# ORLICZ-TYPE INTEGRAL
# INEQUALITIES FOR OPERATORS

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ABSTRACT. We examine Orlicz-type integral inequalities for operators and obtain as a corollary a characterization of such inequalities for the Hardy-Littlewood maximal operator extending the well-known $L^p$-norm inequalities.

## 1. Introduction

Let $f \to Tf$ be an operator and for $j = 1, 2, \cdots$, let

$$T^{(j)}f = \underbrace{T \cdots T}_{j \text{ times}} f$$

be the $j$-times iterated operator $T$. The problem which we will address in this note is to find conditions on $T$ and conditions on $\Phi, \Psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$(1) \qquad \int_{\mathbb{R}^n} \Phi(|T^{(j)}f|) \le c_j \int_{\mathbb{R}^n} \Psi(c_j|f|)$$

with the constant $c_j$ independent of $f$. Conversely, we will also examine what the inequality (1) implies about $\Phi$ and $\Psi$.

In particular, if $T = M$, the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(t)| dt,$$

our results lead to a characterization of (1) with $T = M$. The case $T = M$ and $j = 1$ has been studied extensively when $\Phi = \Psi$ is non-decreasing [1]. In fact

$$(2) \qquad \int_{\mathbb{R}^n} \Phi(Mf) \le c \int_{\mathbb{R}^n} \Phi(c|f|)$$

if and only if

$$\int_0^s \frac{d\Phi(t)}{t} \le \frac{c\Phi(cs)}{s}, \quad 0 \le s < \infty.$$

In the general case, $\Phi \ne \Psi$, the latest attempt to characterize (2) with $\Psi$ in the right side is in [4]. However in [4] the well-known $L^p$-inequalities for $M$ are excluded. Our results will restore the $L^p$-inequalities, and in doing this we were led to study general integral inequalities (1).

In section 2 we examine conditions on $\Phi$ and $\Psi$ which imply (1) and in section 3 we study when these conditions on $\Phi$ and $\Psi$ are implied by (1). All of this will give us a characterization of (1) with $T = M$. Section 4 is devoted to $L \log L$ and $\exp(L)$ inequalities of $T$, and in section 5 we outline similar results for the operator $T_r f = [T(|f|^r)]^{1/r}$, $r > 0$.

## 2. Integral inequalities

We will assume that $\Phi$ and $\Psi$ can be written as

$$\Phi(t) = \int_0^t a(s)ds, \quad \Psi(t) = \int_0^t b(s)\,ds,$$

where $a, b : \mathbb{R}_+ \to \mathbb{R}_+$. Our results, as in [4], will be expressed in terms of $a$ and $b$.

Our first theorem deals with an operator $T$ satisfying the following distributional inequality:

$$(3) \qquad |\{|Tf| > \lambda\}| \le \frac{c_0}{\lambda} \int_{\lambda/c_0}^\infty |\{|f| > s\}|ds,$$

where $c_0 > 1$ is independent of $\lambda > 0$ and $f$.

REMARK. If $T$ is sublinear, then (3) holds if and only if $T$ is weak-type $(1,1)$ and $(\infty, \infty)$ [3, p.91]. In particular, $M$ satisfies (3). In Theorem 2 below we will see another characterization of (3).

We need a version of (3) for $T^{(j)}$ and this is the content of Lemma 1.

LEMMA 1. *If $T$ satisfies (3), then*

$$(4) \qquad |\{|T^{(j)}f| > \lambda\}| \le \frac{c_0^j}{(j-1)!\lambda} \int_{\lambda/c_0^j}^\infty |\{|f| > s\}| \log^{j-1}\left(\frac{c_0^j s}{\lambda}\right) ds$$

*for $j = 1, 2, \cdots$.*

*Proof.* The proof is by induction. The case $j = 1$ is (3). Assume (4) holds up to $j - 1$, and let $c = c_0$. Then

$$|\{|T^{(j)}f| > \lambda\}| \le \frac{c}{\lambda} \int_{\lambda/c}^{\infty} |\{|T^{(j-1)}f| > s\}| ds$$

$$\le \frac{c^j}{(j-2)!\lambda} \int_{\lambda/c}^{\infty} \frac{1}{s} \int_{s/c^{j-1}}^{\infty} |\{|f| > t\}| \log^{j-2}\left(\frac{c^{j-1}t}{s}\right) dt ds$$

$$= \frac{c^j}{(j-2)!\lambda} \int_{\lambda/c^j}^{\infty} \int_{\lambda/c}^{c^{j-1}t} |\{|f| > t\}| \frac{1}{s} \log^{j-2}\left(\frac{c^{j-1}t}{s}\right) ds dt.$$

The integral in $s$ is $\frac{1}{j-1} \log^{j-1}\left(\frac{c^j t}{\lambda}\right)$. $\qquad\qquad\square$

The next theorem gives a characterization of condition (3) in terms of the integral inequality (1).

