DECOMPOSITION OF SOME CENTRAL SEPARABLE ALGEBRAS

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ABSTRACT. If an Azumaya algebra $A$ is a homomorphic image of a finite group ring $RG$ where $G$ is a direct product of subgroups then $A$ can be decomposed into subalgebras $A_i$ which are homomorphic images of subgroup rings of $RG$. This result is extended to projective Schur algebras, and in this case behaviors of 2-cocycles will play major role. Moreover considering the situation that $A$ is represented by Azumaya group ring $RG$, we study relationships between the representing groups for $A$ and $A_i$.

1. Introduction

Let $R$ denote a commutative ring. The Brauer group $B(R)$ is the group of similar classes $[A]$ consisting of an Azumaya (i.e., central separable) $R$-algebra $A$ over $R$ (refer to [4, (2.5)]). An Azumaya $R$-algebra which is a homomorphic image of a group ring $RG$ for some finite group $G$ is called the Schur algebra. The set of similar classes of Schur algebras forms the Schur subgroup $S(R)$ of $B(R)$. In [5], two subgroups of $S(R)$ were introduced; one is $S'(R)$ consisting of elements in $S(R)$ that are represented by cyclotomic algebras $(R(\varepsilon_n)/R, \alpha)$ with 2-cocycle $\alpha$ on $\text{Gal}(R(\varepsilon_n)/R)$ having values in $(\varepsilon_n)$ for $n > 0$. The other is $S''(R)$ consisting of elements in $S(R)$ with a representative which is a homomorphic image of separable group algebra $RG$. The group ring $RG$ is separable if and only if $|G|$ is unit of $R$. The $S'(R)$ and $S''(R)$ need not equal, however if $R = k$ a field then $S''(k) = S'(k) = S(k)$ due to Brauer-Witt theorem [10].

The Schur $k$-algebra was generalized by Lorenz and Opolka (1978) that a finite dimensional central simple $k$-algebra which is a homomorphic image of a twisted group algebra $kG^{\alpha}$ for some finite group $G$ and some $\alpha \in H^2(G, k^*)$ is called the projective Schur algebra over $k$. The set of similar classes of projective Schur algebras forms the projective Schur group $PS(k)$.

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In the paper we study decompositions of some Azumaya algebras such as Schur and projective Schur algebras. In section 2, we prove that if \( A \) is an Azumaya algebra which is an epimorphic image of \( RG \) and if \( G = G_1 \times G_2 \) then \( A \) is decomposed into Azumaya subalgebras \( A_i \) where \( A_i \) is represented by \( RG_i \) (\( i = 1, 2 \)). The similar result can be obtained with respect to projective Schur algebras, and in this case behaviors of 2-cocycles corresponding to representing groups play important roles. In section 3, regarding \( RG \) itself as an Azumaya group ring which represents \( A \), we study interrelationships between the representing Azumaya group rings for \( A \) and \( A_i \).

Throughout the paper, \( R \) will always denote a connected commutative ring. Let \([A] \in B(R)\) denote a similar class of finite dimensional Azumaya \( R \)-algebra \( A \), and for \( A' \in [A] \) we denote \( A' \sim A \). Let \( u(R) \) be the set of units of \( R \), \( k^* \) be the multiplicative subgroup of a field \( k \) and \( \epsilon_n \) (\( n > 0 \)) be a primitive \( n \)-th root of unity. For a field extension \( L/k \), we denote \( H^2(Gal(L/k), L^*) \) by \( H^2(L/k) \).

2. Schur and projective Schur algebras

For Galois extensions of commutative ring \( R \), we may refer to [1] or [4]. Let \( A \) be any \( R \)-algebra and \( B \) be any subalgebra of \( A \). Let \( A^B = \{ a \in A \mid ab = ba \text{ for any } b \in B \} \). Then \( A^B \) is an \( R \)-subalgebra of \( A \) which commutes with \( B \). We remark the following lemma for convenience.

**Lemma 1** [4, (2.4.3)]. Let \( A \) be an Azumaya \( R \)-algebra.

1. If \( B \) is an Azumaya \( R \)-subalgebra of \( A \) then \( A^B \). Moreover \( (A)^{A_a} = B \) and \( B \otimes A^B \cong A \); \( b \otimes a \mapsto ba \) for \( b \in B, a \in A^B \).

