In order to prove the converse, we first replace \( y \) by \( y^{-1} \) in (3) to obtain
\[
(7) \quad f_1(xy^{-1}) + f_2(xy) = f_3(x).
\]
Setting \( y = e \) (the identity of \( G \)) in (7), we get
\[
(8) \quad f_1(x) + f_2(x) = f_3(x).
\]
Adding (3) and (7) and then using (8), we have
\[
(9) \quad f_1(xy) + f_2(xy) + f_1(xy^{-1}) + f_2(xy^{-1}) = 2(f_1(x) + f_2(x)).
\]
Using Lemma 1, from (9) and (4) we obtain
\[
(10) \quad f_1(x) + f_2(x) = 2\phi(x) + a + b.
\]
Subtracting (7) from (3), we get
\[
f_1(xy) - f_2(xy) = f_1(xy^{-1}) - f_2(xy^{-1})
\]
for all \( x, y \in G \). Hence
\[
(11) \quad f_1(x) - f_2(x) = a - b.
\]
From (10), (11) and (8) we obtain (5). This completes the proof of the lemma. \( \square \)

3. General solutions of Eq. (1)

Theorem 1. Let \( G \) be a group in which \( x^2 = y \) has a solution for all \( x, y \in G \). The general solution \( f : G^2 \rightarrow \mathbb{C} \) of the functional equation (1) satisfying
\[
(12) \quad f(x_1y_1, x_2) = f(y_1x_1, x_2) \quad \text{and} \quad f(x_1, x_2y_2) = f(x_1, y_2x_2)
\]
for all \( x_1, x_2, y_1, y_2 \in G \), is given by
\[
(13) \quad f(x_1, x_2) = A(x_1, x_2) + \alpha(x_1) + \beta(x_2),
\]
where \( \alpha, \beta : G \rightarrow \mathbb{C} \) are arbitrary functions satisfying \( \alpha(x_1x_2) = \alpha(x_2x_1) \) and \( \beta(x_1x_2) = \beta(x_2x_1) \) for all \( x_1, x_2 \in G \), and \( A : G^2 \rightarrow \mathbb{C} \) is additive in each variable.

Proof. It is easy to check that (13) satisfies (1) and (12). In order to prove the converse, we rewrite (1) as
\[
(14) \quad f(x_1y_1, x_2y_2) - f(x_1y_1, x_2) - (f(x_1y_1^{-1}, x_2y_2^{-1}) - f(x_1y_1^{-1}, x_2))
\]
\[
= f(x_1, x_2y_2) - f(x_1, x_2y_2^{-1})
\]
and define $F_i : G^3 \to C$ ($i = 1, 2, 3$) by

\begin{align}
F_1(x_1, x_2, y_2) &:= f(x_1, x_2y_2) - f(x_1, x_2), \\
F_2(x_1, x_2, y_2) &:= -f(x_1, x_2y_2^{-1}) + f(x_1, x_2), \\
F_3(x_1, x_2, y_2) &:= f(x_1, x_2y_2) - f(x_1, x_2y_2^{-1}).
\end{align}

The equation (14) can be written as

\[ F_1(x_1y_1, x_2, y_2) + F_2(x_1y_1^{-1}, x_2, y_2) = F_3(x_1, x_2, y_2). \]

From (12), (15) and (16), we get

\begin{align}
F_1(x_1y_1, x_2, y_2) &= F_1(y_1x_1, x_2, y_2), \\
F_2(x_1y_1, x_2, y_2) &= F_2(y_1x_1, x_2, y_2)
\end{align}

for $x_1, x_2, y_1, y_2 \in G$. So by Lemma 2, (15) and (16) we get

\[ f(x_1, x_2y_2) - f(x_1, x_2) = \phi_1(x_1, x_2, y_2) + b(x_2, y_2) \]

and

\[ f(x_1, x_2y_2^{-1}) - f(x_1, x_2) = -\phi_1(x_1, x_2, y_2) + a(x_2, y_2) - b(x_2, y_2), \]

where $\phi_1(x_1, x_2, y_2)$ satisfies

\[ \phi_1(x_1, x_2, y_2) = \phi_1(x_1, x_2, y_2) + \phi_1(y_1, x_2, y_2). \]

