J. Korean Math. Soc. **38** (2001), No. 1, pp. 101–123

# CHARACTERIZATION OF STRICTLY OPERATOR SEMI-STABLE DISTRIBUTIONS

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ABSTRACT. For a linear operator $Q$ from $R^d$ into $R^d$ and $0 < b < 1$, the $(Q, b)$-semi-stability and the strict $(Q, b)$-semi-stability of probability measures on $R^d$ are defined. The $(Q, b)$-semi-stability is an extension of operator stability with exponent $Q$ on one hand and of semi-stability with index $\alpha$ and parameter $b$ on the other. Characterization of strictly $(Q, b)$-semi-stable distributions among $(Q, b)$-semi-stable distributions is made. Existence of $(Q, b)$-semi-stable distributions which are not translation of strictly $(Q, b)$-semi-stable distribution is discussed.

## 1. Introduction

Let $R^d$ be the $d$-dimensional Euclidean space. We understand that $R^d$ is the set of real column vectors with $d$ components with inner product $\langle x, y\rangle = \sum_{j=1}^{d} x_j y_j$ for $x = (x_j)_{1\le j\le d}$, $y = (y_j)_{1\le j\le d}$ and norm $|x| = \sqrt{\langle x, x\rangle}$. Let $End(R^d)$ be the set of linear operators (endomorphisms) from $R^d$ into $R^d$ and $Aut(R^d)$ be the set of invertible linear operator (automorphism) from $R^d$ onto $R^d$. Let $\mathcal{B}(R^d)$ be the collection of Borel sets in $R^d$. Let $\mathcal{P}(R^d)$ be the collection of probability measures (distributions) defined on $\mathcal{B}(R^d)$ and let $I(R^d)$ be the collection of infinitely divisible distributions defined on $\mathcal{B}(R^d)$. The characteristic function of $\mu \in \mathcal{P}(R^d)$ is denoted by $\widehat{\mu}(z)$, $z \in R^d$. The $t$-th convolution power of $\mu \in I(R^d)$ is denoted by $\mu^t$. For $\mu \in \mathcal{P}(R^d)$ and $T \in End(R^d)$, $T\mu \in \mathcal{P}(R^d)$ is defined by $(T\mu)(E) = \mu(\{x : Tx \in E\})$, $E \in \mathcal{B}(R^d)$. The delta distribution at $c$ is denoted by $\delta_c$. Let $M_+(R^d)$ be the class of linear operators on $R^d$ all of whose eigenvalues have positive real parts. For $Q \in End(R^d)$ and $b > 0$, we define $b^Q = \sum_{n=0}^{\infty}(n!)^{-1}(\log b)^n Q^n$. The indicator function of a set $E$ is

Received January 18, 2000.

2000 Mathematics Subject Classification: 60E07.

Key words and phrases: infinite divisibility, operator semi-stability, semi-stability, operator stability.

This work was supported by Korea Research Foundation.

denoted by $I_E(x)$. Let $0 < b < 1$ and $Q \in M_+(R^d)$ in this paper throughout. We call $\mu \in \mathcal{P}(R^d)$ $(Q,b)$-*semi-stable* if $\mu \in I(R^d)$ and there is $c \in R^d$ such that

$$\mu^b = b^Q \mu * \delta_c. \tag{1.1}$$

We call $\mu \in \mathcal{P}(R^d)$ *strictly* $(Q,b)$-*semi-stable* if $\mu \in I(R^d)$ and

$$\mu^b = b^Q \mu. \tag{1.2}$$

Any $\mu \in I(R^d)$ has the Lévy representation $(A, \nu, \gamma)$, which means

$$\widehat{\mu}(z) = exp\left\{ i\langle \gamma, z\rangle - \frac{1}{2}\langle Az, z\rangle + \int_{R^d} G(z,x)\nu(dx) \right\},$$

with $G(z,x) = e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle(1+|x|^2)^{-1}$. Here $\gamma \in R^d$, $A$ is a symmetric nonnegative-definite operator on $R^d$ satisfying $\nu(\{0\}) = 0$ and

$$\int |x|^2(1+|x|^2)^{-1}\nu(dx) < \infty.$$

These $A$, $\nu$, and $\gamma$ are uniquely determined by $\mu$, $A$ is called the Gaussian covariance of $\mu$ and $\nu$ is called the Lévy measure of $\mu$. It is proved in [2] that $\mu$ is $(Q,b)$-semi-stable if and only if the Lévy measure $\nu$ of $\mu$ is expressed as

$$\nu(E) = \int_{S_\Gamma} \lambda(d\xi) \int_0^\infty I_E(u^Q\xi)d\left(\frac{-H_\xi(u)}{u}\right), \quad E \in \mathcal{B}(R^d), \tag{1.3}$$

where

(a) $\lambda$ is a finite measure on $S_\Gamma$ defined in Section 2,
(b) $H_\xi(u)$ is a real-valued function being right-continuous in $u \in (0,\infty)$ and measurable in $\xi \in S_\Gamma$ such that $H_\xi(u)u^{-1}$ is decreasing(in the wide sense allowing flatness), $H_\xi(1) = 1$ and $H_\xi(bu) = H_\xi(u)$ for any $u$ and $\xi$.

The integral in (1.3) with respect to $d\left(\frac{-H_\xi(u)}{u}\right)$ is the Lebesgue- Stieltjes integral in $u$. This $\lambda$ is uniquely determined by $\nu$, and $H_\xi(u)$ is unique for $\lambda$-almost every $\xi$. We call $\lambda$ and $H_\xi(u)$ the *spherical component* and the $Q$-*radial component* of $\nu$ respectively. The representation (1.3) of the Lévy measure is close to Chorny [4]. A different representation of $\nu$ is obtained by Luczak [9], which will be given in Section 3.

The purpose of this paper is to give a characterization of strictly $(Q,b)$-semi-stable distribution and to discuss the relation between $(Q,b)$-semi-stability and strict $(Q,b)$-semi-stability.

In [14], Sharpe states that any operator stable distribution with exponent $Q$ is a translation of some strictly operator stable distribution with exponent $Q$ if 1 is not an eigenvalue of $Q$. But, his statement cannot generalize to operator semi-stable case. This is shown by Example 2.5 provided in Section 2. The following proposition is easily proved.

PROPOSITION 1.1. *Suppose that $b$ is not an eigenvalue of $b^Q$. Then, any $(Q,b)$-semi-stable distribution is a translation of strictly $(Q,b)$-semi-stable distribution.*

*Proof.* Let $\mu$ be a $(Q,b)$-semi-stable distribution on $R^d$ satisfying the hypothesis that $b$ is not an eigenvalue of $b^Q$. Since $b-b^Q$ maps $R^d$ one-to-one and onto $R^d$, there exists a unique $a \in R^d$ satisfying

$$(b-b^Q)a = c \quad \text{for all} \quad c \in R^d,$$

which gives that $\mu * \delta_{-a}$ is strictly $(Q,b)$-semi-stable. □

By Proposition 1.1, a main difficulty in the relation between $(Q,b)$-semi-stability and strict $(Q,b)$-semi-stability lies in the case where $b$ is an eigenvalue of $b^Q$.