2. Suppose \( B \) and \( C \) are subalgebras with \( B \otimes C \cong A \), \( b \otimes c \mapsto bc \) (\( b \in B, c \in C \)). Then \( B, C \) are Azumaya algebras with \( A^B = C, A^C = B \).

Thus for a given Azumaya algebra \( A \), Azumaya subalgebras of \( A \) occur in pairs, each of the pair is the commutator subalgebra of the other and whose tensor product is isomorphic to \( A \).

**Theorem 2.** Suppose that \([A] \in S(R)\) and the Azumaya algebra \( A \) is represented by a finite group ring \( RG \). If \( G = G_1 \times G_2 \) then \( A \) can be decomposed into \( A_1 \otimes A_2 \) where each \( A_i \) is represented by \( RG_i \), thus \([A_i] \in S(R)\) for \( i = 1, 2 \). Furthermore if \([A] \in S''(R)\) then each \([A_i] \in S''(R)\).

**Proof.** Let \( f \) be the surjective homomorphism \( RG \rightarrow A \) and let \( \{ u_g \mid g \in G \} \) denote an \( R \)-basis for \( RG \) with multiplication \( u_g u_x = u_{gx} \) for \( g, x \in G \).

Since \( RG = R(G_1 \times G_2) \cong RG_1 \otimes_R RG_2 \) as \( R \)-algebras [9], for \( g = g_1 g_2 \in G \) \((g_i \in G_i)\), the \( R \)-basis element \( u_g \in RG \) corresponds to \( u_{g_1} \otimes u_{g_2} \) where
$u_g_i$ is an $R$-basis of $RG_i$. Hence we may use the same notation $f$ for the surjection $RG_1 \otimes RG_2 \to A$ defined by $f(u_{g_1} \otimes u_{g_2}) = f(u_g)$.

Let $f_i$ be the restriction of $f$ to $RG_i$ and let $A_i = f_i(RG_i)$ for $i = 1, 2$. Then $f_1 \otimes f_2 : RG \to A_1 \otimes A_2$ maps $u_{g_1} \otimes u_{g_2}$ to $f_1(u_{g_1}) \otimes f_2(u_{g_2})$, this implies that $A_1$ and $A_2$ are $R$-subalgebras of $A$ such that $A_1 \otimes A_2 \to A$, $a_1 \otimes a_2 \mapsto a_1 a_2$ is an $R$-algebra isomorphism. Thus $f = f_1 \otimes f_2$ and $A_i$ is an Azumaya $R$-algebra due to Lemma 1. Clearly $[A_i] \in S(R)$ for $i = 1, 2$.

In particular if $[A] \in S''(R)$ then $|G| \in u(R)$ hence there exists $m \in R$ such that $m|G| = 1_R$. Since $|G_i|$ divides $|G|$, $|G| = |G_i| t_i$ for some $t_i > 0$ and $1_R = t_i m|G_i|$, thus $|G_i|$ is unit in $R$. It thus follows that $[A_i] \in S''(R)$. □

For finite groups $G$ and $H$, if $\alpha \in Z^2(G, u(R))$ and $\beta \in Z^2(H, u(R))$ then $\alpha \times \beta$ defined by $\alpha \times \beta((g_1, h_1), (g_2, h_2)) = \alpha(g_1, g_2)\beta(h_1, h_2)$ with $g_i \in G$, $h_i \in H$ is an element in $Z^2(G \times H, u(R))$. In particular if $G = H$ then $\alpha \beta$ defined by $\alpha \beta(g_1, g_2) = \alpha(g_1, g_2)\beta(g_1, g_2) \in Z^2(G, u(R))$.

**Theorem 3.** Let $[A] \in S'(R)$ and $A$ be a cyclotomic algebra $(R(\epsilon_n)/R, \alpha)$ where $\alpha$ has values in $\langle \epsilon_n \rangle$. If $n$ is divisible by $pq$ with primes $p \neq q$ then $A$ can be decomposed into $A_1 \otimes A_2$ and $[A_i] \in S''(R)$ for $i = 1, 2$.