Adding (18) and (20) we get

\[ f(x_1, x_2y_2) + f(x_1, x_2y_2^{-1}) - 2f(x_1, x_2) = a(x_2, y_2). \]

Setting $x_1 = e$ in (20) we have

\[ a(x_2, y_2) = \beta_1(x_2y_2) + \beta_1(x_2y_2^{-1}) - 2\beta_1(x_2), \]

where $\beta_1(x_2) := f(e, x_2)$. Further, by (12), we also have $\beta_1(x_2) = \beta_1(y_2x_2)$. Substituting (21) into (20), we get

\[ F(x_1, x_2y_2) + F(x_1, x_2y_2^{-1}) = 2F(x_1, x_2), \]

where $F(x_1, x_2) := f(x_1, x_2) - \beta_1(x_2)$ satisfies $F(x_1, x_2y_2) = F(x_1, y_2x_2)$. By Lemma 1, we obtain

\[ F(x_1, x_2) = \phi_2(x_1, x_2) + \alpha(x_2), \]

that is

\[ f(x_1, x_2) = \phi_2(x_1, x_2) + \alpha(x_1) + \beta_1(x_2), \]

where $\phi_2(x_1, x_2)$ satisfies

\[ \phi_2(x_1, x_2y_2) = \phi_2(x_1, x_2) + \phi_2(x_1, y_2). \]
Since $f$ satisfies (12), using (22) we obtain
\begin{equation}
\phi_2(x_1y_1, x_2) + \alpha(x_1y_1) = \phi_2(y_1x_1, x_2) + \alpha(y_1x_1)
\end{equation}
for all $x_1, y_1, x_2 \in G$. Replacing $x_2$ by $x_2y_2$ in (24), we get
\begin{equation}
\phi_2(x_1y_1, x_2y_2) + \alpha(x_1y_1) = \phi_2(y_1x_1, x_2y_2) + \alpha(y_1x_1).
\end{equation}
Hence from (23) and (25), we see that
\begin{align*}
\phi_2(x_1y_1, x_2) + \phi_2(x_1y_1, y_2) + \alpha(x_1y_1) \\
= \phi_2(y_1x_1, x_2) + \phi_2(y_1x_1, y_2) + \alpha(y_1x_1).
\end{align*}
Thus the last equation together with (24) yields
\begin{equation}
\phi_2(x_1y_1, y_2) = \phi_2(y_1x_1, y_2)
\end{equation}
for all $x_1, y_1, y_2 \in G$. From (24) and (26), we see that $\alpha(x_1x_2) = \alpha(x_2x_1)$ for all $x_1, x_2 \in G$. Substituting (22) into (1) and using (23), we get
\begin{equation}
\phi_2(x_1y_1, y_2) + \phi_2(x_1y_1^{-1}, y_2) = 2\phi_2(x_1, y_2).
\end{equation}
From Lemma 1, (27) and (26), we get
\begin{equation}
\phi_2(x_1, x_2) = A(x_1, x_2) + \beta_2(x_2),
\end{equation}
where $A : G^2 \to \mathbb{C}$ satisfies
\begin{equation}
A(x_1y_1, x_2) = A(x_1, x_2) + A(y_1, x_2).
\end{equation}
Rewriting (28), we have
\begin{equation}
A(x_1, x_2) = \phi_2(x_1, x_2) - \beta_2(x_2).
\end{equation}
Using (30) in (29), we obtain
\begin{equation}
\phi_2(x_1y_1, x_2) = \phi_2(x_1, x_2) + \phi_2(y_1, x_2) - \beta_2(x_2).
\end{equation}
Replacing $x_1$ by $x_1y_1$ in (23) and using (31), we see that
\begin{equation}
\beta_2(x_2y_2) = \beta_2(x_2) + \beta_2(y_2).
\end{equation}
From (23), (30) and (32), we obtain
\begin{equation}
A(x_1, x_2y_2) = A(x_1, x_2) + A(x_1, y_2).
\end{equation}
Putting (28) into (22), we get
\begin{equation*}
f(x_1, x_2) = A(x_1, x_2) + \alpha(x_1) + \beta(x_2),
\end{equation*}
where $\beta(x_2) = \beta_1(x_2) + \beta_2(x_2)$ which proves (13). It is easy to check that $\beta$ satisfies $\beta(x_1x_2) = \beta(x_2x_1)$ for all $x_1, x_2 \in G$. This concludes the proof of the theorem. \qed