This paper is organized as follows. In Section 2, the complex representation for strictly $(Q,b)$-semi-stable distributions on $R^d$ and characterization of translations of strictly $(Q,b)$-semi-stable distributions on $R^d$ are given. These results are obtained by using the decomposition of $C^d$ induced by $Q$ and the formula (1.3). Here $C^d$ is the set of complex column vectors with $d$ components. We then show that, in the case where 1 an eigenvalue of $Q$, there exists, for any given $Q$-radial component of Lévy measure, a $(Q,b)$-semi-stable distribution which is not a translation of a strictly $(Q,b)$-semi-stable distribution. We also discuss some examples. In Section 3, using the decomposition of $R^d$ induced by $b^Q$ and the representation for the Lévy measures formulated by Łuczak [9], we give another characterization of strictly $(Q,b)$-semi-stable distributions and translations of strictly $(Q,b)$-semi-stable distributions in real forms, and show that, in the case where $b$ is an eigenvalue of $b^Q$, there exists a $(Q,b)$-semi-stable distribution which is not a translation of a strictly $(Q,b)$-semi-stable distribution.

Operator semi-stable distributions are studied by Jajte [5], Łuczak [9], Chorny [4], and others. They are defined as limit distributions of subsequences via $\{n_j\}$ with $n_j/n_{j+1} \to b$ for some $b \in (0,1)$, of operator normalizations of partial sums $S_n$ of sequences of independent identically distributed random vectors. The operator normalization of $S_n$ here is $T_nS_n + c_n$ with $T_n \in Aut(R^d)$ and $c_n \in R^d$. We say that $\mu \in \mathcal{P}(R^d)$ is full if the support of $\mu$ is not contained in any $(d-1)$ -dimensional hyperplane. Assuming the fullness of $\mu$, they prove that $\mu$ is operator semi-stable if and only if $\mu$ is $(Q, b)$-semi-stable with some $0 < b < 1$ and some $Q \in Aut(R^d)$ satisfying the condition that the real parts of all eigenvalues are more than or equal to 1/2. We can similarly define strictly operator semi-stable distributions, restricting the normalization to $c_n = 0$. In the full case, similar characterization of strict operator semi-stability by strict $(Q, b)$-semi-stable stability is possible. The notion of operator semi-stable distribution is an extension of that of operator stable distribution of Sharpe [14] on the one hand, and of that of semi-stable distributions of Lévy [8], Shimizu [15], and Kruglov [7] on the other. Characterization of strictly operator stable distributions and strictly semi-stable distributions is respectively made by Sato [12] and Choi [1]. *Our results in this paper are extension of results in* [12] and [1]. Some properties of the associated Lévy processes are studied by Choi [1], and Choi and Sato [3], and Watanabe [16], and the role of the strictness in the long time behaviors is analyzed.

## 2. Complex characterization of strictly operator semi-stable distribution

We begin with some notation used in [11,12,13]. Let $\theta_1, \ldots, \theta_{q+2r}$, denote all distinct eigenvalues of $Q$. Let $\theta_j = \alpha_j + i\beta_j$ with $\alpha_j$ and $\beta_j$ are real. Let $q$ be the number of distinct real eigenvalues and $2r$ be the number of distinct non-real eigenvalues of $Q (q \geq 0, r \geq 0)$. We arrange the eigenvalues so that $\theta_1, \ldots, \theta_q$, are real if $q \geq 1$ and that $\theta_{q+1}, \ldots, \theta_{q+2r}$, are non-real and $\theta_j = \overline{\theta_{j+r}}$ for $q + 1 \leq j \leq q + r$ if $r \geq 1$. Let $f(\zeta)$ be the minimal polynomial of $Q$. That is, $f(\zeta)$ is the real polynomial of the smallest degree satisfying $f(Q) = 0$ with 1 as the coefficient of the highest term. Then

$$f(\zeta) = (\zeta - \theta_1)^{n_1} \ldots (\zeta - \theta_{q+2r})^{n_{q+2r}}.$$

Here $n_1, \ldots, n_{q+2r}$ are positive integers satisfying $n_j \leq n'_j$ for $1 \leq j \leq q + 2r$ and $n_j = n_{j+r}$ for $q + 1 \leq j \leq q + r$, where $n'_j$ is the multiplicity of the eigenvalue $\theta_j$ and $n'_1 + \cdots + n'_{q+2r} = d, n'_j = n'_{j+r}$ for $q + 1 \leq j \leq q + r$. Thus

$$f(\zeta) = f_1(\zeta)^{n_1} \ldots f_{q+r}(\zeta)^{n_{q+r}},$$

where

$$f_j(\zeta) = \begin{cases} \zeta - \theta_j & \text{for} \quad 1 \leq j \leq q \\ (\zeta - \alpha_j)^2 + {\beta_j}^2 & \text{for} \quad q+1 \leq j \leq q+r. \end{cases}$$

We write $W_j$ for the kernel of $f_j(Q)^{n_j}$ in $R^d$, $1 \leq j \leq q+r$. The projector onto $W_j$ in the direct sum decomposition

$$R^d = W_1 \oplus \cdots \oplus W_{q+r}$$

is written by $U_j$. We denote the kernel of $(Q - \theta_j)^{n_j}$ in $C^d$, $1 \leq j \leq q+2r$, by $V_j$, that is, $V_j$ is the eigenspace of $Q$ in the wide sense associated with the eigenvalue $\theta_j$ for $1 \leq j \leq q+2r$. Let $T_j$ be the projector onto $V_j$ in the decomposition

$$C^d = V_1 \oplus \cdots \oplus V_{q+2r}.$$

For $x \in R^d$, we see that

$$U_j x = T_j x + T_{j+r} x = T_j x + \overline{T_j x} \quad \text{for} \quad q+1 \leq j \leq q+r$$

and

$$U_j x = T_j x \quad \text{for} \quad 1 \leq j \leq q.$$

We denote

$$D_j = \{(Q - \theta_j)v : v \in V_j\} \quad \text{in} \quad C^d, \quad 1 \leq j \leq q+2r.$$

Let $P_j$ be a projector onto $D_j$ in $C^d$, $\quad 1 \leq j \leq q+2r$. We set

$$\begin{aligned} J &= \{j : 1 \leq j \leq q+2r \quad \text{satisfying} \quad b^{\theta_j} = b \quad \text{and} \quad \alpha_j > 1/2\}, \\ I &= \{j : 1 \leq j \leq q+2r \quad \text{satisfying} \quad b^{\theta_j} \neq b \quad \text{and} \quad \alpha_j > 1/2\}, \\ \Gamma &= \{j : 1 \leq j \leq q+r \quad \text{satisfying} \quad \alpha_j > 1/2\}. \end{aligned}$$

It is easily checked that if $j \in J$, then $\theta_j = 1 + i\frac{2n\pi}{\log b}$ for some integer $n$. Let $W_\Gamma = \oplus_{j \in \Gamma} W_j$, and let

$$S_\Gamma = \{\xi \in W_\Gamma : \mid \xi \mid = 1, \mid u^Q \xi \mid > 1 \quad \text{for all} \quad u > 1\}.$$

Any $x \in W_\Gamma$ is uniquely expressed as $x = u^Q \xi$ with $\xi \in S_\Gamma$ and $u \in (0, \infty)$.