**Proof.** For any $x, y \in \text{Gal}(R(\epsilon_n)/R)$, the order of $\alpha(x, y)$ divides $n$ because $\alpha(x, y) \in \langle \epsilon_n \rangle$. With the prime divisor $p$ of $n$, write $\alpha(x, y) = \alpha(x, y)_p \alpha(x, y)_{p'}$ and $n = npn_{p'}$ where $\alpha(x, y)_p$ [resp. $np$] is the $p$-part and $\alpha(x, y)_{p'}$ [resp. $n_{p'}$] is the $p'$-part of $\alpha(x, y)$ [resp. $n$]. In fact, $\alpha(x, y)_p$ and $\alpha(x, y)_{p'}$ are powers of $\alpha(x, y)$ such that the order of $\alpha(x, y)_p$ is a power of $p$ while the order of $\alpha(x, y)_{p'}$ is prime to $p$. Since $pq \mid n$ for $p \neq q$, $n_{p'} \neq 1$ and $\alpha(x, y)_{p'} \neq 1$. Thus it follows that $\langle \epsilon_n \rangle = \langle \epsilon_{np} \rangle \times \langle \epsilon_{n_{p'}} \rangle$ hence $\alpha(x, y)_p \in \langle \epsilon_{np} \rangle$ and $\alpha(x, y)_{p'} \in \langle \epsilon_{n_{p'}} \rangle$.

Let $\alpha_1(x, y) = \alpha(x, y)_p$ and $\alpha_2(x, y) = \alpha(x, y)_{p'}$. Then it is easy to see that

$$
\alpha_1(x, y)\alpha_2(x, y) = \alpha_1(x, y)\alpha_2(x, y) = \alpha_1(x, y) \alpha_2(x, y) = \alpha_1(x, y) \alpha_2(x, y)
$$

for any $x, y \in \text{Gal}(R(\epsilon_n)/R)$. Thus due to the uniqueness of $p, p'$-part, it follows that $\alpha_1, \alpha_2 \in Z^2(R(\epsilon_n)/R, u(R(\epsilon_n)))$ on which the natural Galois action is defined, and the values of $\alpha_1$, $\alpha_2$ are contained in $\langle \epsilon_{np} \rangle$ and $\langle \epsilon_{n_{p'}} \rangle$ respectively. Consequently $\alpha = \alpha_1 \alpha_2$ and it follows from [4, (4.2.13)] or [7, (29.9)]) that $A = (R(\epsilon_n)/R, \alpha)$ is similar to $(R(\epsilon_n)/R, \alpha_1) \otimes (R(\epsilon_n)/R, \alpha_2)$. Now let $(R(\epsilon_n)/R, \alpha_i) = A_i$. Then $A = A_1 \otimes A_2$ and $[A_i] \in S''(R)$.

The converse of Theorem 2 follows immediately that if an $R$-algebra $A$ is decomposed into $A_1 \otimes A_2$ where $A_i$ are Schur $R$-algebras ($i = 1, 2$) then $[A]$ belongs to $S(R)$. In particular if $[A_i] \in S''(R)$ for $i = 1, 2$ then $[A] \in S''(R)$. For the same question with respect to $S'(R)$, we have the next theorem.
Theorem 4. Let $A = A_1 \otimes A_2$ with $[A_i] \in S'(R)$ ($i = 1, 2$). Then $[A] \in S'(R)$. Moreover if $A_i = (R(\varepsilon_{n_i})/R, \alpha_i)$ for a 2-cocycle $\alpha_i$ and if $(n_1, n_2) = 1$ then $A$ is a cyclotomic $R$-algebra with respect to the 2-cocycle $\alpha_1 \times \alpha_2$.

Proof. We denote the inflation map $H^2(R(\varepsilon_{n_i})/R) \to H^2(R(\varepsilon_{n_1}, \varepsilon_{n_2})/R)$ by $\inf_i$ for $i = 1, 2$, and we consider $\inf_i \alpha_i$ defined by

$$(\inf_i \alpha_i)(\theta_{i1}, \theta_{i2}) = \alpha_i(\tilde{\theta}_{i1}, \tilde{\theta}_{i2}),$$

where $\tilde{\theta}_{ij} = \theta_{ij} \text{Gal}(R(\varepsilon_{n_1}, \varepsilon_{n_2})/R(\varepsilon_i))$. Then following [7, (29.16)], it is easy to see that $(R(\varepsilon_{n_i})/R, \alpha_i)$ is similar to $(R(\varepsilon_{n_1}, \varepsilon_{n_2})/R, \inf_i \alpha_i)$, and it thus follows that

$$A_1 \otimes A_2 = (R(\varepsilon_{n_1})/R, \alpha_1) \otimes (R(\varepsilon_{n_2})/R, \alpha_2) \sim (R(\varepsilon_{n_1}, \varepsilon_{n_2})/R, \inf_1 \alpha_1 \inf_2 \alpha_2).$$