**Remark 1.** Theorem 1 also remains true if $C$ is replaced by a 2-cancellative abelian group $H$. 

4. Stability results

In this section, we first prove the Hyers-Ulam stability of the functional equations (2) and (3) and apply these results to the proof of the Hyers-Ulam stability of the equation (1). Throughout this section, we assume \( G \) to be an abelian group.

**Lemma 3.** Let \( G \) be an abelian group. If a function \( f : G \to \mathbb{C} \) satisfies the functional inequality

\[
|f(xy) + f(xy^{-1}) - 2f(x)| \leq \delta \quad (x, y \in G)
\]

for some given \( \delta \geq 0 \), then there exist an arbitrary constant \( a \in \mathbb{C} \) and a group homomorphism \( \phi \) of \( G \) into the additive group \( (\mathbb{C}, +) \) of \( \mathbb{C} \) such that

\[
|f(x) - \phi(x) - a| \leq 2\delta
\]

for all \( x \) in \( G \).

**Proof.** By \( e \) we will denote the unit element of \( G \). If we define \( \varphi(x) = f(x) - f(e) \), then \( \varphi(e) = 0 \). From (33) we have

\[
|\varphi(xy) + \varphi(xy^{-1}) - 2\varphi(x)| \leq \delta \quad (x, y \in G).
\]

We put \( y = x \) in (34) to get

\[
|\varphi(x^2) - 2\varphi(x)| \leq \delta \quad (x \in G).
\]

Substituting \( x^{-1}y^{-1} \) for \( y \) in (34), we have

\[
|\varphi(y^{-1}) + \varphi(x^2y) - 2\varphi(x)| \leq \delta \quad (x, y \in G).
\]

By (35) and (36), we obtain

\[
|\varphi(y^{-1}) + \varphi(x^2y) - \varphi(x^2)|
\leq
|\varphi(y^{-1}) + \varphi(x^2y) - 2\varphi(x)| + |2\varphi(x) - \varphi(x^2)|
\leq
2\delta \quad (x, y \in G).
\]

If we put \( u = y^{-1} \) and \( v = x^2y \) in (37), then we have

\[
|\varphi(u) + \varphi(v) - \varphi(uv)| \leq 2\delta \quad (u, v \in G).
\]

By a theorem of Rätz [9] (or see also Hyers and Rassias [6], Forti [4], Jung [7] or Hyers, Isac and Rassias [5]), there exists a group homomorphism \( \phi \) of \( G \) into the additive group \( (\mathbb{C}, +) \) of \( \mathbb{C} \) such that

\[
|\varphi(x) - \phi(x)| \leq 2\delta \quad (x \in G),
\]

and hence

\[
|f(x) - \phi(x) - a| \leq 2\delta \quad (x \in G),
\]

where \( a = f(e) \). \( \square \)
Lemma 4. Let $G$ be an abelian group in which $x^2 = y$ has a solution for all $x, y \in G$. If a function $f : G \to \mathbb{C}$ satisfies the functional inequality
\begin{equation}
|f_1(xy) + f_2(xy^{-1}) - f_3(x)| \leq \delta \quad (x, y \in G)
\end{equation}
for some $\delta \geq 0$, then there exist constants $a, b \in \mathbb{C}$ and a group homomorphism $\phi : G \to (\mathbb{C}, +)$ such that
\begin{align*}
|f_1(x) - \phi(x) - a| &\leq 5\delta, \\
|f_2(x) - \phi(x) - b| &\leq 5\delta, \\
|f_3(x) - 2\phi(x) - a - b| &\leq 11\delta
\end{align*}
for all $x \in G$.