THEOREM 2. *The following are equivalent for an operator* $f \to Tf$.
(A) *The integral inequality (1) holds for every* $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying*

$$(5) \qquad \int_0^s \frac{a(t)}{t} \log^{j-1}\left(\frac{s}{t}\right) dt \le c'b(c''s), 0 \le s < \infty.$$

(B) $T$ *satisfies condition (3).*

REMARK. It is easy to verify that the left side of (5) is a $j$-times iterated integral:

$$(j-1)! \int_0^s \frac{1}{s_{j-1}} \int_0^{s_{j-1}} \cdots \frac{1}{s_1} \int_0^{s_1} \frac{a(t)}{t} dt ds_1 \cdots ds_{j-1}.$$

*Proof.* (B)$\Rightarrow$(A). From Lemma 1

$$\int_{\mathbb{R}^n} \Phi(|T^{(j)}f|) = \int_0^{\infty} |\{|T^{(j)}f| > t\}| a(t) dt$$

$$\le c_j \int_0^{\infty} \frac{a(t)}{t} \int_{t/c_0^j}^{\infty} |\{|f| > s\}| \log^{j-1}\left(\frac{c_0^j s}{t}\right) ds dt$$

$$= c_j \int_0^{\infty} |\{|f| > s\}| \int_0^{c_0^j s} \frac{a(t)}{t} \log^{j-1}\left(\frac{c_0^j s}{t}\right) dt ds$$

$$c_j c' \int_0^{\infty} |\{|f| > s\}| b(c''c_0^j s) ds = C_j \int_{\mathbb{R}^n} \Psi(c''c_0^j|f|).$$

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Since $c_j = c_0^j/(j-1)!$, the constant $C_j = c'/(c''(j-1)!)$.

(A)$\Rightarrow$(B). First we show that (5) implies

$$\int_0^s \frac{a(t)}{t}dt \le c'b(c''es), 0 \le s < \infty.$$

Simply note that the integral on the left is

$$\le \int_0^s \frac{a(t)}{t} \log^{j-1}\left(\frac{es}{t}\right) dt \le \int_0^{es} \frac{a(t)}{t} \log^{j-1}\left(\frac{es}{t}\right) dt \le c'b(c''es).$$

Thus (A) implies (1) with $j = 1$, i.e.,

$$L \equiv \int_0^\infty |\{|Tf| > \lambda\}|a(\lambda)d\lambda \le c\int_0^\infty |\{c|f| > \lambda\}|b(\lambda)d\lambda \equiv R,$$

whenever $a, b$ satisfy (5) with $j = 1$. For $0 < \lambda_0 < \infty$ and $h > 0$, let $a(\lambda) = \frac{1}{h}\chi_{[\lambda_0, \lambda_0+h]}(\lambda)$ and define

$$b(s) = \int_0^s \frac{a(t)}{t}dt = \begin{cases} 0, & 0 \le s \le \lambda_0 \\ \frac{1}{h}\log(s/\lambda_0), & \lambda_0 < s < \lambda_0 + h \\ \frac{1}{h}\log\frac{\lambda_0+h}{\lambda_0}, & s \ge \lambda_0 + h. \end{cases}$$

Then

$$L = \frac{1}{h}\int_{\lambda_0}^{\lambda_0+h} |\{|Tf| > \lambda\}|d\lambda \to |\{|Tf| > \lambda_0\}|,$$

as $h \to 0$. The right side

$$R = c\frac{1}{h}\int_{\lambda_0}^{\lambda_0+h} |\{c|f| > \lambda\}|\log(\lambda/\lambda_0)d\lambda$$
$$+c\frac{1}{h}\log\left(\frac{\lambda_0+h}{\lambda_0}\right)\int_{\lambda_0}^\infty |\{c|f| > \lambda\}|d\lambda.$$