Thus $A$ is similar to $(R(\varepsilon_{n_1}, \varepsilon_{n_2})/R, \beta)$ for $\beta = \inf_1 \alpha_1 \inf_2 \alpha_2$, hence $[A] \in S'(R)$.

We now suppose that $(n_1, n_2) = 1$. Then for any $\theta_i \in \text{Gal}(R(\varepsilon_{n_1}, \varepsilon_{n_2})/R)$, $\theta_i$ can be written as $(\sigma_i, \tau_i)$ where $\sigma_i \in \text{Gal}(R(\varepsilon_{n_1})/R)$ and $\tau_i \in \text{Gal}(R(\varepsilon_{n_2})/R)$. Thus $\inf_1 \alpha_1(\theta_1, \theta_2) = \alpha_1(\sigma_1, \sigma_2)$ and $\inf_2 \alpha_2(\theta_1, \theta_2) = \alpha_2(\tau_1, \tau_2)$, this shows that

$$\inf_1 \alpha_1 \inf_2 \alpha_2(\theta_1, \theta_2) = \alpha_1 \times \alpha_2((\sigma_1, \tau_1), (\sigma_2, \tau_2)),$$

and $\beta = \inf_1 \alpha_1 \inf_2 \alpha_2 = \alpha_1 \times \alpha_2$.

This completes the proof. $\square$

Let $\alpha$ be a 2-cocycle in $Z^2(G, u(R))$ with trivial $G$-action on $R$ and let $\{u_g | g \in G\}$, $u_1 = 1$ denote an $R$-basis for the twisted group ring $RG^\alpha$ with multiplication $(ru_x)(su_y) = rs\alpha(x, y)u_{xy}$ and $\alpha(x, 1) = \alpha(1, x) = 1$ for all $r, s \in R, x, y \in G$. An Azumaya $R$-algebra $A$ is called a projective Schur $R$-algebra if it is a homomorphic image of $RG^\alpha$ for finite group $G$, and the set of similar classes of projective Schur algebras forms a group $PS(R)$.

Theorem 5. Suppose that $[A] \in PS(R)$ and $A$ is represented by a twisted group ring $RG^\alpha$. Assume $G = G_1 \times G_2$ with $|G_1|, |G_2| = 1$. Then $A$ can be decomposed into $A_1 \otimes A_2$ where $A_i$ is represented by $RG_i^\alpha_i$ for a 2-cocycle $\alpha_i \in Z^2(G_i, u(R))$, thus $[A_i] \in PS(R)$ for $i = 1, 2$.

Proof. Let $f$ be the surjective homomorphism $RG^\alpha \to A$, and let $\alpha_i \in Z^2(G_i, u(R))$ be the restrictions of $\alpha$ to $G_i$ for $i = 1, 2$. Since $|G_1|, |G_2| = 1$, it follows from [6, (2.3.14)] that the pairing $\phi_\alpha : G_1 \times G_2 \to u(R)$ defined by $\phi_\alpha(a, b) = \alpha(a, b)\alpha(b, a)^{-1}$ for $a \in G_1, b \in G_2$ is trivial, thus $\alpha(a, b) = \alpha(b, a)$ and $H^2(G_1 \times G_2, u(R))$ is isomorphic to $H^2(G_1, u(R)) \times H^2(G_2, u(R))$ that makes $\alpha$ correspond to $(\alpha_1, \alpha_2)$.
For any $g = g_1g_2 \in G$ ($g_i \in G_i$), we define

$$\psi : R G^{\alpha_1 \times \alpha_2} \rightarrow R G^\alpha, \quad \psi(w_g) = \alpha(g_1, g_2)u_g$$

