COMPLETE SYSTEM OF FINITE ORDER FOR
CR MAPPINGS BETWEEN REAL ANALYTIC
HYPERSURFACES OF DEGENERATE LEVI FORM

SUNG-YEON KIM

ABSTRACT. We prove that the germ of a CR mapping $f$ between real analytic real hypersurfaces has a holomorphic extension and satisfies a complete system of finite order if the source is of finite type in the sense of Bloom-Graham and the target is $k$-nondegenerate under certain generic assumptions on $f$.

Introduction

This paper is concerned with construction of a complete system for CR mappings and with the real analyticity and the finiteness of CR mappings between real analytic CR manifolds of degenerate Levi form.

Let $M$ and $M'$ be germs of real analytic($C^\omega$) real hypersurfaces in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{N+1}$, $1 \leq n \leq N$, respectively, and $F = (f^1, \cdots, f^{N+1}) : M \to M'$ be a continuously differentiable CR mapping. Then $F$ is a solution of an overdetermined system

\[
\begin{align*}
L_i f_j &= 0 \quad i = 1, \ldots, n, \quad j = 1, \ldots, N + 1 \\
r' \circ F &= 0
\end{align*}
\]

(1)

where $\{L_i\}_{i=1,\ldots,n}$ is a basis of the CR structure bundle $H^{1,0}(M) := T^{1,0}(\mathbb{C}^{n+1}) \cap CT(M)$ of $M$ and $r'$ is a $C^\omega$ defining function of $M'$.

It is well known that if $M$ and $M'$ are Levi-nondegenerate hypersurfaces in $\mathbb{C}^{n+1}$ and $F : M \to M'$ is a CR equivalence, then $F$ extends holomorphically to a neighborhood of $M([12], [14], [16])$.

Moreover, $F$ is determined by 2-jet at a point. This follows from the fact that $F$ preserves the complete set of Chern-Moser invariants and thus

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$F$ satisfies the complete system of third order in the sense of Definition 5, see [5] and [7].

Let $r$ be a $C^ω$ defining function of $M$ such that $dr \neq 0$ on $M$ and let $\{L_j\}_{j=1,2,\ldots,n}$ be a $C^ω$ basis of $H^{1,0}(M)$. For an $n$-tuple of integers $\alpha = (\alpha_1, \ldots, \alpha_n)$ let $L^\alpha := L_1^{\alpha_1} \cdots L_n^{\alpha_n}$. We say that $M$ is $k$-nondegenerate at $p \in M$ if the vectors $\{L^\alpha r_Z(p) : |\alpha| \leq k\}$ span $\mathbb{C}^{n+1}$, where $r_Z = \left( \frac{\partial r}{\partial z_1}, \ldots, \frac{\partial r}{\partial z_{n+1}} \right)$.

The smallest such integer $k$ does not depend on the choice of the basis $L_1, \ldots, L_n$ and the defining function $r$. $M$ is 1-nondegenerate at $p$ if and only if $M$ is of nondegenerate Levi form at $p$.

In this paper we study the analyticity and finite determination of CR mappings to $C^ω$ hypersurface which is $k$-nondegenerate at a reference point. Our main results are the following:

**Theorem 1.** Let $M$ and $M'$ be $C^ω$ real hypersurfaces through the origin of $\mathbb{C}^{n+1}$ and $\mathbb{C}^{N+1}$, $1 \leq n \leq N$, respectively, and let $F : M \to M'$ be a CR mapping such that $F(0) = 0$. Let $\{L_j\}_{j=1,\ldots,n}$ be a $C^ω$ basis of $H^{1,0}(M)$. Suppose that $M$ is of finite type at 0 in the sense of Bloom-Graham and $M'$ is $k$-nondegenerate at 0. Suppose further that there exists a positive integer $K$ such that

$$\{L^\gamma (r'_Z \circ F)(0) : |\gamma| \leq K\}$$

span $\mathbb{C}^{N+1}$. Then $F$ extends holomorphically to a neighborhood of $0 \in M$ if $F \in C^K$.

**Theorem 2.** Let $M$ and $M'$ be $C^ω$ real hypersurfaces in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{N+1}$, $1 \leq n \leq N$, respectively as in Theorem 1 and let $F : M \to M'$ be a CR mapping as in Theorem 1. Then $F$ is determined by $4K$-jet at 0. Moreover, $F$ satisfies a complete system of order $4K + 1$.