Proof. If we replace $y$ by $y^{-1}$ in (38), then we have
\begin{equation}
|f_1(xy^{-1}) + f_2(xy) - f_3(x)| \leq \delta.
\end{equation}
Put $y = e$ in (38) to get
\begin{equation}
|f_1(x) + f_2(x) - f_3(x)| \leq \delta.
\end{equation}
This inequality, together with (38) and (39), yields
\begin{align*}
&|f_1(xy) + f_2(xy) + f_1(xy^{-1}) + f_2(xy^{-1}) - 2f_1(x) - 2f_2(x)| \\
&\leq |f_1(xy) + f_2(xy^{-1}) - f_3(x)| + |f_1(xy^{-1}) + f_2(xy) - f_3(x)| \\
&\quad + 2|f_3(x) - f_1(x) - f_2(x)| \\
&\leq 4\delta.
\end{align*}
By Lemma 3, we can conclude that there exists a group homomorphism $\phi : G \to (\mathbb{C}, +)$ such that
\begin{equation}
|f_1(x) + f_2(x) - 2\phi(x) - f_1(e) - f_2(e)| \leq 8\delta
\end{equation}
for any $x$ in $G$. By (38) and (39), we get
\begin{align*}
&|f_1(xy) - f_2(xy) - f_1(xy^{-1}) + f_2(xy^{-1})| \\
&\leq |f_1(xy) + f_2(xy^{-1}) - f_3(x)| + |f_3(x) - f_1(xy^{-1}) - f_2(xy)| \\
&\leq 2\delta.
\end{align*}
Putting $y = x$ in the above inequality, we obtain
\begin{equation}
|f_1(x^2) - f_2(x^2) - f_1(e) + f_2(e)| \leq 2\delta.
\end{equation}
Since the equation $x^2 = y$ is solvable for all $x$ and $y$ in $G$, the last inequality can be rewritten as
\begin{equation}
|f_1(x) - f_2(x) - f_1(e) + f_2(e)| \leq 2\delta
\end{equation}
for each \( x \in G \). It then follows from (41) and (42) that
\[
2 |f_1(x) - \phi(x) - f_1(e)| \\
\leq |f_1(x) + f_2(x) - 2\phi(x) - f_1(e) - f_2(e)| \\
+ |f_1(x) - f_2(x) - f_1(e) + f_2(e)| \\
\leq 10\delta,
\]
that is,
\[
|f_1(x) - \phi(x) - f_1(e)| \leq 5\delta
\]
for all \( x \in G \). Analogously, we have
\[
|f_2(x) - \phi(x) - f_2(e)| \leq 5\delta.
\]
By (40), (43) and (44), we see
\[
|f_3(x) - 2\phi(x) - f_1(e) - f_2(e)| \\
\leq |f_3(x) - f_1(x) - f_2(x)| + |f_1(x) - \phi(x) - f_1(e)| \\
+ |f_2(x) - \phi(x) - f_2(e)| \\
\leq 11\delta
\]
which completes the proof. \( \square \)

We are now ready to prove the main theorem of this section concerning the Hyers-Ulam stability of the functional equation (1).

**Theorem 2.** Let \( G \) be an abelian group in which \( x^2 = y \) has a solution for all \( x, y \in G \). If a function \( f : G^2 \to C \) satisfies the functional inequality
\[
|f(x_1 y_1, x_2 y_2) + f(x_1 y_1^{-1}, x_2) + f(x_1, x_2 y_2^{-1}) \\
- f(x_1 y_1^{-1}, x_2 y_2^{-1}) - f(x_1 y_1, x_2) - f(x_1, x_2 y_2)| \leq \delta
\]
for some \( \delta \geq 0 \) and for all \( x_1, y_1, x_2, y_2 \in G \), then there exist arbitrary functions \( \alpha, \beta : G \to C \) and \( A : G^2 \to C \) additive in each variable such that
\[
|f(x_1, x_2) - A(x_1, x_2) - \alpha(x_1) - \beta(x_2)| \leq 14982\delta
\]
for all \( x_1, x_2 \in G \).