where $w_g, u_g$ are bases for $RG^{\alpha_1 \times \alpha_2}$ and $RG^\alpha$ respectively. For $x = x_1x_2 \in G$, since

$$\psi(w_g)\psi(w_x) = \alpha(g_1, g_2)\alpha(x_1, x_2)\alpha(g_1g_2, x_1x_2)u_{gx} = \alpha(g_1, g_2x_1x_2)\alpha(g_2, x_1x_2)\alpha(x_1, x_2)u_{gx} = \alpha(g_1, x_1)\alpha(g_2, x_2)\alpha(g_1x_1, g_2x_2)u_{gx} = \alpha_1(g_1, x_1)\alpha_2(g_2, x_2)\alpha(g_1x_1, g_2x_2)u_{gx} = (\alpha_1 \times \alpha_2)(g, x)\alpha(g_1x_1, g_2x_2)u_{gx} = \psi((\alpha_1 \times \alpha_2)(g, x)w_{gx}) = \psi(w_gw_x),$$

it follows that $\psi$ is an isomorphism. Moreover due to [6, (5.1.1)], we have $RG^\alpha \cong R G^{\alpha_1 \times \alpha_2} = R(G_1 \times G_2)^{\alpha_1 \times \alpha_2} \cong R G_1^{\alpha_1} \otimes R G_2^{\alpha_2}$. Using the same notation $f$, write $f : R G_1^{\alpha_1} \otimes R G_2^{\alpha_2} \rightarrow A$ abusively and let $f_i$ be the restrictions of $f$ to $R G_i^{\alpha_i}$ and $A_i = f_i(R G_i^{\alpha_i})$. Then $A_1 \otimes A_2 = f(R G_1^{\alpha_1} \otimes R G_2^{\alpha_2}) \cong A$ which maps $a_1 \otimes a_2 \in A_1 \otimes A_2$ to $a_1a_2 \in A$. This implies that $A$ is a homomorphic image of $R(G_1 \times G_2)^{\alpha_1 \times \alpha_2}$, and $A_1$ and $A_2$ are Azumaya algebras because of Lemma 1. Hence it follows that $[A_i] \in PS(R)$. \hfill \Box

Moreover, the next corollary follows immediately.

**Corollary 6.** If $A = A_1 \otimes A_2$ with $[A_i] \in PS(R)$ and if $A_i$ is an image of $RG_i^{\alpha_i}$ for $i = 1, 2$, then $[A] \in PS(R)$ and $A$ is the image of $R(G_1 \times G_2)^{\alpha_1 \times \alpha_2}$.

### 3. Azumaya algebras

Consider projective Schur algebras which are epimorphic images of twisted group rings $RG^\alpha$ for $\alpha \in Z^2(G, u(R))$ that are separable algebras, i.e., $|G| \in u(R)$. Then the set $PS''(R)$

$$PS''(R) = \{[A] \in PS(R) \mid RG^\alpha \rightarrow A, |G| \in u(R), \bar{\alpha} \in H^2(G, u(R))\}$$

forms a subgroup of $PS(R)$ ([9, 1.2(7)]), and $S''(R) < PS''(R)$.

We restrict our attention to the situation that $[A] \in PS''(R)$ where $A$ is a homomorphic image of $RG^\alpha$ such that $G = G_1 \times G_2$. Then by Theorem 5 we have that $A = A_1 \otimes A_2$ where $A_i$ is represented by $RG_i^{\alpha_i}$ for $\alpha_i = res_0$, and moreover $[A_i] \in PS''(R)$. In [9], it was studied the situation that an Azumaya algebra $A$ that is represented by an epimorphic image of $RG^\alpha$ may also be obtained as the image of such a group ring which is moreover itself an Azumaya algebra.
In this section we find relationships between the Azumaya twisted group rings $RG^\alpha$ and $RG_i^\alpha$ that represents $A$ and $A_i$ respectively. We use Lemma 1 to discuss commutator subalgebras of each other. As an application we study a situation that when a Schur algebra is represented by certain twisted group ring which is itself an Azumaya algebra.