If $M$ and $M'$ are of same dimension and $k$-nondegenerate, then a CR equivalence $F$ between $M$ and $M'$ extends holomorphically to a neighborhood of $M$ if $F$ is sufficiently differentiable([6],[2]) and is determined by $(k^2 + k)$-jet at a point([7]). A basic idea in [7] is to construct, by differentiating (1) repeatedly, a complete system of finite order, which determines all the derivatives of $F$ of order greater than or equal to $k^2 + k + 1$. More recently Zaitsev showed that $F$ is determined by $4k$-jet at a point by using the Segre varieties([17]).
Suppose $M$ and $M'$ are in normal coordinates at 0 (see §2). Then (2) span $\mathbb{C}^{N+1}$ if and only if the image

$$\{(a_1^K(z), \ldots, a_N^K(z)) : z \in \mathbb{C}^n\}$$

is not contained in a hyperplane of $\mathbb{C}^N$, where $a_j^K$, $j = 1, \ldots, N$, are $K$-th order Taylor series expansion of $\frac{\partial r^j}{\partial z_j}(F(z, 0), \overline{F(0)})$, $j = 1, \ldots, N$.

In [10], Hayashimoto showed using the method of complete system that if $M$ and $M'$ are real hypersurfaces in $\mathbb{C}^{n+1}$ and if $M'$ is of nondegenerate Levi form, then $F$ extends holomorphically to a neighborhood of $M$ and is determined by a finite jet at a point under the condition that the image

$$\{(a_1^K(z), \ldots, a_N^K(z)) : z \in \mathbb{C}^n\}$$

is not contained in a hyperplane of $\mathbb{C}^n$, which is equivalent to our hypotheses in Theorem 1.

In [2], Baouendi, Jacobowitz, and Treves replace the holomorphic structure on a neighborhood of $M$ by a new one whose real analytic structure is the same as the standard one. Then they extend each $f^j$ as a collection of holomorphic functions (in one variable in the case of hypersurface) to a wedge with edge $M$ using some identity that involves CR vector fields and a defining function of $M$. By the edge of the wedge theorem $F$ is real analytic on $M$ and hence extends holomorphically to a neighborhood of $M$ under the original holomorphic structure.

In this paper, we express $F$ in terms of the derivatives of $\overline{F}$ on $M$. We use this identity to prove Theorem 1 by the same argument as in §3 of [2]. To prove Theorem 2 we use the method of Segre variety as in [15], [1] and [17].

Holomorphic continuation of a CR mapping to a neighborhood of $C^\omega$ CR submanifold has been studied by many authors. In [3], Baouendi and Rothschild showed the holomorphic continuation of a CR mapping between $C^\omega$ real hypersurfaces of same dimension under certain nondegeneracy conditions.

To state their result we fix notations and definitions first:

Let $M = \{r = 0\} \subset \mathbb{C}^{n+1}$ be in normal coordinates. We can write $r((z, 0), (\overline{z}, 0)) = \sum a_\alpha(z)\overline{z}^\alpha$, where $z \in \mathbb{C}^n$. Then $M$ is said to be essentially finite at 0 if the $\mathbb{C}$-vector space $\mathcal{O}[z]/(a_\alpha(z))$ is of finite dimension, where $(a_\alpha(z))$ is the ideal generated by $\{a_\alpha(z)\}$ in $\mathcal{O}[z]$. The essential type of $M$ at 0 is the dimension of the complex vector space $\mathcal{O}[z]/(a_\alpha(z))$.

Suppose that $F : M \to M'$ is a $C^K$, $K \in \mathbb{N} \cup \{\infty\}$, CR mapping between $C^\infty$ real hypersurfaces in $\mathbb{C}^{n+1}$. Then there exists a (formal) holomorphic change of coordinates on a neighborhood of $M$ such that $F = J(Z) + O(|Z|^{K+1})$ if $K < \infty$ and $F = J(Z) + O(|Z|^{|l+1|}$ for all $l$ if $K = \infty$, where
$Z = (z, z_{n+1}) \in \mathbb{C}^{n+1}$ and $J(Z) = (j_1(Z), ..., j_{n+1}(Z))$ is an $(n+1)$-tuple of (formal) holomorphic functions in $Z$. We say that $F$ is of finite multiplicity at $0$ if $\mathcal{O}[z]/(J(z,0))$ is of finite dimension. The multiplicity of $F$ at $0$ is defined by the dimension of the complex vector space $\mathcal{O}[z]/(J(z,0))$.