**Proof.** Define \( F_i : G^3 \to C \) \((i = 1, 2, 3)\) by (15), (16) and (17). Then, the inequality (45) can be written as
\[
|F_1(x_1 y_1, x_2, y_2) + F_2(x_1 y_1^{-1}, x_2, y_2) - F_3(x_1, x_2, y_2)| \leq \delta
\]
for any \( x_1, y_1, x_2, y_2 \in G \). According to Lemma 4, by considering (15), (16) and (17), the last inequality implies that there exists a function \( \phi_1 : G^3 \to \)
C such that
\[
\phi_1(x_1y_1, x_2, y_2) = \phi_1(x_1, x_2, y_2) + \phi_1(y_1, x_2, y_2),
\]
(47) \[|f(x_1, x_2y_2) - f(x_1, x_2) - \phi_1(x_1, x_2, y_2) - b(x_2, y_2)| \leq 5\delta\]
and
(48) \[|f(x_1, x_2y_2^{-1}) - f(x_1, x_2) + \phi_1(x_1, x_2, y_2) - a(x_2, y_2) + b(x_2, y_2)| \leq 5\delta\]
for all \(x_1, y_1, x_2, y_2 \in \mathbb{G}\). Combining (47) and (48), we get
(49) \[|f(x_1, x_2y_2) + f(x_1, x_2y_2^{-1}) - 2f(x_1, x_2) - a(x_2, y_2)| \leq 10\delta\]
for any \(x_1, x_2, y_2 \in \mathbb{G}\). Putting \(x_1 = e\) and setting \(\beta_1(x_2) = f(e, x_2)\) in (49) yield
(50) \[|a(x_2, y_2) - \beta_1(x_2y_2) - \beta_1(x_2y_2^{-1}) + 2\beta_1(x_2)| \leq 10\delta.\]
By defining \(F(x_1, x_2) = f(x_1, x_2) - \beta_1(x_2)\), it follows from (49) and (50) that
(51) \[|F(x_1, x_2y_2) + F(x_1, x_2y_2^{-1}) - 2F(x_1, x_2)| \leq 20\delta\]
for any \(x_1, x_2, y_2 \in \mathbb{G}\). Hence, by Lemma 3, there exist an arbitrary function \(\alpha : \mathbb{G} \to \mathbb{C}\) and a function \(\phi_2 : \mathbb{G}^2 \to \mathbb{C}\) such that
(52) \[\phi_2(x_1, x_2y_2) = \phi_2(x_1, x_2) + \phi_2(x_1, y_2)\]
and
(53) \[|f(x_1, x_2) - \phi_2(x_1, x_2) - \alpha(x_1) - \beta_1(x_2)| \leq 40\delta\]
for all \(x_1, x_2, y_2 \in \mathbb{G}\). Using (45), (51) and (52), we obtain
\[
|\phi_2(x_1y_1, y_2) - \phi_2(x_1y_1^{-1}, y_2)| + 2\phi_2(x_1, y_2)|
\leq |f(x_1y_1, x_2y_2) - \phi_2(x_1y_1, x_2y_2) - \alpha(x_1y_1) - \beta_1(x_2y_2)|
+ |f(x_1y_1^{-1}, x_2) - \phi_2(x_1y_1^{-1}, x_2) - \alpha(x_1y_1^{-1}) - \beta_1(x_2)|
+ |f(x_1, x_2y_2^{-1}) - \phi_2(x_1, x_2y_2^{-1}) - \alpha(x_1) - \beta_1(x_2y_2^{-1})|
+ |f(x_1y_1^{-1}, x_2y_2^{-1}) + \phi_2(x_1y_1^{-1}, x_2y_2^{-1}) + \alpha(x_1y_1^{-1}) + \beta_1(x_2y_2^{-1})|
+ |f(x_1, x_2y_2) + \phi_2(x_1, x_2y_2) + \alpha(x_1) + \beta_1(x_2y_2)|
+ |f(x_1y_1, x_2y_2) - f(x_1y_1^{-1}, x_2) - f(x_1, x_2y_2^{-1})|
+ |f(x_1y_1^{-1}, x_2y_2^{-1}) + f(x_1y_1, x_2) + f(x_1, x_2y_2)|
\leq 241\delta.
\]
By Lemma 3 again, there exist an arbitrary function \(\beta_2 : \mathbb{G} \to \mathbb{C}\) and a function \(A_1 : \mathbb{G}^2 \to \mathbb{C}\) such that
(54) \[A_1(x_1y_1, x_2) = A_1(x_1, x_2) + A_1(y_1, x_2)\]
and
\begin{equation}
|\phi_2(x_1, x_2) - A_1(x_1, x_2) - \beta_2(x_2)| \leq 482\delta
\end{equation}
for all \(x_1, y_1, x_2 \in G\). By considering (53), it follows from (54) that
\begin{equation}
|\phi_2(x_1 y_1, x_2) - \phi_2(x_1, x_2) - \phi_2(y_1, x_2) + \beta_2(x_2)|
\leq |\phi_2(x_1 y_1, x_2) - A_1(x_1 y_1, x_2) - \beta_2(x_2)|
+ | - \phi_2(x_1, x_2) + A_1(x_1, x_2) + \beta_2(x_2)|
+ | - \phi_2(y_1, x_2) + A_1(y_1, x_2) + \beta_2(x_2)|
\leq 1446\delta.
\end{equation}