When $h \to 0$, the first term goes to 0 and the second term goes to $\frac{c}{\lambda_0}$ $\int_{\lambda_0}^\infty |\{c|f| > \lambda\}| d\lambda$. Hence

$$|\{|Tf| > \lambda\}| \le \frac{c^2}{\lambda_0}\int_{\lambda/c}^\infty |\{|f| > s\}|ds.$$

This is (3) with constant $c^2$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

REMARKS. (i) In (A)⇒(B) we have shown that condition (5) implies the same condition with $j = 1$. The converse is not true as $a(t) = \log^+ t$ and $b(s) = \int_0^s \frac{a(t)}{t} dt$ shows. However, if $a = b$ and (5) holds for $j = 1$, then it also holds for $j = 2, 3, \cdots$. This can be seen from the Remark after the statement of Theorem 2. (ii) This is the special case $j = 1$ of Theorem 2. If $T$ satisfies condition (3) with constant $c_0$, and if

$$\int_0^s \frac{a(t)}{t} dt \leq c'b(c''s), \quad 0 \leq s < \infty,$$

then

$$\int_{\mathbb{R}^n} \Phi(|Tf|) \leq \frac{c'}{c''} \int_{\mathbb{R}^n} \Psi(c''c_0|f|).$$

Before we examine to what extent (1) implies (5) it will be useful to make an observation. Let us assume that $T$ satisfies the following reverse weak-type inequality:

(6) $$\frac{c}{\lambda} \int_{\{|f|>\lambda\}} |f| \leq |\{|Tf| > \lambda\}|,$$

with $c$ independent of $\lambda > 0$ and $f$.

Again we need a version of (6) for $T^{(j)}$ and the next two lemmas deal with this.

LEMMA 3. *Assume $T$ satisfies (6) and $0 \leq k < \infty$. Then there exists $0 < c < \infty$ such that*

$$c \int_{\{|f|>\lambda\}} |f| \log^{k+1} \left( \frac{|f|}{\lambda} \right) \leq \int_{\{|Tf|>\lambda\}} |Tf| \log^k \left( \frac{|Tf|}{\lambda} \right).$$

*Proof.* Let $\Phi(t) = 0, 0 \leq t \leq \lambda$, and $= t \log^k(t/\lambda), t > \lambda$. Then $\Phi(t) = \int_0^t a(s) ds$, where $a(s) = 0, 0 \leq s \leq \lambda$, and $= \log^k(s/\lambda) + k \log^{k-1}(s/\lambda)$, $s > \lambda$. Let now $b(s) = \int_0^s \frac{a(t)}{t} dt$. Then $b(s) = 0, 0 \leq s \leq \lambda$, and is $\geq c \log^{k+1}(s/\lambda)$, $s > \lambda$. Replace in (6) $\lambda$ by $s$, multiply by $a(s)$ and integrate in $s$ from $\lambda$ to $\infty$. Interchange the order of integration to obtain the inequality. $\square$

LEMMA 4. *Assume $T$ satisfies (6). Then there exists $0 < c_j < \infty$ such that*

$$\frac{c_j}{\lambda} \int_{\{|f|>\lambda\}} |f| \log^{j-1}\left(\frac{|f|}{\lambda}\right) \leq |\{|T^{(j)}f| > \lambda\}|$$

*for $j = 1, 2, \cdots$.*

*Proof.* (6) gives us

$$|\{|T^{(j)}f| > \lambda\}| \geq \frac{c}{\lambda} \int_{\{|T^{(j-1)}f|>\lambda\}} |T^{(j-1)}f|$$

and by Lemma 3 the last integral is

$$\geq c_1 \int_{\{|T^{(j-2)}f|>\lambda\}} |T^{(j-2)}f| \log\left(\frac{|T^{(j-2)}f|}{\lambda}\right)$$

$$\geq \cdots c_{j-2} \int_{\{|f|>\lambda\}} |f| \log^{j-1}\left(\frac{|f|}{\lambda}\right). \qquad \square$$