**Theorem 3.** ([3]) Let $F : M \to M'$ be a smooth CR mapping, where $M$ and $M'$ are $C^\omega$ hypersurfaces in $\mathbb{C}^{n+1}$. Let $0 \in M$ and $F(0) = 0$. If either one of the following two conditions is satisfied, then $F$ is the restriction of a holomorphic mapping from a neighborhood of $0$ in $\mathbb{C}^{n+1}$ into $\mathbb{C}^{n+1}$.

i) The mapping $H$ is of finite multiplicity at $0$, and $M'$ is essentially finite at $0$.

ii) $M$ is essentially finite at $0$ and $F$ satisfies

$$dF(\mathcal{C}T_0 M) \not\subset H_0^{1,0}(M') \oplus H_0^{0,1}(M')$$ (Hopf Lemma property).

From Theorem 1 and Theorem 2 we have the following

**Corollary 4.** Let $F : M \to M'$ be a CR mapping, where $M$ and $M'$ are $C^\omega$ hypersurfaces in $\mathbb{C}^{n+1}$. Let $F(0) = 0$. Suppose $M'$ is $k$-nondegenerate at $0$. Then $F$ satisfies a complete system of finite order if one of the following conditions is satisfied:

i) The mapping $F$ is of finite multiplicity at $0$.

ii) $M$ is essentially finite at $0$ and $F$ satisfies

$$dF(\mathcal{C}T_0 M) \not\subset H_0^{1,0}(M') \oplus H_0^{0,1}(M').$$ (5)

In case i) $F$ satisfies a complete system of order $4k \cdot (\text{mult } F_0) + 1$ and in case ii) $F$ satisfies a complete system of order $4k \cdot (\text{ess type } M_0) + 1$, where $(\text{mult } F_0)$ is the multiplicity of $F$ at $0$ and $(\text{ess type } M_0)$ is the essential type of $M$ at $0$.

After finishing this paper, the author was informed of the B. Lamel’s result[11], in which he proved the real analyticity of $F$ in Theorem 1 in more general situation(generic CR manifolds) using ideas similar to ours.

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1. E. Cartan’s equivalence problem and the complete systems

In this section we briefly explain E. Cartan’s equivalence problem and the notion of complete system.

For a $C^\infty$ manifold $M$ with a geometric structure, construct a principal fiber bundle $P$ with the structure group $G$ over $M$ such that any structure preserving map $f$ lifts to $\tilde{f}$ for which the following diagram commutes:

\[
\begin{array}{ccc}
P_1 & \xrightarrow{f} & P_2 \\
\pi_{M_1} & \downarrow & \pi_{M_2} \\
M_1 & \xrightarrow{f} & M_2
\end{array}
\]

(6)

E. Cartan’s equivalence problem is to find necessary and sufficient conditions for the existence of $\tilde{f}$.

Suppose there exists a unique torsion-free connection $\omega$ on $M$. Then there is a unique vector-valued 1-form

\[
\omega : T(P) \to \mathbb{R}^K
\]

(7)

which is an isomorphism at each point, where $K = \text{dim}M + \text{dim}G = \text{dim}P$, such that there exists a local structure preserving map $f : M_1 \to M_2$ if and only if $\tilde{f}_* (\omega_2) = \omega_1$. Such $\omega$ is called a complete set of invariants for the equivalence problem. In this case, $f$ satisfies

\[
\frac{\partial^2 f^a}{\partial x^i \partial x^j} = h^a_{ij} \left( x, f, \frac{\partial f^b}{\partial x^k} : b, k = 1, \ldots, n \right)
\]

for all $i, j = 1, \ldots, n$, where $h^a_{ij}$ is a $C^\infty$ function in its arguments.

The concept of complete system is the generalization of the equation (3). We define the notion of complete system in jet-theoretical setting using the same notations as in [13].

Let $J^q(M, \mathbb{R}^N)$ be the $q$-th order jet space of $M \times \mathbb{R}^N$. Consider a system of differential equations of order $q$ for unknown functions $u = (u^1, \ldots, u^N)$ of independent variables $x = (x^1, \ldots, x^n)$

\[
\Delta_\lambda(x, u^{(q)}) = 0, \ \lambda = 1, \ldots, l,
\]

where $u^{(q)}$ is the $q$-th jet of $u$.

A complete system of order $k$ is defined as follows.