We define

$$a(b, u, \xi) = \frac{1}{1 + |u^Q \xi|^2} - \frac{1}{1 + |(\frac{u}{b})^Q \xi|^2}.$$

We use $C_j, 0 \le j \le 7$, for constants independent of $\xi \in S_\Gamma$ and $u$. By calculation from the form of $a(b,u,\xi)$, we obtain that

$$|a(b,u,\xi)| \le C_0 |u^Q \xi|^2. \tag{2.1}$$

We write, for $x \ne 0$ in $R^d$, $\alpha(x) = \min\{\alpha_j : 1 \le j \le q+2r, T_j x \ne 0\}$. For $j$ such that $T_j x \ne 0$, we set $n(x,j) = \max\{n \ge 0 : (Q-\theta_j)^n T_j x \ne 0\}$. For $x \ne 0$ in $R^d$, we define $n(x) = \max\{n(x,j) : 1 \le j \le q+r, T_j x \ne 0, \alpha_j = \alpha(x)\}$ and $N = \max\{n_j : 1 \le j \le q+2r\}$. Let $\alpha^+ = \min\{\alpha_j : 1 \le j \le q+2r, \alpha_j > \frac{1}{2}\}$ and $\alpha^{++} = \max\{\alpha_j : 1 \le j \le q+2r, \alpha_j > \frac{1}{2}\}$. Then $\alpha^+ \le \alpha(\xi) \le \alpha^{++}$ for $\xi \in S_\Gamma$. From Lemma 4.1 in [12] (see Lemma 5.1 in [13] or Lemma 5.6 in [11]), we get that

$$|u^Q \xi| \le C_1 u^{\alpha(\xi)} |\log u|^{N-1} \quad \text{for} \quad 0 < u \le 1/e. \tag{2.2}$$

Hence, by (2.1) it follows that

$$|a(b,u,\xi)| \le C_2 u^{2\alpha(\xi)} |\log u|^{2N-2} \quad \text{for} \quad 0 < u \le 1/e. \tag{2.3}$$

Set

$$g_{j,k}(b,u,\xi) = u^{\theta_j} (k!)^{-1} (\log u)^k a(b,u,\xi)$$

for $1 \le j \le q+2r, k \ge 0$. For $\xi \in S_\Gamma$, define

(2.4)

$$g_0(b,\xi) = \sum_{j \in I} \int_0^\infty \sum_{k=0}^{n_j-1} (Q-\theta_j)^k T_j \xi g_{j,k}(b,u,\xi) d\left(\frac{-H_\xi(u)}{u}\right)$$

(2.5)

$$g_1(b,\xi) = \sum_{j \in J} \int_0^\infty \sum_{k=0}^{n_j-1} (Q-\theta_j)^k T_j \xi g_{j,k}(b,u,\xi) d\left(\frac{-H_\xi(u)}{u}\right),$$

using componentwise integrals of vector-valued functions.

LEMMA 2.1. *The functions $g_0(b,\xi)$ and $g_1(b,\xi)$ are well-defined, $R^d$-valued, bounded, and measurable on $S_\Gamma$.*

*Proof.* From (2.2) and (2.3), we get that

$$\begin{aligned}
&\int_0^{\frac{1}{e}} |u^Q \xi| |a(b,u,\xi)| d\left(\frac{-H_\xi(u)}{u}\right) \\
&\le C_3 \int_0^{\frac{1}{e}} u^{3\alpha(\xi)} |\log u|^{3N-3} d\left(\frac{-H_\xi(u)}{u}\right)
\end{aligned}$$

for any $\xi \in S_\Gamma$. We see that

$$\begin{aligned}&\int_{\frac{1}{e}b^{n+1}}^{\frac{1}{e}b^n} u^{3\alpha(\xi)}|\log u|^{3N-3} d\left(\frac{-H_\xi(u)}{u}\right)\\&\le |(n+1)\log b - 1|^{3N-3}\int_{\frac{1}{e}b^{n+1}}^{\frac{1}{e}b^n} u^{3\alpha(\xi)} d\left(\frac{-H_\xi(u)}{u}\right)\end{aligned}$$

for $n = 0, 1, \cdots$ . Let $L$ be a positive integer satisfying $1 \le e^{-1}b^{-L}$. Then we have that

$$\begin{aligned}&\int_{\frac{1}{e}b^{n+1}}^{\frac{1}{e}b^n} u^{3\alpha(\xi)} d\left(\frac{-H_\xi(u)}{u}\right)\\&= b^{3(n+1+L)(\alpha(\xi)-\frac{1}{3})}\int_{\frac{1}{e}b^{-L}}^{\frac{1}{e}b^{-L-1}} u^{3\alpha(\xi)} d\left(\frac{-H_\xi(u)}{u}\right)\\&\le b^{3(n+1+L)(\alpha^+-\frac{1}{3})}\int_{1}^{b^{-L-1}} u^{3\alpha(\xi)} d\left(\frac{-H_\xi(u)}{u}\right)\\&\le b^{2(n+1+L)(\alpha^+-\frac{1}{3})}b^{-3(L+1)\alpha(\xi)}\int_{1}^{\infty} d\left(\frac{-H_\xi(u)}{u}\right)\\&\le b^{2(n+1+L)(\alpha^+-\frac{1}{3})-3(L+1)\alpha^{++}}.\end{aligned}$$

Hence we get that, for any $\xi \in S_\Gamma$,

$$\begin{aligned}&\int_{0}^{\frac{1}{e}} u^{3\alpha(\xi)}|\log u|^{3N-3} d\left(\frac{-H_\xi(u)}{u}\right)\\&= \sum_{n=0}^{\infty}\int_{\frac{1}{e}b^{n+1}}^{\frac{1}{e}b^n} u^{3\alpha(\xi)}|\log u|^{3N-3} d\left(\frac{-H_\xi(u)}{u}\right)\\&\le C_4.\end{aligned}\tag{2.6}$$

Putting $l(u) = \frac{u}{1+u^2}$, we see that $l(u) \le 1/2$. This gives that, for any $\xi \in S_\Gamma$,

$$\begin{aligned}&\int_{\frac{1}{e}}^{\infty} |u^Q\xi||a(b,u,\xi)| d\left(\frac{-H_\xi(u)}{u}\right)\\&\le C_4\int_{\frac{1}{e}}^{\infty} d\left(\frac{-H_\xi(u)}{u}\right)\\&< C_5,\end{aligned}\tag{2.7}$$

because
$$\int_{\frac{1}{e}}^{\infty} d\left(\frac{-H_\xi(u)}{u}\right) = \frac{H_\xi(e^{-1})}{e^{-1}} \leq \frac{H_\xi(b^L)}{b^L} = b^{-L}.$$

It follows from (2.6) and (2.7) that

$$\int_0^{\infty} |u^Q\xi||a(b,u,\xi)|d\left(\frac{-H_\xi(u)}{u}\right) < C_6 \quad \text{for} \quad \xi \in S_\Gamma. \tag{2.8}$$

Since
$$u^Q T_j\xi = u^{\theta_j}\sum_{k=0}^{n_j-1}(k!)^{-1}(\log u)^k(Q-\theta_j)^k T_j\xi,$$

we have

$$\begin{aligned}\int_0^{\infty} &|u^Q T_j\xi||a(b,u,\xi)|d\left(\frac{-H_\xi(u)}{u}\right) \\ &= \int_0^{\infty} |\sum_{k=0}^{n_j-1} g_{j,k}(b,u,\xi)(Q-\theta_j)^k T_j\xi| d\left(\frac{-H_\xi(u)}{u}\right).\end{aligned} \tag{2.9}$$

Since $u^Q T_j = T_j u^Q$, we have $|u^Q T_j\xi| \leq C_7|u^Q\xi|$. Hence, using (2.8) and (2.9), we see that $g_0(b,\xi)$ and $g_1(b,\xi)$ are well-defined bounded functions on $S_\Gamma$. Their measurability is obvious. Since $T_j\xi = \overline{T_{j+r}\xi}$ for $q+1 \leq j \leq q+r$, we see that $g_0(b,\xi) = \overline{g_0(b,\xi)}$ and $g_1(b,\xi) = \overline{g_1(b,\xi)}$. That is, $g_0(b,\xi)$ and $g_1(b,\xi)$ are $R^d$-valued. □

For $j \in J$, $\xi \in S_\Gamma$ and $T_j\xi \neq 0$, we write,

$$g_{j,0}(b,\xi) = \int_0^{\infty} g_{j,0}(b,u,\xi)d\left(\frac{-H_\xi(u)}{u}\right).$$

This is well-defined, since the proof of Lemma 2.1 shows that

$$\int_0^{\infty} |g_{j,0}(b,u,\xi)|d\left(\frac{-H_\xi(u)}{u}\right) < \infty.$$

For $j \in J$, we see that if $\xi \in S_\Gamma$, then

$$\begin{aligned} g_{j,0}(b,\xi) &= \int_0^{\infty} u\cos(\frac{2n\pi}{\log b}\log u)a(b,u,\xi)d\left(\frac{-H_\xi(u)}{u}\right) \\ &\quad + i\int_0^{\infty} u\sin(\frac{2n\pi}{\log b}\log u)a(b,u,\xi)d\left(\frac{-H_\xi(u)}{u}\right)\end{aligned}$$

for some integer $n$. In fact, if $j \in J$, then $\theta_j = 1 + i\frac{2n\pi}{\log b}$ for some integer $n$.