THEOREM 5. *Assume $T$ satisfies (6). If $a : \mathbb{R}_+ \to \mathbb{R}_+$ and $b(s) = \int_0^s \frac{a(t)}{t} \log^{j-1}\left(\frac{s}{t}\right) dt$, then*

$$c \int_{\mathbb{R}^n} |f| b(|f|) \leq \int_{\mathbb{R}^n} \Phi(|T^{(j)}f|),$$

*where as before $\Phi(t) = \int_0^t a(s)ds$.*

*Proof.* Multiply the inequality in Lemma 4 by $a(\lambda)$ and integrate in $\lambda$ from 0 to $\infty$. The right side is what we want, and the left side is obtained by an interchange of the order of integration. $\qquad \square$

COROLLARY 6. *Assume $T$ satisfies both condition (3) and (6). If $a, b$ are as in Theorem 5 and $\Phi(t) = \int_0^t a(s)ds$, $\Psi(t) = \int_0^t b(s)ds$, then*

$$c \int_{\mathbb{R}^n} |f| b(|f|) \leq \int_{\mathbb{R}^n} \Phi(|T^{(j)}f|) \leq C \int_{\mathbb{R}^n} \Psi(C|f|).$$

*Proof.* The left side is Theorem 5 and the right side is Theorem 2. $\qquad \square$

REMARK. (i) The Corollary is especially interesting in the case where the left and right sides are essentially the same. An example is $a(t) = t^\alpha, \alpha > 0$. Other examples will be examined in section 4. (ii) An example of an operator satisfying both (3) and (6) is $M$ as we shall point out in the next section.

## 3. (1) implies (5)

For this implication we need to assume that the function $b : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (i) $b(t) > 0, t > 0$, and (ii) $b$ is *quasi-increasing* , i.e., there is a constant $0 < c_0 < \infty$ such that $b(t') \leq c_0 b(c_0 t'')$ for $0 \leq t' \leq t''$.

REMARK. In [4] $b(t)$ was assumed to be positive for $t \geq 0$ thus not allowing $b(t) = t^\alpha, \alpha > 0$. (ii) is the same as in [4].

THEOREM 7. *Assume* $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $b$ *satisfying (i) and (ii). If* $T$ *satisfies the condition (6), then the integral inequality (1) implies (5).*

*Proof.* We deny (5) and have then $0 < s_k < \infty$ such that

$$\int_0^{s_k} \frac{a(t)}{t} \log^{j-1} \left( \frac{s_k}{t} \right) dt > 2^k b(2^k k s_k), k = 1, 2, \cdots .$$

Let $\{Q_k\}$ be a disjoint collection of cubes with

$$|Q_k| = \frac{1}{2^k \Psi(2^k s_k)}.$$

Let $0 < c_2 < \infty$ be given. We claim that there is $f : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\int_{\mathbb{R}^n} \Psi(c_2 f) < \infty \text{ and } \int_{\mathbb{R}^n} \Phi(|T^{(j)} f|) = \infty.$$

Define

$$f(x) = \frac{1}{c_2} \sum 2^k s_k \chi_{Q_k}(x).$$

Then

$$\int_{\mathbb{R}^n} \Psi(c_2 f) = \sum \int_{Q_k} \Psi(c_2 f) = \sum \Psi(2^k s_k) |Q_k| = \sum \frac{1}{2^k} < \infty.$$

Next, by Lemma 4,

$$\int_{\mathbb{R}^n} \Phi(|T^{(j)} f|) = \int_0^\infty |\{|T^{(j)} f| > \lambda\}| a(\lambda) d\lambda \geq$$

$$c \int_0^\infty \left( \int_{\{|f| > \lambda\}} \log^{j-1} \left( \frac{f(x)}{\lambda} \right) dx \right) \frac{a(\lambda)}{\lambda} d\lambda =$$

$$c \int_{\mathbb{R}^n} f(x) \int_0^{f(x)} \frac{a(\lambda)}{\lambda} \log^{j-1} \left( \frac{f(x)}{\lambda} \right) d\lambda dx =$$

$$\frac{c}{c_2} \sum 2^k s_k \left( \int_0^{2^k s_k / c_2} \frac{a(\lambda)}{\lambda} \log^{j-1} \left( \frac{2^k s_k}{c_2 \lambda} \right) d\lambda \right) |Q_k| \geq$$

$$\frac{c}{c_2} \sum_{k \geq k_0} 2^k s_k 2^k b(2^k k s_k) \frac{1}{2^k \Psi(2^k s_k)} \equiv L,$$