**Definition 5.** We say that (4) satisfies a complete system of order $k$ if there exist $C^\infty$ functions $H^q_j(x, u^{(p)} : p < k)$ in their arguments such that for any $C^k$ solution $u$ of (4),

\[
u_j^q = H^q_j(x, u^{(p)} : p < k)
\]

(10)
for all \(a = 1, \ldots, N\) and for all multi-indices \(J\) with \(|J| = k\).

Let \(\phi^a_I = du^a_I - \sum_{j=1}^n u^a_{i,j}dx^j\), \(a = 1, \ldots, N\), \(|I| \leq k - 2\), be the contact 1-forms defined on \(J^{k-1}(M, \mathbb{R}^N)\) and \(S_\Delta \subseteq J^{k-1}(M, \mathbb{R}^N)\) be the zero set of (4) and the derivatives of (4) in the space of partial derivatives of \(u\) up to order \(k - 1\). If (4) satisfies a complete system of order \(k\), then \(f\) is a solution of (4) if and only if \(x \to (\frac{\partial^{|I|}}{\partial x^{|I|}}(x), |I| \leq k - 1)\) is a maximal integral manifold of the distribution
\[
\phi^a_I = 0, \quad a = 1, \ldots, N, \quad |I| \leq k - 2
\]
and
\[
du^a_I - \sum_{j=1}^n H^a_{i,j}dx^j = 0, \quad |I| = k - 1,
\]
where \(H^a_{i,j} = D_j H^a_i\). In particular, we have

**Proposition 6.** Suppose (4) satisfies a complete system of order \(k\), then a solution \(f\) of (4) is uniquely determined by \((k - 1)\)-jet at a point and is \(C^\infty\) if \(f \in C^k\). Furthermore, if (4) is \(C^\omega\), then each \(H^a_j\) is \(C^\omega\) and \(f \in C^\omega\).

2. Proof of theorems and corollary

Let \(M, M'\) and \(F\) be as in Theorem 1.

In this section we use \(\alpha, \beta, \gamma, \cdots\) for \(n\)-tuples of integers and \(\alpha', \beta', \gamma'\cdots\) for \(N\)-tuples of integers.

We say that \(M\) is in normal coordinates if \(M\) is defined by
\[
z_{n+1} = R(z, \bar{z}) + \bar{z}_{n+1}P(z, \bar{z}, \bar{z}_{n+1})
\]
where \(z \in \mathbb{C}^n\) and \(R, P\) are holomorphic in their arguments such that
\[
R(z, 0) \equiv R(0, \bar{z}) \equiv 0
\]
and
\[
P(z, 0, \bar{z}_{n+1}) \equiv P(0, \bar{z}, \bar{z}_{n+1}) \equiv 1. \quad ([3])
\]

Since the smallest integer \(K\) which satisfies the hypotheses of Theorem 1 is independent of choice of \(\{L_i\}_{i=1, \ldots, n}\) and defining function \(r'\), we may assume that \(M\) and \(M'\) are in normal coordinates.

Now assume that \(M'\) is defined by
\[
\zeta_{N+1} = R'(\zeta, \bar{\zeta}) + \bar{\zeta}_{N+1}P'(\zeta, \bar{\zeta}, \zeta_{N+1}),
\]
where \(\zeta \in \mathbb{C}^N\) and \(R', P'\) are holomorphic in their arguments such that
\[
R'(\zeta, 0) \equiv R'(0, \bar{\zeta}) \equiv 0
\]
and
\[
P'(\zeta, 0, \zeta_{N+1}) \equiv P'(0, \bar{\zeta}, \zeta_{N+1}) \equiv 1. \quad ([3])
\]
where $\zeta \in \mathbb{C}^N$. Write

$$ R'(\zeta, \overline{\zeta}) = \sum_{j=1}^N a_j(\overline{\zeta}) \zeta_j + \sum_{|\alpha'| \geq 2} a_{\alpha'}(\overline{\zeta}) \zeta^{\alpha'} .$$

**Lemma 7.** There exist $\Phi_j$, $j = 1, \ldots, N+1$, which are holomorphic in their arguments such that

$$ f^j = \Phi_j(\overline{L'}F, |\gamma| \leq K) $$

for all $j = 1, \ldots, N+1$.