For $\alpha \in Z^2(G, u(R))$, an element $x \in G$ is said to be $\alpha$-regular if $\alpha(g, x) = \alpha(x, g)$ for all $g \in C_G(x)$ the centralizer of $x$. The set $Z(G)_\alpha = \{ x \in Z(G) \mid x \text{ is } \alpha\text{-regular} \}$ is called a root group of $G$ with respect to $\alpha$, and this group plays an important role for $RG^\alpha$ to be central. In fact, a necessary condition for $RG^\alpha$ to be central is that $Z(G)_\alpha$ is trivial. For an abelian group, this condition is also sufficient.

**Lemma 7 ([9, Theorem 2.2]).** For $[A] \in PS(R)$, we may assume that it is given by an epimorphism $RG^\alpha \to A$ where $Z(G)_\alpha = 1$. Hence if $[A] \in PS''(R)$ then $RG^\alpha$ itself is an Azumaya algebra.

**Theorem 8.** Let $G = G_1 \times G_2$ with $(|G_1|, |G_2|) = 1$. If $\alpha \in Z^2(G, u(R))$ and $\alpha_i \in Z^2(G_i, u(R))$ is the restriction of $\alpha$, then $\alpha$ is cohomologous to $\alpha_1 \times \alpha_2$ so that the corresponding root groups are equal. Moreover we have the following.

1. Let $x = x_1x_2 \in G$ with $x_i \in G_i$ ($i = 1, 2$). Then $x$ is $\alpha$-regular if and only if $x_i$ is $\alpha_i$-regular for $i = 1, 2$.
2. $Z(G)_\alpha = Z(G_1)_{\alpha_1} \times Z(G_2)_{\alpha_2}$ and $G/Z(G)_\alpha \cong G_1/Z(G_1)_{\alpha_1} \times G_2/Z(G_2)_{\alpha_2}$.

**Proof.** Define a map $t : G \to u(R)$ by $g \mapsto \alpha(g_1, g_2)$ for $g = g_1g_2$. Then
\[
\alpha(g, x)t(g)t(x) = \alpha(g_1, g_2x_1x_2)\alpha(g_2, x_1x_2)\alpha(x_1, x_2) = \alpha(g_1, x_1g_2x_2)\alpha(x_1g_2, x_2)\alpha(x_1, g_2) = t(g_2)(\alpha_1 \times \alpha_2)(g, x),
\]
where the second equality holds because of trivial pairing of $G_1$ and $G_2$ [6, (2.3.14)]. Thus $\alpha$ is cohomologous to $\alpha_1 \times \alpha_2$, and the $\alpha$-regularity is equal to the $\alpha_1 \times \alpha_2$-regularity by [6, (3.6.1)], so that $Z(G)_\alpha$ corresponds to $Z(G)_{\alpha_1 \times \alpha_2}$.

For $x = x_1x_2$, if $x$ is $\alpha$-regular and if $a \in C_{G_1}(x_1)$ then $xa = x_1ax_2 = ax_1x_2 = ax$, thus $\alpha(x, a) = \alpha(a, x)$. Moreover we have that
\[
\alpha_1(x_1, a)\alpha(x, x_2^{-1}) = \alpha(x_2^{-1}, a)\alpha(x, x_2^{-1}) = \alpha_1(a, x_1)\alpha(x, x_2^{-1}),
\]
hence $x_1$ is $\alpha_1$-regular, and similarly we get $x_2$ is $\alpha_2$-regular.

Conversely, assume that $x_i$ is $\alpha_i$-regular and choose any $g \in G$ such that $xg = gx$. Then $x_ig_i = g_ix_i$ and $c_i(x_i, g_i) = \alpha_i(g_i, x_i)$ for $i = 1, 2$. Thus $(\alpha_1 \times \alpha_2)(x, g) = \alpha_1(x_1, g_1)\alpha_2(x_2, g_2) = (\alpha_1 \times \alpha_2)(g, x)$, which proves that $x$ is $(\alpha_1 \times \alpha_2)$-regular, so that $x$ is $\alpha$-regular, this proves (1).
If $ab \in Z(G_1)_{\alpha_1} \times Z(G_2)_{\alpha_2}$ for $a \in Z(G_1)_{\alpha_1}$ and $b \in Z(G_2)_{\alpha_2}$, then we have $a \in Z(G_1)$, $b \in Z(G_2)$ and for any $g_1 \in G_1$ and $x_2 \in G_2$, $\alpha_1(a, g_1) = \alpha_1(g_1, a)$ and $\alpha_2(b, x_2) = \alpha_2(x_2, b)$. Thus $ab \in Z(G_1) \times Z(G_2) = Z(G)$. Furthermore $ab$ is $\alpha$-regular because