LEMMA 2.2. *Let $j \in J$. Then there is an integer $n$ such that*

$$g_{j,0}(b,\xi) = \int_{[b,1)} u\cos(\frac{2n\pi}{\log b}\log u)d\left(\frac{-H_\xi(u)}{u}\right) + i\int_{[b,1)} u\sin(\frac{2n\pi}{\log b}\log u)d\left(\frac{-H_\xi(u)}{u}\right)$$

*for $\xi \in S_\Gamma$.*

*Proof.* Choose $n$ such that $\theta_j = 1 + i\frac{2n\pi}{\log b}$. Let $N_1$, $N_2$ be integers such that $-\infty < N_1 < N_2 < \infty$. Then we see that

$$\begin{aligned}
&\int_{[b^{N_2+1},b^{N_1})} u\cos(\frac{2n\pi}{\log b}\log u)a(b,u,\xi)d\left(\frac{-H_\xi(u)}{u}\right) \\
&= \sum_{k=N_1}^{N_2}\int_{[b^{k+1},b^k)} u\cos(\frac{2n\pi}{\log b}\log u)a(b,u,\xi)d\left(\frac{-H_\xi(u)}{u}\right) \\
&= \int_{[b,1)} \frac{u\cos(\frac{2n\pi}{\log b}\log u)}{1+|b^{N_2Q}u^Q\xi|^2}d\left(\frac{-H_\xi(u)}{u}\right) \\
&- \int_{[b,1)} \frac{u\cos(\frac{2n\pi}{\log b}\log u)}{1+|b^{(N_1-1)Q}u^Q\xi|^2}d\left(\frac{-H_\xi(u)}{u}\right)
\end{aligned}$$

since for any integer $k$, we have that

$$\begin{aligned}
&\int_{[b^{k+1},b^k)} u\cos(\frac{2n\pi}{\log b}\log u)a(b,u,\xi)d\left(\frac{-H_\xi(u)}{u}\right) \\
&= \int_{[b,1)} u\cos(\frac{2n\pi}{\log b}\log u)a(b,b^ku,\xi)d\left(\frac{-H_\xi(u)}{u}\right).
\end{aligned}$$

Hence, by letting $N_1 \longrightarrow -\infty$ and $N_2 \longrightarrow \infty$, it follows that

$$\begin{aligned}
&\int_0^\infty u\cos(\frac{2n\pi}{\log b}\log u)a(b,u,\xi)d\left(\frac{-H_\xi(u)}{u}\right) \\
&= \int_{[b,1)} u\cos(\frac{2n\pi}{\log b}\log u)d\left(\frac{-H_\xi(u)}{u}\right),
\end{aligned}$$

because that $|b^{(N_1-1)Q}u^Q\xi|^2 \longrightarrow \infty$ as $N_1 \longrightarrow -\infty$ and $|b^{N_2Q}u^Q\xi|^2 \longrightarrow 0$ as $N_2 \longrightarrow \infty$. A similar argument shows that

$$\int_0^\infty u\sin(\frac{2n\pi}{\log b}\log u)a(b,u,\xi)d\left(\frac{-H_\xi(u)}{u}\right)$$
$$= \int_{[b,1)} u\sin(\frac{2n\pi}{\log b}\log u)d\left(\frac{-H_\xi(u)}{u}\right). \qquad \square$$

THEOREM 2.1. *Let $\mu$ be a $(Q,b)$-semi-stable distribution on $R^d$ with Lévy representation $(A,\nu,\gamma)$. Let $\lambda$ and $H_\xi(u)$ be the spherical component and the $Q$-radial component of $\nu$. Then, $\mu$ is strictly $(Q,b)$-semi-stable if and only if*

$$(2.10) \qquad (b-b^Q)T_j\gamma = b\int T_j(g_0+g_1)(b,\xi)\lambda(d\xi) \quad \text{for} \quad 1 \le j \le q+2r.$$

*Proof.* Let $\mu$ be a $(Q,b)$-semi-stable distribution on $R^d$ satisfying (1.1) with Lévy representation $(A,\nu,\gamma)$. Then, $\mu$ is strictly $(Q,b)$-semi-stable if and only if

$$(2.11) \qquad b\gamma = b^Q\gamma + b\int_{S_\Gamma}\lambda(d\xi)\int_0^\infty u^Q\xi a(b,u,\xi)d\left(\frac{-H_\xi(u)}{u}\right).$$

As in (2.9), (2.11) is equivalent to

$$b\gamma = b^Q\gamma + b\int_{S_\Gamma}(g_0+g_1)(b,\xi)\lambda(d\xi).$$

Hence we get (2.10). $\square$

REMARK 2.1. In an operator stable case (see [12]), the above functions $g_0(b,\xi)$ and $g_1(b,\xi)$ are written in the following forms:

$$g_1(b,\xi) = \sum_{j\in J}\sum_{k=0}^{n_j-1}(Q-\theta_j)^k T_j\xi\{\sum_{l=0}^{k}(1-\theta_j)^{-l-1}$$
$$\times\int_0^\infty u^{\theta_j-1}((k-l)!)^{-1}(\log u)^{k-l}\frac{d}{du}\left(1+|u^Q\xi|^2\right)^{-1}du$$
$$-\sum_{m=0}^{k}(m!)^{-1}b^{\theta_j-1}(\log b)^m\sum_{l=m}^{k}(1-\theta_j)^{-l+m-1}$$
$$\times\int_0^\infty u^{\theta_j-1}((k-l)!)^{-1}(\log u)^{k-l}\frac{d}{du}\left(1+|u^Q\xi|^2\right)^{-1}du\},$$

and $g_0(b,\xi)$ equals the right-hand side of the above with $\sum_{j\in J}$ replaced by $\sum_{j\in I}$.

THEOREM 2.2. *Let $\mu$ be as in Theorem 2.1. Then $\mu$ is a translation of a strictly $(Q,b)$-semi-stable distribution if and only if*

$$\int_{S_\Gamma}(I-P_j)g_{j,0}(b,\xi)T_j\xi\lambda(d\xi)=0 \quad \text{for} \quad j\in J. \tag{2.12}$$

REMARK 2.2. The condition (2.12) in Theorem 2.2 is equivalent to

$$\int_{S_\Gamma}T_jg_1(b,\xi)\lambda(d\xi)\in D_j \quad \text{for} \quad j\in J. \tag{2.13}$$

*Proof of Theorem 2.2.* Assume that, for some $c\in R^d$, $\mu*\delta_c$ is strictly $(Q,b)$-semi-stable. Then by Theorem 2.1, we have that

$$(b-b^Q)\sum_{j\in I}T_j(\gamma+c)=b\int_{S_\Gamma}g_0(b,\xi)\lambda(d\xi)$$

and

$$(b-b^Q)\sum_{j\in J}T_j(\gamma+c)=b\int_{S_\Gamma}g_1(b,\xi)\lambda(d\xi).$$

If $j\in J$, then $(b-b^Q)T_j=-b\sum_{k=1}^{n_j-1}(k!)^{-1}(\log b)^k(Q-\theta_j)^kT_j$. Therefore (2.13) is a necessary condition for a translation of a strictly $(Q,b)$-semi-stable distribution.