**Proof.** Let $F = (f, g) = (f^1, \ldots, f^N, g)$. Then we have

$$ g = \sum_{j=1}^N a_j(\overline{f}) f^j + \sum_{|\alpha'| \geq 2} a_{\alpha'}(\overline{f}) f^{\alpha'} + \overline{g}' P(f, \overline{f}, \overline{g}) $$

Applying $\overline{L'}$, $|\gamma| > 0$, to (15) we have

$$ 0 = \sum_{j=1}^N \overline{L'} a_j(\overline{f}) f^j + \sum_{|\alpha'| \geq 2} \overline{L'} a_{\alpha'}(\overline{f}) f^{\alpha'} + \overline{L'} \left( \overline{g}' P(f, \overline{f}, \overline{g}) \right) . $$

Since $\overline{L'} \overline{g}(0) = 0$ for all $\gamma$, we have

$$ \overline{L'} \left( r'_Z \circ F \right)(0) = (\overline{L'} a_1(\overline{f})(0), \ldots, \overline{L'} a_N(\overline{f})(0), 0) $$

for all $\gamma$ with $|\gamma| > 0$.

By the hypothesis of Theorem 1, there exist $\gamma_l$, $l = 1, \ldots, N$, such that $|\gamma_l| \leq K$ and $\{\overline{L}'(r'_Z \circ F)(0)\}_{l=1,\ldots,N}$ together with $r'_Z \circ F(0) = (0, \ldots, 0, 1)$ span $\mathbb{C}^{N+1}$. Then by the implicit function theorem we can solve the system

$$ g = \sum_{j=1}^N a_j(\overline{f}) f^j + \sum_{|\alpha'| \geq 2} a_{\alpha'}(\overline{f}) f^{\alpha'} + \overline{g}' P(f, \overline{f}, \overline{g}) $$

$$ 0 = \sum_{j=1}^N \overline{L}' a_j(\overline{f}) f^j + \sum_{|\alpha'| \geq 2} \overline{L}' a_{\alpha'}(\overline{f}) f^{\alpha'} + \overline{L}' \left( \overline{g}' P(f, \overline{f}, \overline{g}) \right), $$

$l = 1, \ldots, N$, for $f^j$, $j = 1, \ldots, N$, and $g = f^{N+1}$ in terms of $\overline{L}'F$, $|\gamma| \leq K$. This implies that there exist $\Phi_j$, $j = 1, \ldots, N+1$, which are holomorphic in their arguments such that

$$ f^j = \Phi_j(\overline{L}'F, |\gamma| \leq K) $$

for all $j = 1, \ldots, N+1$. \hfill \Box
Proof of Theorem 1
In [4], Baouendi and Treves showed that if $M$ is of finite type in the sense of Bloom-Graham, then there is one side of $M$ to which every CR distribution extends as a holomorphic function. Then by Lemma 7 together with Lemma 2.2 and Lemma 2.4 of [2] $F$ is $C^\omega$ on $M$ and hence extends holomorphically to a neighborhood of $M$.

Proof of Theorem 2
Let $\Phi = (\Phi_1, \cdots, \Phi_{N+1})$ and $Q(z, \bar{z}, \bar{z}_{n+1}) = R(z, \bar{z}) + \bar{z}_{n+1}P(z, \bar{z}, \bar{z}_{n+1})$. Since $F$ is holomorphic on a neighborhood of $M$, we can write (14) as
\begin{equation}
F(z, Q(z, \bar{z}, z_{n+1})) = \Phi(j^K \bar{F}(z, \bar{z}_{n+1}), j^{K+1}Q(z, \bar{z}, \bar{z}_{n+1}))
\end{equation}
\begin{equation}
:= \Phi(z, \bar{z}, z_{n+1}, j^K \bar{F}(z, \bar{z}_{n+1})).
\end{equation}

Let $\bar{z} = \chi$ and $z_{n+1} = \chi_{n+1}$. Then we can extend (19) as
\begin{equation}
F(z, Q(z, \chi, \chi_{n+1})) = \Phi(z, \chi, \chi_{n+1}, j^K \bar{F}(\chi, \chi_{n+1})).
\end{equation}

Passing to the $K$-th jet and taking its complex conjugate, we have
\begin{equation}
J^K \bar{F}(\chi, Q(\chi, z, z_{n+1})) = \Phi^K(\chi, z, z_{n+1}, j^{2K} F(z, z_{n+1})),
\end{equation}
where $\Phi^K$ is holomorphic in its arguments.

Substituting for $J^K \bar{F}$ in (20), we have
\begin{equation}
F(w, Q(w, \chi, Q(\chi, z, z_{n+1}))) = \Psi(z, z_{n+1}, \chi, w, j^{2K} F(z, z_{n+1})),
\end{equation}
where $\Psi$ is holomorphic in its arguments.