$$(\alpha_1 \times \alpha_2)(ab, l) = \alpha_1(a, l_1)\alpha_2(b, l_2) = \alpha_1(l_1, a)\alpha_2(l_2, b) = (\alpha_1 \times \alpha_2)(l, ab)$$

for any $l = l_1l_2 \in G$ with $l_i \in G_i$. Hence $ab \in Z(G)_{\alpha_1 \times \alpha_2} = Z(G)_{\alpha}$.

On the other hand, if $g \in Z(G)_{\alpha_1 \times \alpha_2}$ then $g \in Z(G)$ and $g$ is $(\alpha_1 \times \alpha_2)$-regular. Clearly $g = g_1g_2 \in Z(G_1) \times Z(G_2)$, $g$ is $\alpha$-regular and $g_i$ is $\alpha_i$-regular due to (1) hence it concludes that $g = g_1g_2 \in Z(G_1)_{\alpha_1} \times Z(G_2)_{\alpha_2}$. The remaining of (2) is clear. □

**Theorem 9.** Let $[A] \in PS''(R)$ where $A$ is represented by $RG^\alpha$. If $G = G_1 \times G_2$ such that $(|G_1|, |G_2|) = 1$ then $A = A_1 \otimes A_2$ where $A$ and $A_i$ are represented by Azumaya twisted group rings $RN^\beta$ and $RN_i^\beta$ respectively for some finite groups $N$ and $N_i$. Moreover $(RN^\beta)_{RN^\beta_1} = RN_2^\beta$ and $(RN^\beta)_{RN^\beta_2} = RN_1^\beta$.

**Proof.** It was proved in Theorem 5 that $RG^\alpha \cong R(G_1 \times G_2)^{\alpha_1 \times \alpha_2} \cong RG_1^{\alpha_1} \otimes RG_2^{\alpha_2}$ where $\alpha_i$ is the restriction of $\alpha$ to $G_i$ ($i = 1, 2$). And $A = A_1 \otimes A_2$ where each $A_i$ ($i = 1, 2$) is given by epimorphism $RG_i^{\alpha_i} \rightarrow A_i$. Moreover since $[A] \in PS''(R)$, each $[A_i]$ belongs to $PS''(R)$ for $i = 1, 2$. Thus due to Lemma 7 we may consider that the algebras $A$ and $A_i$ are epimorphic images of some twisted group rings which are Azumaya algebras.

In fact since $[A] \in PS''(R)$ we may assume that the representing twisted group ring $RG^\alpha$ for $A$ is separable, i.e., $|G| \in u(R)$. Thus if $Z(G)_{\alpha}$ is trivial then $RG^\alpha$ itself is central so that Azumaya. In case that $Z(G)_{\alpha}$ is not trivial, we consider the quotient group $G/Z(G)_{\alpha}$ following the idea in [9, (2.2)]. Then there is a 2-cocycle $\beta \in Z^2(G/Z(G)_{\alpha}, u(R))$ such that $A$ is given by epimorphism $R(G/Z(G)_{\alpha})^\beta \rightarrow A$ by Lemma 7. Replacing the representing pair $(G, \alpha)$ by $(G/Z(G)_{\alpha}, \beta)$ and continuing this process, we get a representation with trivial root group with respect to $\beta$. If we denote $G/Z(G)_{\alpha}$ by $N$ then $A$ is an epimorphic image of $RN^\beta$, $Z(N)_{\beta} = 1$ and $RN^\beta$ is an Azumaya algebra.

Since $G = G_1 \times G_2$ with $(|G_1|, |G_2|) = 1$, it follows from Theorem 8 that $G/Z(G)_{\alpha} \cong G_1/Z(G_1)_{\alpha_1} \times G_2/Z(G_2)_{\alpha_2}$. Thus if we let $N_i = G_i/Z(G_i)_{\alpha_i}$ then $N = N_1 \times N_2$ with $(|N_1|, |N_2|) = 1$. Regarding $N_i$ as a subgroup of $N$, let $\beta_i$ be the restriction of $\beta$ to $N_i$. Then it follows from Theorem 8 that $\beta$ is cohomologous to $\beta_1 \times \beta_2$ and $Z(N)_\beta = Z(N_1)_{\beta_1} \times Z(N_2)_{\beta_2}$. 
Because $Z(N)_\beta$ is trivial, so are $Z(N_i)_\beta = 1$ hence $RN^\beta_i$ is an Azumaya algebra. Moreover $RN^\beta_1 \otimes RN^\beta_2 \cong RN^\beta$ and the epimorphism $RN^\beta \to A$ gives rise to epimorphisms $RN^\beta_i \to A_i$, as is required.