where $k_0 > c_0^2$ is chosen so large that $2^{k_0} \geq c_2$. Since $b$ is quasi-increasing we see that

$$\Psi(2^k s_k) = \int_0^{2^k s_k} b(s)ds \leq c_0 b(c_0 2^k s_k) 2^k s_k.$$

Hence

$$L \geq \frac{c}{c_0 c_2} \sum_{k \geq k_0} \frac{b(2^k k s_k)}{b(c_0 2^k s_k)} = \infty,$$

since for $k \geq c_0^2$, $c_0 2^k s_k \leq (k 2^k s_k)/c_0$, from which we get $b(c_0 2^k s_k) \leq c_0 b(k 2^k s_k)$.                                                       □

REMARK. The proof of Theorem 4 is along the lines of [4](see also [1, p.14]).

Theorems 2 and 7 imply the following characterization of (1) for $T = M$, the Hardy-Littlewood maximal operator.

THEOREM 8. *Assume* $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $b$ *satisfying (i) and (ii). Then*

$$\int_{\mathbb{R}^n} \Phi(M^{(j)} f) \leq c_j \int_{\mathbb{R}^n} \Psi(c_j |f|)$$

*if and only if*

$$\int_0^s \frac{a(t)}{t} \log^{j-1}\left(\frac{s}{t}\right) dt \leq c' b(c'' s), 0 \leq s < \infty.$$

*Proof.* Since $M$ is sublinear, weak-type $(1,1)$, and $(\infty, \infty)$, $M$ satisfies the condition (3)[3, p.91]. Hence Theorem 2 gives the sufficiency. For the necessity, simply observe that $M$ also satisfies the condition (6). For $f \in L^1(\mathbb{R}^n)$, this follows immediately from a Calderon-Zygmund decomposition at height $\lambda$ [2, p.23]. For general $f \in L^1_{loc}(\mathbb{R}^n)$, approximate by $f_N(x) = f(x)\chi_{\{|x| \leq N\}}(x)$.                                                       □

## 4. LlogL and exp(L) inequalities

In this section we will present some more examples for which the left and right sides in Corollary 6 are essentially the same. This leads then to a characterization for $\int_{\mathbb{R}^n} \Phi(|T^{(j)} f|)$ to be finite.

For the next lemma it is useful to use the notation $f(s) \sim g(s)$ on $E$ to mean: there are constants $0 < c < C < \infty$ such that $c \leq f(s)/g(s) \leq C, s \in E$.

LEMMA 9. *For $0 < k < \infty$ and $j$ a positive integer, let*

$$\phi(s) = \int_0^s \frac{\log^k(1+t)}{t} \log^{j-1}\left(\frac{s}{t}\right) dt.$$

*Then (i) $\phi(s) \sim s^k$ on $[0,1]$ and (ii) $\phi(s) \sim \log^{k+j}(1+s)$ on $[1,\infty)$.*

*Proof.* (i) This follows from the fact that $\phi(s)$ can be written as a $j$-times iterated integral of $(\log^k(1+t))/t$ (Remark after Theorem 2) and $\log(1+t) \sim t$ on $[0,1]$. For (ii) simply note that $\phi(s) > 0$ and continuous for $s \geq 1$. Thus all we have to verify is that $\phi(s)/\log^{k+j}(1+s) \to L$ as $s \to \infty$ and $0 < L < \infty$. This follows from a $j$-times repeated application of L'Hôpital's rule. $\square$

Let for $0 < k < \infty$ and $j = 1, 2, \cdots$,

$$L_{kj}(f) = \int_{\{|f|\leq 1\}} |f|^{k+1} + \int_{\{|f|>1\}} |f| \log^{k+j}(1+|f|),$$

$$K_{kj}(f) = \int_{\mathbb{R}^n} |T^{(j)}f| \log^k(1+|T^{(j)}f|).$$

THEOREM 10. *Let $0 < k < \infty$. If $T$ satisfies the conditions (3) and (6), then there are constants $0 < c_{kj} < C_{kj} < \infty$ such that*