Also, we have
\begin{equation}
J^{2K} F(w, Q(w, \chi, Q(\chi, z, z_{n+1}))) = \Psi^{2K}(z, z_{n+1}, \chi, w, j^{4K} F(z, z_{n+1})),
\end{equation}
where $\Psi^{2K}$ is holomorphic in its arguments.

On the other hand, we have
\begin{equation}
F(u, Q(u, \tau, Q(\tau, w, w_{n+1}))) = \Psi(w, w_{n+1}, \tau, u, j^{2K} F(w, w_{n+1})),
\end{equation}
where $u \in \mathbb{C}^n$.

Lemma 8. There exist $(p, p_{n+1}) \in \mathbb{C}^{n+1}$ sufficiently close to 0 and holomorphic functions $\chi = \chi(z, z_{n+1})$, $\tau = \tau(u, u_{n+1})$ defined on a neighborhood $V$ of 0 such that
\begin{equation}
p_{n+1} = Q(p, \chi, \bar{Q}(\chi, z, z_{n+1}))
\end{equation}
and
\begin{equation}
u_{n+1} = Q(u, \tau, \bar{Q}(\tau, p, p_{n+1}))
\end{equation}
on $V$. 


Proof. It's enough to show that there exist \((p, p_{n+1}) \in \mathbb{C}^{n+1}\) and \(\chi^0, \tau^0 \in \mathbb{C}^n\) which are sufficiently small such that

\[
(27) \quad \frac{\partial}{\partial \chi_j} \left[ Q(p, \chi, \overline{Q}(\chi, z, z_{n+1})) \right] \bigg|_{(\chi^0, 0)} = \frac{\partial Q}{\partial \chi_j}(p, \chi^0, 0) \neq 0
\]

for some \(j = 1, \ldots, n\) and

\[
(28) \quad \frac{\partial}{\partial \tau_j} \left[ Q(u, \tau, \overline{Q}(\tau, p, p_{n+1})) \right] \bigg|_{(0, \tau^0)} = \frac{\partial Q}{\partial \tau_j}(\tau^0, p, p_{n+1}) \neq 0
\]

for some \(j = 1, \ldots, n\). Then by implicit function theorem we can prove the lemma.

But

\[
\frac{\partial Q}{\partial \chi_j}(p, \chi^0, 0) = \frac{\partial R}{\partial \chi_j}(p, \chi^0)
\]

and

\[
\frac{\partial Q}{\partial \tau_j}(\tau^0, p, p_{n+1}) = \frac{\partial R}{\partial \tau_j}(\tau^0, p) + p_{n+1} \frac{\partial P}{\partial \tau_j}(\tau^0, p, p_{n+1}).
\]

Since \(M\) is of finite type in the sense of Bloom-Graham, \(R \neq 0\). Hence we can choose \((p, p_{n+1}) \in \mathbb{C}^{n+1}\) and \(\chi^0, \tau^0 \in \mathbb{C}^n\) sufficiently close to 0 which satisfy the above conditions.

Then substituting for \(\chi = \chi(z, z_{n+1})\) and \(\tau = \tau(u, u_{n+1})\) in (23) and (24), respectively, and substituting for \(J^{2K}F(p, p_{n+1})\) in (24), we have

\[
(29) \quad F(u, u_{n+1}) = H(J^{4K}F(z, z_{n+1}), z, z_{n+1}, \overline{z}, \overline{z}_{n+1}, u, u_{n+1}, \overline{u}, \overline{u}_{n+1}),
\]

where \(H\) is holomorphic in its arguments.

Passing through \((4K + 1)\)-jet and taking \((u, u_{n+1}) = (z, z_{n+1}) \in M\), we have

\[
(30) \quad J^{4K+1}F(z, z_{n+1}) = H'(J^{4K}F(z, z_{n+1}), z, z_{n+1}, \overline{z}, \overline{z}_{n+1}),
\]

where \(H'\) is holomorphic in its arguments.