As Azumaya algebras $RN^\beta$ and $RN^\beta_i$ ($i = 1, 2$), the isomorphism $RN^\beta_1 \otimes RN^\beta_2 \cong RN^\beta$ is defined by $u_{n_1} \otimes u_{n_2} \mapsto u_{n_1} u_{n_2} = \beta(n_1, n_2) u_n$ for $n = n_1 n_2 \in N$ by Theorem 5. Hence due to Lemma 1, the commutator subalgebras are

$$(RN^\beta)^{RN^\beta_1} = RN^\beta_2 \quad \text{and} \quad (RN^\beta)^{RN^\beta_2} = RN^\beta_1,$$

this completes the proof. \qed

Finally we study a relationship between Schur and projective Schur algebras. Clearly a Schur algebra is a projective Schur algebra with trivial 2-cocycle. Besides the trivial case, we may regard a Schur algebra as a projective Schur algebra with respect to non-trivial 2-cocycle $\alpha$ in the bijective correspondence between projective representations of a finite group $G$ and ordinary representations of the covering group $H$ of $G$. In this case $\alpha$ is determined by a group extension $H$ by $G$, however the values of $\alpha$ may not contained in the ring $R$.

**Theorem 10.** Let $[A] \in S(R)$ and $f : RG \to A$ be an epimorphism with finite group $G$. Assume the center $Z(G) \neq 1$. Then there is a finite group $H$, 2-cocycle $\alpha \in Z^2(H, u(R))$ such that $A$ is an epimorphic image of $RH^\alpha \to A$. Moreover if $G = G_1 \times G_2$ with $(|G_1|, |G_2|) = 1$ then we have the following.

1. $RH^\alpha \cong RH^\alpha_1 \otimes RH^\alpha_2$ for subgroups $H_i$ of $H$ and restrictions $\alpha_i$ of $\alpha$. Thus $A_i$ is an epimorphic image of $RH^\alpha_i$ such that $A = A_1 \otimes A_2$.
2. Furthermore if $[A] \in S''(R)$ then from $RH^\alpha \to A$, we may assume $RH^\alpha$ is central, if necessary, by taking quotient $H/Z(H)_\alpha$ until $Z(H)_\alpha$ is trivial. Hence $RH^\alpha_i$ are assumed to be Azumaya.

**Proof.** Consider a central group extension $E : 1 \to Z(G) \to G \to G/Z(G) \to 1$. Then due to the well known correspondence between ordinary representations of $G$ and projective representations of $G/Z(G)$, there is an epimorphism on twisted group ring of $G/Z(G)$ over $R$ with respect to a factor set $\lambda$ of the extension $E$. The factor set $\lambda \in Z^2(G/Z(G), Z(G))$, however the values of $\lambda$ need not belong to $R$.

Regarding $\lambda$ as an element in $Z^2(G/Z(G), u(RZ(G)))$, it was proved in [9] that $RZ(G)(G/Z(G))^\lambda$ is isomorphic to $RG$, thus the same notation $f$ can be used for the map $f : RZ(G)(G/Z(G))^\lambda \to A$. 
For any $x \in Z(G)$, $f(x)$ is central in $A$, hence in $u(R)$. If we let $\alpha = f\lambda$ then $\alpha$ belongs to $Z^2(G/Z(G), u(R))$ thus we have a surjective homomorphism $R(G/Z(G))^{\alpha} \to A$.

By setting $G/Z(G) = H$, (1) follows immediately from Theorem 5. Moreover if $[A] \in S''(R)$ then the representing group algebra $RG$ satisfies $|G| \in u(R)$, thus by regarding $[A]$ as in $PS''(R)$ (2) follows from Theorem 9.

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