Conversely, suppose that (2.13) holds. Set $\widehat{V}_j=\text{Kernel}(Q-\theta_j), j=1,\cdots,q+2r$. Let $\widetilde{V}_j$ be the orthogonal complement of $\widehat{V}_j$ in $V_j$. Then $V_j=\widetilde{V}_j\oplus\widehat{V}_j$. The restriction of $Q-\theta_j$ to $\widetilde{V}_j$ has image $D_j$ and kernel $\{0\}$, which implies $dim(D_j)=dim(\widetilde{V}_j)$. If $v_j\in V_j$ and $v_j\neq 0$, then there is a nonnegative integer $n_0(v_j)$ such that $(Q-\theta_j)^{n_0(v_j)}v_j=0$ and $(Q-\theta_j)^lv_j\neq 0$ for all $l=0,\cdots,n_0(v_j)-1$. We have $1\le n_0(v_j)\le n_j$ and $\{(Q-\theta_j)^lv_j, l=0,\cdots,n_0(v_j)-1\}$ is a linearly independent system.

We assert that $b-b^Q$ maps $\widetilde{V}_j$ one-to-one and onto $D_j$ for $j\in J$.

To show the assertion, let $j\in J$, $\widetilde{v}_j\neq 0$ and $\widetilde{v}_j\in\widetilde{V_j}$. Denote $n_0(\widetilde{v}_j)=l_j$. Suppose that $\widetilde{v}_j\neq 0$ and $(b-b^Q)\widetilde{v}_j=0$. Then we see that $2\le l_j\le n_j$ and

$$(b-b^Q)\widetilde{v}_j=-b\sum_{k=1}^{l_j-1}(k!)^{-1}(\log b)^k(Q-\theta_j)^k\widetilde{v}_j=0.$$

This contradicts the linear independence of $(Q-\theta_j)^l\widetilde{v}_j, l=0,\cdots,l_j-1$. Hence, $b-b^Q$ is one-to-one as a map from $\widetilde{V_j}$ to $D_j$. Since $dim(D_j) = dim(\widetilde{V}_j)$, we see that $b-b^Q$ maps $\widetilde{V}_j$ onto $D_j$ for $j\in J$ as desired. Hence, for $j\in J$, we can choose a unique $\gamma_j\in\widetilde{V}_j$ such that

$$(b-b^Q)\gamma_j = bT_j\int_{S_\Gamma} g_1(b,\xi)\lambda(d\xi) \quad \text{for} \quad j\in J.$$

Given $j\notin J$, let $v_j\in V_j$, $v_j\neq 0$, and $l_j = n_0(v_j)$. Then we see that $(b-b^{\theta_j})v_j\neq 0$, which implies that

$$(b-b^Q)v_j = (b-b^{\theta_j})v_j - b^{\theta_j}\sum_{k=1}^{n_j-1}(k!)^{-1}(\log b)^k(Q-\theta_j)^k(v_j)\neq 0$$

by the linear independence of $(Q-\theta_j)^l v_j, l=0,\cdots,l_j-1$. Hence we see that, for $j\notin J$, $b-b^Q$ maps $V_j$ one-to-one and onto $V_j$. Thus, we can choose a unique $\gamma_j\in V_j$ such that

$$(b-b^Q)\gamma_j = bT_j\int_{S_\Gamma} g_0(b,\xi)\lambda(d\xi) \quad \text{for} \quad j\notin J.$$

We set $c=\sum_{j=1}^{q+2r}\gamma_j$. Noticing that, for $q+1\le j\le q+r$, $g_{j+r,k}(b,u,\xi) = \overline{g_{j,k}(b,u,\xi)}$ and $T_{j+r}x = \overline{T_j x}$ for $x\in R^d$, and using the uniqueness of $\gamma_j$, we see that, for $q+1\le j\le q+r$, $\gamma_{j+r} = \overline{\gamma_j}$. Hence $c\in R^d$. We have

$$(b-b^Q)c = (b-b^Q)\sum_{j=1}^{q+2r}\gamma_j = b\int_{S_\Gamma}(g_0+g_1)(b,\xi)\lambda(d\xi).$$

Notice that $\gamma_j = 0$ for $j\notin J\cup I$. We conclude that $\mu * \delta_{-\gamma+c}$ is strictly $(Q,b)$-semi-stable by Theorem 2.1. Hence (2.13) is a sufficient condition for a translation of a strictly $(Q,b)$-semi-stable distribution.

In order to complete the proof, it is enough to show that the condition (2.13) is equivalent to (2.12). We see that the condition (2.13) is equivalent to

$$\int_{S_\Gamma}(I-P_j)T_j g_1(b,\xi)\lambda(d\xi) = 0.$$

Since $T_j g_1(b,\xi) - g_{j,0}(b,\xi)T_j\xi\in D_j$, we see that

$$(I-P_j)T_j g_1(b,\xi) = (I-P_j)g_{j,0}(b,\xi)T_j\xi.$$

Hence the condition (2.13) in Remark 2.2 is equivalent to (2.12). The proof is complete. $\square$

THEOREM 2.3. *Assume that* 1 *is an eigenvalue of* $Q$. *Suppose that we are given a real-valued function* $H_\xi(u)$ *of* $u \in (0,\infty)$ *and* $\xi \in S_\Gamma$ *satisfying the condition* $(b)$. *Then, there exists a* $(Q,b)$*-semi-stable distribution on* $R^d$ *such that the* $Q$*-radial component of the Lévy measure of* $\mu$ *is the given* $H_\xi(u)$ *and that* $\mu$ *is not a translation of any strictly* $(Q,b)$*-semi-stable distribution.*

*Proof.* We may assume, without loss of generality, that $\theta_1 = 1$. We have $\Gamma \neq \emptyset$, since $1 \in \Gamma$. We choose $x_0 \in W_1$ so that $(Q-1)^{n_1-1}x_0 \neq 0$. Put $u_0 = \sup\{u > 0 : u^Q x_0 \leq 1\}$, then $0 < u_0 < \infty$. Setting $u_0^Q x_0 = \xi_0$, we see that $\xi_0 \in W_1$, $\xi_0 \in S_\Gamma$ and $(Q-1)^{n_1-1}\xi_0 \neq 0$. For $\xi \in S_\Gamma$, we have that

$$g_{1,0}(b,\xi) = \int_{[b,1)} ud\left(\frac{-H_\xi(u)}{u}\right)$$

by Lemma 2.2. Hence we see that $g_{1,0}(b,\xi_0) \neq 0$, which implies that

$$(Q-1)^{n_1-1}U_1 g_1(b,\xi_0) = g_{1,0}(b,\xi_0)(Q-1)^{n_1-1}\xi_0 \neq 0.$$

Hence we have that $U_1 g_1(b,\xi_0) \notin D_1$. Choose a $\delta$-distribution at $\xi_0$ as $\lambda$. Suppose that $\nu$ satisfies (1.3). Let $\mu$ be an infinitely divisible distribution with Lévy representation $(0,\nu,0)$. We see that $\mu$ is $(Q,b)$-semi-stable. On the other hand, we see that

$$U_1 \int_{S_\Gamma} g_1(b,\xi)\lambda(d\xi) = U_1 g_1(b,\xi_0) \notin D_1,$$

which implies that $\mu$ is not a translation of a strictly $(Q,b)$-semi-stable distribution by Theorem 2.2. $\square$

REMARK 2.3. Theorem 2.3 is not true if we make the assumption that $b$ is an eigenvalue of $b^Q$ instead of the assumption that 1 is an eigenvalue of $Q$. The following Example 2.1 shows it.