$$c_{kj} L_{kj}(f) \leq K_{kj}(f) \leq C_{kj} L_{kj}(f).$$

*Proof.* Let $\Phi(t) = t \log^k(1+t)$. Then $\Phi(t) = \int_0^t a(s)ds$, where

$$a(s) = \log^k(1+s) + \frac{ks}{1+s} \log^{k-1}(1+s).$$

Let $b(s) = \int_0^s \frac{a(t)}{t} \log^{j-1}\left(\frac{s}{t}\right) dt$. By Corollary 6,

$$c \int_{\mathbb{R}^n} |f(x)| b(|f(x)|) \leq \int_{\mathbb{R}^n} |T^{(j)}f| \log^k(1+|T^{(j)}f|) \leq C \int_{\mathbb{R}^n} \Psi(C|f|),$$

where $\Psi(t) = \int_0^t b(s)ds$ and where we may take $C > 1$. By Lemma 9 the left side is

$$\geq c\left(\int_{\{|f|\leq 1\}} |f|^{k+1} + \int_{\{|f|>1\}} |f| \log^{k+j}(1+|f|)\right).$$

Again from Lemma 9 we see that $\Psi(t) \sim t^{k+1}$ on $[0,1]$ and $\Psi(t) \sim t \log^{k+j}(1+t)$ on $[1,\infty)$. Since for $C > 1$, $\log(1+Ct) \le C\log(1+t)$, the right side is $\le C'' \int_{\mathbb{R}^n} \Psi(|f|) \le C'' L_{kj}(f)$. $\qquad\qquad\qquad\qquad\qquad\square$

The $k = 0$ case reads as follows:

THEOREM 11. *Assume $T$ satisfies the conditions (3) and (6). Then there are constants $0 < c_j < C_j < \infty$ and there is a constant $0 < c_0 < 1$ such that*

$$c_j \int_{\{|f| \ge 1\}} |f| \log^{j-1} |f| \log(1 + |f|) \le \int_{\{|T^{(j)}f| \ge 1\}} |T^{(j)}f|$$

$$\le C_j \int_{\{|f| > c_0\}} |f| \log^{j-1}\left(\frac{|f|}{c_0}\right) \log\left(1 + \frac{|f|}{c_0}\right).$$

*Proof.* Let $a(t) = \chi_{[1,\infty)}(t)$. Then $\Phi(t) = \int_0^t a(s)ds = 0, 0 \le t \le 1$, and $= t - 1, t > 1$. If $b(s) = \int_0^s \frac{a(t)}{t} \log^{j-1}\left(\frac{s}{t}\right) dt$, then $b(s) = 0, 0 \le s \le 1$, and $= (\log^j s)/j, s > 1$. Hence from Corollary 6

$$c \int_{\mathbb{R}^n} |f| b(|f|) \le \int_{\{|T^{(j)}f| > 1\}} (|T^{(j)}f| - 1) \le C \int_{\mathbb{R}^n} \Psi(C|f|),$$

where again $\Psi(t) = \int_0^t b(s)ds$. Add $|\{|T^{(j)}f| > 1\}|$ to the inequality and let $L$ be the left side and $R$ the right side. By Lemma 4,

$$L \ge c \left( \int_{\{|f| \ge 1\}} |f| \log^j |f| + \int_{\{|f| \ge 1\}} |f| \log^{j-1} |f| \right)$$

$$\ge c \int_{\{|f| \ge 1\}} |f| \log^{j-1} |f| (1 + \log |f|)$$

$$\ge c \int_{\{|f| \ge 1\}} |f| \log^{j-1} |f| \log(1 + |f|).$$

By Lemma 1,

$$R \le C \left( \int_{\mathbb{R}^n} \Psi(C|f|) + \int_{c'}^{\infty} |\{|f| > s\}| \log^{j-1}\left(\frac{s}{c'}\right) ds \right)$$

for some $0 < c' < 1$. Note that $\Psi(s) = 0, 0 \le s \le 1$ and $= j^{-1} \int_1^s \log^j t\, dt \le cs \log^j s, s > 1$. Hence

$$\int_{\mathbb{R}^n} \Psi(C|f|) \le C \int_{\{|f|>1/C\}} |f| \log^j (C|f|).$$

If we set $\Phi_j(s) = \int_{c'}^s \log^{j-1}\left(\frac{t}{c'}\right) dt$ for $s \ge c'$ and $= 0$ for $0 \le s \le c'$, then the second integral in $R$ is

$$\int_{c'}^\infty |\{|f| > s\}| d\Phi_j(s) = \int_0^\infty |\{\Phi_j(|f|) > t\}| dt = \int_{\{|f|>c'\}} \Phi_j(|f|).$$