By replacing \(x_2\) by \(y_2\) respectively by \(x_2 y_2\) in (55) and combining (55) and the resulting inequalities and then considering (51), we have
\begin{equation}
|\beta_2(x_2) + \beta_2(y_2) - \beta_2(x_2 y_2)| \leq 4338\delta
\end{equation}
for any \(x_2, y_2 \in G\). By a theorem of Forti [3] or Rätz [9] (see also Forti [4], Hyers and Isac and Rassias [5], Hyers and Rassias [6] and Jung [7]), there exists an additive function \(A_2 : G \to \mathbb{C}\) such that
\begin{equation}
|\beta_2(x_2) - A_2(x_2)| \leq 4338\delta
\end{equation}
for all \(x_2 \in G\). By (54) and (56), we have
\begin{equation}
|\phi_2(x_1, x_2) - A_1(x_1, x_2) - A_2(x_2)| \leq 4820\delta
\end{equation}
for any \(x_1, x_2 \in G\). By replacing \(x_2\) by \(y_2\) and by \(x_2 y_2\) separately in (57), and combining (57) and the resulting inequalities, and then considering (51) and the fact that \(A_2\) is additive, we get
\begin{equation}
|A_1(x_1, x_2 y_2) - A_1(x_1, x_2) - A_1(x_1, y_2)| \leq 14460\delta
\end{equation}
for any \(x_1, x_2, y_2 \in G\). According to a theorem of Forti [3] or Rätz [9] again, if we define a function \(A : G^2 \to \mathbb{C}\) by
\begin{equation*}
A(x_1, x_2) = \lim_{n \to \infty} 2^{-n} A_1(x_1, x_2^{2^n})
\end{equation*}
for any \(x_1, x_2 \in G\), then \(A\) is additive in the second variable and also in the first variable, since \(A_1\) has this property (see (53)). Furthermore,
\begin{equation}
|A_1(x_1, x_2) - A(x_1, x_2)| \leq 14460\delta
\end{equation}
for all \(x_1, x_2 \in G\). By (52), (54), (58) and putting \(\beta(x_2) = \beta_1(x_2) + \beta_2(x_2)\), we see that the inequality (46) holds true. \(\square\)
References


Jukang K. Chung  
Department of Applied Mathematics  
South China University of Technology  
Guanzhou, P. R. China

Soon-Mo Jung  
Mathematics Section  
College of Science and Technology  
Hong-Ik University  
Chochiwon 339-701, Korea

Prasanna K. Sahoo  
Department of Mathematics  
University of Louisville  
Louisville, Kentucky 40292, USA