EXAMPLE 2.1. Let $d = 2$ and let $Q = \begin{pmatrix} 1 & \frac{2\pi}{\log b} \\ -\frac{2\pi}{\log b} & 1 \end{pmatrix}$. Put $V_1 = \text{Kernel}(Q-(1-i\frac{2\pi}{\log b}))$ and $V_2 = \text{Kernel}(Q-(1+i\frac{2\pi}{\log b}))$. Then we have that $D_1 = \{(Q-$

$(1-i\frac{2\pi}{\log b}))v : v \in V_1\} = \{0\}$ and $D_2 = \{(Q-(1+i\frac{2\pi}{\log b}))v : v \in V_2\} = \{0\}$. Let us take $A > 0$ and $0 < b < 1$ such that $\mid \frac{4}{\log b} \mid < A$. For any integer $n$, let

$$R_{2n} = \{u : b^{(2n+1)/4} < u \leq b^{(2n)/4}\}$$

and

$$R_{2n+1} = \{u : b^{(2n+2)/4} < u \leq b^{(2n+1)/4}\}.$$

For any $\xi \in S_\Gamma$, we define $H_\xi(u)$ for $u > 0$ by the formula

$$H_\xi(u) = \begin{cases} \frac{1}{A+1}(-\frac{4}{\log b}\log u + 2n + 1 + A) & \text{for} \quad u \in R_{2n} \\ \frac{1}{A+1}(\frac{4}{\log b}\log u - 2n - 1 + A) & \text{for} \quad u \in R_{2n+1} \end{cases}$$

for any $n$. Then we see that $H_\xi(u)$ is continuous for $u > 0$ satisfying the condition (b). By calculation from the form of $H_\xi(u)u^{-1}$, we obtain that

$$\int_{[b,1)} u\cos(\frac{-2\pi}{\log b}\log u)d\left(\frac{-H_\xi(u)}{u}\right) = 0$$

and

$$\int_{[b,1)} u\sin(\frac{-2\pi}{\log b}\log u)d\left(\frac{-H_\xi(u)}{u}\right) = 0.$$

According to Lemma 2.2, this gives that $g_{1,0}(b,\xi) = 0$ and $g_{2,0}(b,\xi) = 0$. This shows that any $(Q,b)$-semi-stable distribution with Lévy representation $(A,\nu,\gamma)$ is a translation of a strictly $(Q,b)$-semi-stable distribution by Theorem 2.2 if the above $H_\xi(u)$ is the $Q$-radial component of the Lévy measure $\nu$.

EXAMPLE 2.2. Let $j \in J$. Assume that the multiplicity of $\theta_j$ in the minimal polynomial of $Q$ is 1. In this case, from the fact that $D_j = \{0\}$, the condition (2.13) for this $j$ in Remark 2.2 is written as

$$\int g_{j,0}(b,\xi)T_j\xi\lambda(d\xi) = 0.$$

EXAMPLE 2.3. For $Q = I$, we have that

$$\int_S g_{1,0}(b,\xi)\xi\lambda(d\xi) = 0$$

as the condition (2.12), where $S$ is the unite sphere. In this case, by Lemma 2.2, the above condition is equivalent to

$$\int_S \xi\lambda(d\xi)\int_{[b,1)} ud\left(\frac{-H_\xi(u)}{u}\right) = 0,$$

which is the condition for the Lévy measure of a strictly semi-stable distribution with exponent 1.

EXAMPLE 2.4. Assume that $H_\xi(u) = 1$. Then this is the operator stable case, and if $j \in J$, then (2.12) gives

$$\int_{S_\Gamma}(I-P_j)T_j\xi\lambda(d\xi)\int_b^1 \frac{\cos(\frac{2n\pi}{\log b}\log u) + i\sin(\frac{2n\pi}{\log b}\log u)}{u}du = 0$$

for some integer $n$ by Lemma 2.2. Assume that 1 is an eigenvalue of $Q$ and let $\theta_1 = 1$. Since

$$\int_b^1 u^{-1}\{\cos(\frac{2n\pi}{\log b}\log u) + i\sin(\frac{2n\pi}{\log b}\log u)\}du = 0$$

for $n \neq 0$, the condition (2.12) gives

$$\int_{S_\Gamma}(I-P_1)T_1\xi\lambda(d\xi)\int_b^1 u^{-1}du = 0.$$

This is written as $\int_{S_\Gamma}(I-P_1)U_1(\xi)\lambda(d\xi) = 0$.

EXAMPLE 2.5. Let $d = 2$ and $Q$, $V_1$, $V_2$ be as in Example 2.1. Put $\xi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then we have that $T_1\xi_0 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2}i \end{pmatrix}$, $T_2\xi_0 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2}i \end{pmatrix}$. We can choose a positive number $A$ such that $A(|\frac{2\pi}{\log b}|+1) < 1$. Suppose that

$$\nu(B) = \int_0^\infty I_B(u^Q\xi_0)d\left(\frac{-H_{\xi_0}(u)}{u}\right), \quad B \in \mathcal{B}(R^d)$$

and $H_{\xi_0}(u) = A\cos(\frac{2\pi}{\log b}\log u) + 1$. Then the condition (b) is satisfied. We consider $\mu$ with Lévy representation $(0, \nu, 0)$. Direct computation shows that

$$\int_b^1 u\cos(\frac{-2\pi}{\log b}\log u)d\left(\frac{-H_{\xi_0}(u)}{u}\right) = \frac{A}{2}(-\log b)$$

and

$$\int_b^1 u\sin(\frac{-2\pi}{\log b}\log u)d\left(\frac{-H_{\xi_0}(u)}{u}\right) = A\pi.$$

Thus by Lemma 2.2, we conclude that

$$\begin{aligned} &T_1 g_1(b,\xi_0) \\ &= \int_b^1 u\{\cos(\frac{-2\pi}{\log b}\log u) + i\sin(\frac{-2\pi}{\log b}\log u)\}T_1\xi_0 d\left(\frac{-H_{\xi_0}(u)}{u}\right) \\ &\neq 0. \end{aligned}$$

This means that

$$T_1 g_1(b,\xi_0) \notin D_1.$$

Hence, by Theorem 2.2, $\mu$ is not a translation of a strictly $(Q,b)$-semi-stable distribution.