Proof of Corollary 4

Let \(M\) and \(M'\) be as in Corollary 4. Suppose \(M\) is essentially finite at 0 and \(F : M \to M'\) satisfies

\[
(31) \quad dF(CT_0M) \not\subseteq H^{0,0}_0(M') \oplus H^{0,1}_0(M') \quad \text{(Hopf Lemma property)}.
\]

In [3], Baouendi and Rothschild showed that \(F\) is of finite multiplicity at 0 and

\[
(32) \quad (\text{ess type } M_0) = (\text{mult } F_0) \cdot (\text{ess type } M'_0).
\]
If $M'$ is $k$-nondegenerate at 0, then
\begin{equation}
\mathcal{O}[\zeta]/(a_\alpha(\zeta)) = \mathcal{O}[\zeta]/(\zeta_1, \cdots, \zeta_n).
\end{equation}
Hence (ess type $M'_0) = 1$ and (mult $F_0) = (\text{ess type } M_0$).

Thus to prove Corollary 4, it's enough to show that if $F$ is of finite multiplicity at 0, then $M$ is of finite type and
\begin{equation}
\{ L^\gamma (r'_Z \circ F)(0) : |\gamma| \leq K \}
\end{equation}
span $\mathbb{C}^{n+1}$, where $K = k \cdot (\text{mult } F_0)$.

Let $F = (f, g) = (f^1, \cdots, f^n, g)$ and $(z)$ be the ideal of $\mathcal{O}[z]$ generated by $z$.

**Lemma 9.** If $F$ is of finite multiplicity at 0, then
\begin{equation}
det \left( \frac{\partial}{\partial z_i} h^j(z, 0) \right)_{i,j=1,\ldots,n} \neq 0,
\end{equation}
where $h^j$, $j = 1, \ldots, n$, are the (mult $F_0$)-th order Taylor series expansion of $f^j$.

**Proof.** Since we only deal with the Taylor series expansion of $F$, we may regard that $F$ is smooth.

Since $M$ and $M'$ are in normal coordinates, $\frac{\partial^{\mu}g}{\partial z^\alpha}(0) = 0$ for all $\alpha$. Hence $F$ is of finite multiplicity at 0 if and only if
\begin{equation}
dim_{\mathbb{C}} \mathcal{O}[z]/(f^1(z, 0), \cdots, f^n(z, 0)) = d < \infty,
\end{equation}
where $d = (\text{mult } F_0)$.

Now let $z^{\alpha} \in (z)^d$. We denote $\beta = (b_1, \cdots, b_n) < \alpha = (a_1, \cdots, a_n)$ if $b_j \leq a_j$ for all $j = 1, \ldots, n$ and $\beta \neq \alpha$.

If $|\alpha| \geq d$, then we can choose $\beta_l, l = 1, \ldots, d$, such that $0 < \beta_1 < \beta_2 \cdots < \beta_d = \alpha$. Suppose $z^{\alpha} \notin (f^1(z, 0), \cdots, f^n(z, 0))$. Then
\begin{equation}
sp < \{1, z^{\beta_l} : l = 1, \ldots, d\} > \cap (f^1(z, 0), \cdots, f^n(z, 0)) = \{0\},
\end{equation}
where $sp < \{1, z^{\beta_l} : l = 1, \ldots, d\} >$ is the $\mathbb{C}$-vector space spanned by $\{1, z^{\beta_l} : l = 1, \ldots, d\}$. Thus
\[
d = \dim_{\mathbb{C}} \mathcal{O}[z]/(f^1(z, 0), \cdots, f^n(z, 0))
= \dim_{\mathbb{C}} sp < \{1, z^{\beta_l} : l = 1, \ldots, d\} >
+ \dim_{\mathbb{C}} sp < \{z^\gamma : \gamma \neq \beta_l, l = 1, \ldots, d\} > / (f^1(z, 0), \cdots, f^n(z, 0))
\geq d + 1.
\]
Hence we conclude that
\begin{equation}
(z)^d \subset (f^1(z, 0), \cdots, f^n(z, 0)).
\end{equation}
Then we have
\[(h^1(z,0), \ldots, h^n(z,0)) \subset (f^1(z,0), \ldots, f^n(z,0)) + (z)^{d+1}\]
\[\subset (f^1(z,0), \ldots, f^n(z,0))\]
and
\[(39) \quad f^j(z,0) - h^j(z,0) \in (z)^{d+1} \subset (z) \cdot (f^1(z,0), \ldots, f^n(z,0))\]
for all \(j = 1, \ldots, n\). Thus by Nakayama's Lemma (see [3])
\[(40) \quad (h^1(z,0), \ldots, h^n(z,0)) = (f^1(z,0), \ldots, f^n(z,0)),\]
which implies
\[(41) \quad \dim_{\mathbb{C}} \mathcal{O}[z]/ (h^1(z,0), \ldots, h^n(z,0)) < \infty.\]
But in [3], it is proved that (35) holds if (41) holds. \(\square\)