Since $\Phi_j(s) \le s \log^{j-1}(s/c'), s \ge c'$, we see that

$$\int_{c'}^\infty |\{|f| > s\}| d\Phi_j(s) \le \int_{\{|f|>c'\}} |f| \log^{j-1}\left(\frac{|f|}{c'}\right).$$

Let now $c_0 = \min(1/C, c')$. Then

$$R \le C \int_{\{|f|>c_0\}} |f| \log^{j-1}\left(\frac{|f|}{c_0}\right)\left(1 + \log\left(\frac{|f|}{c_0}\right)\right).$$

Since $1 + \log s \le 2\log(1+s), s \ge 0$, the proof is complete.  □

REMARKS. (i) If $T$ only satisfies condition (6), then in Theorems 10 and 11 we only get the left inequality, whereas if $T$ only satisfies condition (3) we get only the right inequality. (ii) Since $M$ satisfies both (3) and (6), Theorems 10 and 11 are valid for $T = M$, and for $j = 1$ are the $\mathbb{R}^n$- versions of the well-known fact: if $B$ is a finite ball and $k \ge 0$, then $Mf \log^k(1 + Mf) \in L^1(B)$ if and only if $|f| \log^{k+1}(1 + |f|) \in L^1(B)$ [2, p.23].

The set-up for the $\exp(L)$ inequalities is the following: $a : [1,\infty) \to \mathbb{R}_+$, $b : \mathbb{R}_+ \to \mathbb{R}_+$, $\Phi(t) = \int_1^t a(s)ds$, and $\Psi(t) = \int_0^t b(s)ds$.

THEOREM 12. If

$$(*) \qquad \int_1^{e^s} \frac{a(t)}{\log t} dt \le c'b(c''s), \quad 0 \le s < \infty,$$

and if $T$ satisfies condition (3) with constant $c_0$, then

$$\int_{\mathbb{R}^n} \Phi(e^{|Tf|}) \le \frac{c'}{c''} \int_{\mathbb{R}^n} \Psi(c''c_0|f|).$$

*Proof.* The substitution $t = e^u$ in $(*)$ gives

$$\int_0^s \frac{a(e^u)e^u}{u}\,du \leq c'b(c''s).$$

Hence from Remark (ii) after Theorem 2,

$$\int_{\mathbb{R}^n} \Phi_*(|Tf|) \leq \frac{c'}{c'} \int_{\mathbb{R}^n} \Psi(c''c_0|f|),$$

where $\Phi_*(t) = \int_0^t a(e^u)e^u du = \int_1^{e^t} a(s)ds = \Phi(e^t)$.                    $\square$

We will now briefly discuss two examples illustrating Theorem 12. In both examples we assume that $T$ satisfies condition (3) with constant $c_0$.

(I) If $1 < p < \infty$, then

$$\int_{\mathbb{R}^n} \left(e^{|Tf|} - 1\right)^p \leq C_p \left\{ \int_{\{|f| \leq 1/c_0\}} |f|^p + \int_{\{|f| > 1/c_0\}} \left(e^{c_0|f|} - 1\right)^p \right\}.$$

Let $a(t) = (t-1)^{p-1}, t \geq 1$, and let $b(s) = \int_1^{e^s} \frac{(t-1)^{p-1}}{\log t}\,dt$. The left side is what we want, and for the right side note first that $b(s) \leq c'_p s^{p-1}, 0 \leq s \leq 1$, and $b(s) \leq c''_p (e^s - 1)^p, 1 \leq s < \infty$. This can be seen by showing that

$$\lim_{s \to 0} \frac{b(s)}{s^{p-1}} = 1, \quad \lim_{s \to \infty} \frac{b(s)}{(e^s - 1)^p} = 0.$$

From this we see that $\Psi(t) \leq c'_p t^p/p, 0 < t \leq 1$, and for $t > 1$,

$$\Psi(t) \leq c'_p/p + c''_p \int_1^t (e^s - 1)^p\,ds.$$

Since $\int_1^t (e^s - 1)^p ds/(e^t - 1)^p \to 1/p$ as $t \to \infty$, the last integral is $\leq c'''_p(e^t - 1)^p$. Consequently, $\Psi(t) \leq c_p(e^t - 1)^p, t > 1$ for some constant $c_p$.