## 3. Real characterization of strictly operator semi-stable distribution

From now on we will use a norm $|\cdot|_Q$ which is defined by

$$|\, x\,|_Q = \int_0^1 \frac{|\, u^Q x\,|}{u} du, \quad x \in R^d.$$

This norm is used in [6]. Let $B = b^Q$, where $Q$ and $b$ are as in the preceding section. Then, we note that $b^{\theta_1}, \ldots, b^{\theta_{q+2r}}$, are eigenvalues of $B$, which are not necessarily distinct. Following [10], let $\eta_1, \ldots, \eta_{p+2s}, (1 \le j \le p+2s)$ be all distinct eigenvalues of $B$, where $p$ is the number of distinct real eigenvalues, $2s$ is the number of distinct non-real eigenvalues of $B(p \ge 0, s \ge 0)$ and $\eta_{j+r} = \overline{\eta_j}$ for $p+1 \le j \le p+s$. Let $\eta_j = \sigma_j + i\rho_j$ with $\sigma_j$ and $\rho_j$ being real. Let $h(\zeta)$ be the minimal polynomial of $B$ and

$$h(\zeta) = (\zeta - \eta_1)^{m_1} \ldots (\zeta - \eta_{p+2s})^{m_{p+2s}}.$$

Thus

$$h(\zeta) = h_1(\zeta)^{m_1} \ldots h_{p+s}(\zeta)^{m_{p+s}},$$

where

$$h_j(\zeta) = \begin{cases} \zeta - \eta_j & \text{for} \quad 1 \le j \le p \\ (\zeta - \sigma_j)^2 + {\rho_j}^2 & \text{for} \quad p+1 \le j \le p+s. \end{cases}$$

We write $Y_j$ for the kernel of $h_j(B)^{m_j}$ in $R^d$, $1 \leq j \leq p+s$. The projector onto $Y_j$ in the direct sum decomposition

$$R^d = Y_1 \oplus \cdots \oplus Y_{p+s}$$

is written by $\Phi_j$. Denote

$$Z_j = \mathrm{Kernel}(B - \eta_j)^{m_j} \quad \text{in} \quad C^d, \quad 1 \leq j \leq p+2s.$$

We have

$$C^d = Z_1 \oplus \cdots \oplus Z_{p+2s}.$$

Let $\Psi_j$ be the projector onto $Z_j$ in the above decomposition. We set

$$\Lambda = \{j : 1 \leq j \leq p+s \quad \text{satisfying} \quad |\eta_j| < b^{1/2}\}.$$

Let $Y_\Lambda = \oplus_{j\in\Lambda} Y_j$, let

$$S_\Lambda = \{x \in Y_\Lambda : \mid x \mid_Q \leq 1 \quad \text{and} \quad \mid B^{-1}x \mid_Q > 1\}$$

and $\mathcal{B}(S_\Lambda)$ as the class of Borel sets in $S_\Lambda$. Assume that $b$ is an eigenvalue of $B$. For convenience, let $\eta_1 = b$. Then we have

$$Y_1 = \{x \in R^d : (B-b)^{m_1} x = 0\}.$$

We denote

$$G_1 = \{(B-b)v : v \in Y_1\}.$$

In [9] Łuczak shows that $\mu$ is $(Q, b)$-semi-stable if and only if the Lévy measure $\nu$ of $\mu$ has the form

$$\nu(E) = \int_{S_\Lambda} \sum_{n=-\infty}^{\infty} b^n I_E(B^{-n}x)\nu_0(dx), \quad E \in \mathcal{B}(R^d), \tag{3.1}$$

where $\nu_0$ is a finite Borel measure on $S_\Lambda$. For any integer $n$, we define

$$a_n(x) = \frac{1}{1+|B^{n+1}x|_Q^2} - \frac{1}{1+|B^n x|_Q^2}, \tag{3.2}$$

$$\phi_0(b,x) = \sum_{j=2}^{p+2s} \sum_{n=-\infty}^{\infty} b^{-n} a_n(x) B^{n+1} \Psi_j x$$

and

$$\phi_1(b,x) = \sum_{n=-\infty}^{\infty} b^{-n} a_n(x) B^{n+1} \Phi_1 x.$$

Since $B$ is invertible, for $1 \leq j \leq p+2s$, $Z_j$ are invariant under $B^{-1}$. Thus for any integer $n$, we see that $\Psi_j B^n = B^n \Psi_j$.

Lemma 3.1. *The functions $\phi_0(b,x)$ and $\phi_1(b,x)$ are $R^d$-valued, bounded, and measurable on $S_\Lambda$.*

*Proof.* It is obvious that

$$|a_n(x)|_Q|B^{n+1}x|_Q \le |B^{n+1}x|_Q|B^n x|_Q^2 + |B^{n+1}x|_Q^3. \tag{3.3}$$

For $x \neq 0$ in $R^d$, let $\eta(x) = \max\{|\eta_j| : 1 \le j \le p+2s, \Psi_j x \neq 0\}$ and let $m(x,j) = \max\{m \ge 0 : (B-\eta_j)^m \Psi_j x \neq 0, \Psi_j x \neq 0\}$. For $x \neq 0$ in $R^d$, we define $m(x) = \max\{m(x,j) : 1 \le j \le p+2s,\ \Psi_j x \neq 0, |\eta_j| = \eta(x)\}$ and $M = \max\{m_j : 1 \le j \le p+2s\}$. Let $x \in S_\Lambda$. Note that, for any positive integer $k$,

$$\begin{aligned} B^k\Psi_j x &= (\eta_j + (B-\eta_j))^k \Psi_j x \\ &= \eta_j^k \sum_{l=0}^{m(x,j)\wedge k} \binom{k}{l} \eta_j^{-l}(B-\eta_j)^l \Psi_j x. \end{aligned} \tag{3.4}$$

This leads to

$$|B^k x|_Q \le C_0 \eta(x)^k k^{m(x)} \quad \text{for} \quad k \ge 1, \tag{3.5}$$

where $C_0$ is a positive constant. Further discussion is given in [10, Lemma 4.1 ]. By (3.3) and (3.5), we get that

$$\begin{aligned} |a_n(x)|_Q|B^{n+1}x|_Q \le{}& C_0\eta(x)^{3n+1} n^{2m(x)}(n+1)^{m(x)} \\ &+ C_0\eta(x)^{3n+1}(n+1)^{3m(x)} \end{aligned}$$

for any nonnegative integer $n$. Thus

$$\sum_{n=0}^{\infty} b^{-n}|a_n(x)|_Q|B^{n+1}x|_Q$$

is bounded on $S_\Lambda$, because $\eta(x) \le b^{\frac{1}{2}}$ and $m(x) < M$. There is a positive constant $C_1$ such that, for any nonnegative integer $n$,

$$|a_{-n}(x)|_Q|B^{-n+1}x|_Q \le C_1.$$

This gives that

$$\sum_{n=0}^{\infty} b^n|a_{-n}(x)|_Q|B^{-n+1}x|_Q$$

is bounded on $S_\Lambda$. Hence we see that

$$\sum_{n=-\infty}^{\infty} b^{-n}|a_n(x)|_Q|B^{n+1}x|_Q$$

is bounded on $S_\Lambda$. Since $B\Psi_j = \Psi_j B$, we see that $\phi_0(b,x)$ and $\phi_1(b,x)$ are bounded. Their measurability is obvious. Since $\Psi_{j+s}x = \overline{\Psi_j x}$ for $p+1 \le j \le p+s$ and $x \in R^d$, $\phi_0(b,x)$ and $\phi_1(b,x)$ are $R^d$-valued. □

THEOREM 3.1. *Let $\mu$ be a $(Q,b)$-semi-stable distribution on $R^d$ with Lévy representation $(A,\nu,\gamma)$. Then, $\mu$ is a strictly $(Q,b)$-semi-stable if and only if*

$$(b-B)\Phi_1\gamma = b\int_{S_\Lambda}\phi_1(b,x)\nu_0(dx), \tag{3.6}$$

*and*

$$(b-B)\gamma - b\int_{S_\Lambda}\phi_0(b,x)\nu_0(dx) \in Y_1 \tag{3.7}$$

*Proof.* The distribution $\mu$ is strictly $(Q,b)$-semi-stable if and only if

$$b\gamma = B\gamma + b\int_{S_\Lambda}\sum_{n=-\infty}^{\infty} b^{-n}a_n(x)B^{n+1}x\nu_0(dx). \tag{3.8}$$