Let \(h = (h^1, \ldots, h^n)\). By Lemma 9 we can show by following the same argument of the proof of Theorem 2 of [3] with \(h\) in place of \(F\) and with \(n = \text{modulo } (\xi)^{k(\text{mult } F_0)+1} \cdot (z)^{\text{mult } F_0+1}\) in place of \(n = \text{modulo } (\xi)^{k(\text{mult } F_0)+1}\) that \(M\) is essentially finite at 0 and hence of finite type at 0.

Now suppose there is a vector \(s = (s_1, \ldots, s_n) \in \mathbb{C}^n\) such that
\[(42) \quad \sum_{j=1}^{n} s_j a_j(h)(z,0) \equiv 0.\]
By Lemma 9 there exists \(z_0 \in \mathbb{C}^n\) sufficiently close to 0 and a neighborhood \(U\) of \(z_0\) such that \(h(\cdot,0): U \rightarrow h(U,0) \subset \mathbb{C}^n\) is a biholomorphic map onto an open set \(h(U,0)\) of \(\mathbb{C}^n\). Thus
\[(43) \quad \sum_{j=1}^{n} s_j a_j(\zeta) \equiv 0\]
for all \(\zeta \in h(U,0)\). But \(\sum_{j=1}^{n} s_j a_j(\zeta)\) is holomorphic in \(\zeta\), \(\sum_{j=1}^{n} s_j a_j(\zeta) \equiv 0\) on \(\mathbb{C}^n\).

Let
\[(44) \quad L'_j = \frac{\partial}{\partial \zeta_j} - \frac{r'_j}{r'_{n+1}} \frac{\partial}{\partial \zeta_{n+1}}, \quad j = 1, \ldots, n,\]
where \(r'_j = \frac{\partial r'}{\partial \zeta_j}, \quad j = 1, \ldots, n + 1\). Since \(M'\) is in normal coordinates, we have
\[(45) \quad r'_2'(0) = (0, \ldots, 0, 1)\)
and
\[
L'_{\gamma}(r_Z'(0)) = \left( \frac{\partial |\gamma| a_1}{\partial \zeta'}(0), \ldots, \frac{\partial |\gamma| a_n}{\partial \zeta'}(0), 0 \right)
\]
for all $|\gamma| > 0$.
This implies that $M'$ is $k$-nondegenerate at 0 if and only if
\[
\sum_{j=1}^{n} \bar{s}_j a_j(\zeta) \neq 0
\]
for all $\bar{s} = (\bar{s}_1, \ldots, \bar{s}_n) \neq 0$. Hence we conclude that
\[
\sum_{j=1}^{n} s_j a_j(h)(z, 0) \equiv 0
\]
if and only if $s = 0$.

Now let
\[
a_j(f)(z, 0) = \sum_{\alpha} c_{\alpha} z^{\alpha}
= \sum_{|\alpha| = m_j} c_{\alpha} z^{\alpha} + \sum_{|\alpha| > m_j} c_{\alpha} z^{\alpha},
\]
where $\sum_{|\alpha| = m_j} c_{\alpha} z^{\alpha} \neq 0$. Then $a_j(f)(z, 0) \equiv a_j(h)(z, 0)$ modulo $T^{m_j+1}$.
Hence if $\sum_{j=1}^{n} s_j a_j(h)(z, 0) \neq 0$, then $\sum_{j=1}^{n} s_j a_j(f)(z, 0) \neq 0$ modulo $T^{m+1}$, where $m = \max(m_1, \ldots, m_n) \leq k \cdot (\text{mult } F_0)$, which implies that the image
\[
\left\{ (a_1^K(z), \ldots, a_n^K(z)) : z \in \mathbb{C}^n \right\}
\]
is not contained in a hyperplane of $\mathbb{C}^n$ for $K = k \cdot (\text{mult } F_0)$ or equivalently
\[
\left\{ \mathcal{T}^{\gamma} (r_Z' \circ F)(0) : |\gamma| \leq K \right\}
\]
span $\mathbb{C}^{n+1}$, where $K = k \cdot (\text{mult } F_0)$.

References


Department of Mathematics
Pohang University of Science and Technology
Pohang 790-784, Korea
E-mail: sykim@euclid.postech.ac.kr