(II) There is a version of the inequality in (I) for $0 < p \leq 1$ and it reads as follows:

$$\int_{\mathbb{R}^n} \left(e^{Tf|} - 1\right)^{p-1} \left(e^{|Tf|}(|Tf| - 1) + 1\right)$$

$$\leq C_p \left\{ \int_{\{|f| \leq 1/c_0\}} |f|^{p+1} + \int_{\{|f| > 1/c_0\}} \left(e^{c_0|f|} - 1\right)^p \right\}.$$

To obtain this inequality we let $a(t) = (t-1)^{p-1}\log t, t \ge 1$, and

$$b(s) = \int_1^{e^s} (t-1)^{p-1}\,dt = \frac{1}{p}(e^s-1)^p.$$

Since $(e^s-1)^p \le c'_p s^p$ on $[0,1]$, we see that $\Psi(t) \le c'_p t^{p+1}$ on $[0,1]$, and for $t > 1$, $\Psi(t) \le c_p(e^t-1)^p$ as in (I) above. This gives us the right side of the inequality. For the left side, since $p-1 < 0$, $\Phi(t) \ge (t-1)^{p-1}\int_1^t \log s\,ds = (t-1)^{p-1}(t\log t - t + 1)$.

## 5. The operator $T_r$

For $0 < r < \infty$ let $T_r f(x) = [T(|f|^r)(x)]^{1/r}$, and let $T_r^{(j)}f = [T^{(j)}(|f|^r)]^{1/r}$ be the operator $T_r$ $j$-times iterated. Theorem 2 for $T_r^{(j)}$ reads as follows:

THEOREM 13. *Assume that $T$ satisfies condition (3) and that*

$$(7) \qquad \int_0^s \frac{a(t)}{t^r}\log^{j-1}\left(\frac{s}{t}\right)dt \le \frac{c'b(c''s)}{s^{r-1}}, \quad 0 \le s < \infty.$$

*If $\Phi(t) = \int_0^t a(s)ds$ and $\Psi(t) = \int_0^t b(s)ds$, then*

$$(8) \qquad \int_{\mathbb{R}^n} \Phi(|T_r^{(j)}f|) \le c_{jr}\int_{\mathbb{R}^n}\Psi(c_{jr}|f|).$$

*Proof.* The reader will have no difficulty to adapt the proof of Theorem 2, (B)$\Rightarrow$(A), to $T_r^{(j)}$. □

REMARK. The version of Theorem 2, (A)$\Rightarrow$(B), for $T_r^{(j)}$ reads as follows: If (8) holds whenever $a,b : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (7), then $T$ satisfies condition (3) for non-negative functions. The proof is the same by first reducing (7) to $j = 1$, and then letting $a(\lambda) = \frac{1}{h}\chi_{[\lambda_0,\lambda_0+h]}(\lambda)$, and $b(s) = s^{r-1}\int_0^s \frac{a(t)}{t^r}dt$.

THEOREM 14. *Assume $a,b : \mathbb{R}_+ \to \mathbb{R}_+$ with $b$ satisfying (i) and (ii) of section 3. Let $\Phi$ and $\Psi$ be as in Theorem 13. If $T$ satisfies the condition (6) and if (8) holds, then we have (7).*

*Proof.* With the choice of

$$|Q_k| = \frac{1}{2^{kr}\Psi(2^k s_k)}$$

the proof is, apart from obvious modifications, the same as the proof of Theorem 7. □

Since the Hardy-Littlewood maximal operator satisfies the conditions (3) and (6) we have:

COROLLARY 15. *If* $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $b$ *satisfying* (i) *and* (ii) *of section 3. If* $\Phi$ *and* $\Psi$ *are as in Theorem 12, then*

$$\int_{\mathbb{R}^n} \Phi(M_r^{(j)} f) \leq c_{jr} \int_{\mathbb{R}^n} \Psi(c_{jr}|f|)$$

*if and only if*

$$\int_0^s \frac{a(t)}{t^r} \log^{j-1}\left(\frac{s}{t}\right) dt \leq \frac{c'b(c''s)}{s^{r-1}}, \quad 0 \leq s < \infty.$$

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