This condition (3.8) is equivalent to the following (3.9) and (3.10):

$$b\Phi_1\gamma = B\Phi_1\gamma + b\int_{S_\Lambda}\phi_1(b,x)\nu_0(dx), \tag{3.9}$$

$$b\Psi_j\gamma = B\Psi_j\gamma + b\int_{S_\Lambda}\Psi_j\phi_0(b,x)\nu_0(dx) \quad \text{for} \quad 2 \le j \le p+2s. \tag{3.10}$$

The condition (3.10) is written as

$$\Psi_j\left(b\gamma - B\gamma - b\int_{S_\Lambda}\phi_0(b,x)\nu_0(dx)\right) = 0 \quad \text{for} \quad 2 \le j \le p+2s, \tag{3.11}$$

which means the condition (3.7). For the converse, we see that (3.7) is equivalent to (3.11), that is, to (3.10). Hence (3.6) and (3.7) together give (3.8). Thus we complete the proof. □

THEOREM 3.2. *Let $\mu$ be as in Theorem 3.1. Then $\mu$ is a translation of a strictly $(Q,b)$-semi-stable distribution if and only if*

$$\int_{S_\Lambda}\phi_1(b,x)\nu_0(dx) \in G_1. \tag{3.12}$$

*Proof.* Assume that, for some $c$, $\mu * \delta_c$ is strictly $(Q, b)$-semi-stable. Then by Theorem 3.1, we have that

$$(b - B)\Psi_j(\gamma + c) = b\Psi_j \int_{S_\Lambda} \phi_0(b, x)\nu_0(dx) \quad \text{for} \quad 2 \leq j \leq p + 2s$$

and

$$(b - B)\Phi_1(\gamma + c) = b \int_{S_\Lambda} \phi_1(b, x)\nu_0(dx).$$

Hence we get (3.12). Conversely, assume that the condition (3.12) holds. Set

$$\widehat{Y}_1 = \text{Kernel}(b - B).$$

Then there exists a subspace $\widetilde{Y}_1$ of $Y_1$ such that $Y_1 = \widehat{Y}_1 \oplus \widetilde{Y}_1$. We see that the map $b - B$ restricted to $\widetilde{Y}_1$ is one-to-one and onto $G_1$. Hence, we can choose $\gamma_1 \in \widetilde{Y}_1$ such that

$$(b - B)\gamma_1 = b \int_{S_\Lambda} \phi_1(b, x)\nu_0(dx).$$

Let $2 \leq j \leq p + s$. If $v_j \in Y_j$ and $v_j \neq 0$, then $(b - B)v_j \neq 0$. Hence $b - B$ maps $Y_j$ one-to-one and onto $Y_j$. Thus, we can choose $\gamma_j \in Y_j$ such that

$$(b - B)\gamma_j = b \int_{S_\Lambda} \Phi_j\phi_0(b, x)\nu_0(dx).$$

Let $c = \sum_{j=1}^{p+s} \gamma_j$. Then $c \in R^d$. We have

$$\begin{aligned} &(b - B)c - b \int_{S_\Lambda} \phi_0(b, x)\nu_0(dx) \\ =&b \int_{S_\Lambda} \phi_1(b, x)\nu_0(dx) + b \sum_{j=2}^{p+s} \int_{S_\Lambda} \Phi_j\phi_0(b, x)\nu_0(dx) \\ &-b \int_{S_\Lambda} \phi_0(b, x)\nu_0(dx) = b \int_{S_\Lambda} \phi_1(b, x)\nu_0(dx) \in Y_1, \end{aligned}$$

since $\Phi_1\phi_0(b, x) = 0$. This shows that $\mu * \delta_{-\gamma+c}$ is strictly $(Q, b)$-semi-stable by Theorem 3.1. □

THEOREM 3.3. *We continue to assume that $b$ is an eigenvalue of $b^Q$. Then, there exists a $(Q, b)$-semi-stable distribution on $R^d$ such that $\mu$ is not a translation of any strictly $(Q, b)$-semi-stable distribution.*

*Proof.* We have $\Lambda \neq \emptyset$, since $b \in \Lambda$. We take an $x_0 \in Y_1$ such that

$$(B-b)^{m_1-1}x_0 \neq 0.$$

Denoting by $M_0$ the minimum integer such that $|B^n x_0|_Q \leq 1$, we get that $B^{M_0}x_0 \in S_\Lambda$ and $(B-b)^{m_1-1}B^{M_0}x_0 \neq 0$. We choose the $\delta$-distribution at $B^{M_0}x_0$ as $\nu_0$. Define $\nu$ by (3.1). Then there is an infinitely divisible distribution $\mu$ with Lévy representation $(0, \nu, 0)$. This $\mu$ is $(Q, b)$-semi-stable. For any integer $n$, we have that $(B-b)^{m_1-1}\ B^{M_0+n+1}x_0 \neq 0$ and

$$\int_{S_\Lambda} \phi_1(b,x)\nu_0(dx) = \phi_1(b, B^{M_0}x_0)$$
$$= \sum_{n=-\infty}^{\infty} b^{-n}a_n(B^{M_0}x_0)B^{M_0+n+1}x_0.$$

Let us show that

$$\phi_1(b, B^{M_0}x_0) \notin G_1 \tag{3.13}$$

Let $-\infty < N_1 < N_2 < \infty$ and $x \in Y_1$. By (3.4), we have

$$\sum_{n=N_1}^{N_2} b^{-n}a_n(x)B^n x = \sum_{n=0}^{N_2-N_1} b^{-n-N_1}a_{n+N_1}(x)B^{n+N_1}x$$
$$= b^{-N_1}B^{N_1}\sum_{n=0}^{N_2-N_1} a_{n+N_1}(x) \sum_{l=0}^{(m_1-1)\wedge n} \binom{n}{l} b^{-l}(B-b)^l x.$$

Therefore

$$(B-b)^{m_1-1}\sum_{n=N_1}^{N_2} b^{-n}a_n(x)B^n x$$
$$= b^{-N_1}B^{N_1}(B-b)^{m_1-1}x\sum_{n=0}^{N_2-N_1} a_{n+N_1}(x)$$
$$= (B-b)^{m_1-1}x\sum_{n=N_1}^{N_2} a_n(x),$$

since $B^n(B-b)^{m_1-1}x = b^n(B-b)^{m_1-1}x$ for any nonnegative integer $n$. Let $N_1 \to -\infty$ and $N_2 \to \infty$. Then

$$\sum_{n=N_1}^{N_2} a_n(x) = \frac{1}{1+|B^{N_2+1}x|_Q^2} - \frac{1}{1+|B^{N_1}x|_Q^2} \longrightarrow 1.$$

Letting $x = B^{M_0}x_0$, we get

$$(B-b)^{m_1-1}\phi_1(b, B^{M_0}x_0) = (B-b)^{m_1-1}B^{M_0}x_0 \neq 0.$$

Thus (3.13) is true. Hence, no translation of $\mu$ is strictly $(Q, b)$-semi-stable by Theorem 3.2. □

ACKNOWLEDGEMENT. The author would like to express her utmost gratitude to Professor K. Sato for valuable advice in calculation of Lemma 2.2. Also Professor K. Sato carefully read the manuscript and gave proof of (3.13) and valuable comments in the presentation of this paper. Also she expresses deep thanks to Professor B. D. Choi for helpful suggestion and encouragement during the research of this paper.

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