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NONLINEAR ERGODIC THEOREMS FOR ALMOST-ORBITS OF ASYMPTOTICALLY NONEXPANSIVE TYPE MAPPINGS IN BANACH SPACES

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Abstract. The purpose of this paper is to study the nonlinear ergodic theorems and asymptotic behavior for an almost-orbit of asymptotically nonexpansive type mappings in a uniformly convex Banach space with Opial's condition.

1. Introduction

Ergodic theory today is a large and rapidly developing subject in applied mathematics. The recent developments in the ergodic theory of nonlinear operators in a Hilbert space started with the result of Baillon [1]. Baillon considered a nonexpansive mapping T of a real Hilbert space H into itself. He proved that if T have fixed points in H, then for every $x \in H$, the Cesàro mean

$$S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to a fixed point of T for every $x \in H$, as $n \to \infty$. Corresponding theorems for a strongly continuous one parameter semigroup

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$\{S(t) : t \geq 0\}$ of nonexpansive mappings were given soon after Baillon's work by Baillon ([2]), Baillon-Brezis [3] and Pazy [21]. That is,

$$A_\lambda x = \frac{1}{\lambda} \int_0^\lambda S(t) x dt$$

converges weakly to a common fixed point of $S(t),\ t \geq 0$ as $\lambda \to \infty$.

This theorem was extended to uniformly convex Banach spaces with Fréchet differentiable norm by Bruck [5], Hirano [8], Reich [23, 24], and Hirano-Kido-Takahashi [10, 11]. And also, Hirano [9] first gave an ergodic theorem for nonexpansive mappings and semigroups in a uniformly convex Banach space with Opial's condition which was introduced by Opial [20]; see Oka [18] for the case of asymptotically nonexpansive mappings. These theorems also were extended to the almost-orbits of nonexpansive semigroups by Miyadera-Kobayasi [17] and to the almost-orbit of asymptotically nonexpansive mappings by Oka [19] and Kim-Li [12]. Recently, Li-Ma [15] proved the nonlinear ergodic theorems for the almost-orbit of a non-lipschitzian semigroup of type (γ) in a uniformly convex Banach space which has Fréchet differentiable norm or Opial's condition. However, it has remained open for a few years whether Baillon's theorem is valid for asymptotically nonexpansive type mappings and semigroup in uniformly convex Banach spaces with Opial's condition.

The purpose of this paper is to study the nonlinear ergodic theorems for an almost-orbit of asymptotically nonexpansive type mappings in a uniformly convex Banach space with Opial's condition. And also we will give a weak convergence theorem for the almost-orbit of asymptotically nonexpansive type mappings.

2. Preliminaries and Notations

Let X be a uniformly convex Banach space and C a nonempty bounded closed convex subset of X. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* ([14]) if there exists a sequence $\{\alpha_n\}$ of nonnegative real numbers with $\lim\limits_{n\to\infty} \alpha_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + \alpha_n)\|x - y\|$$

for all $x,\ y \in C$. In particular if $\alpha_n = 0$ for all $n \geq 1$, then T is said to be *nonexpansive*. T is said to be *an asymptotically nonexpansive type* *mapping* if there exists a sequence $\{\alpha_n\}$ of nonnegative real numbers with $\lim_{n\to\infty} \alpha_n = 0$ such that

$$\|T^n x - T^n y\| \le \|x - y\| + \alpha_n$$

for all x, $y \in C$. It is easily seen that nonexpansive mapping is asymptotically nonexpansive mapping, which in turn is asymptotically nonexpansive type mapping.

Some rudiments in the geometry of Banach spaces are necessary for the proof of the main theorems in this paper. In the sequel, we give the notation : "$w - \lim_{n\to\infty}$" for weak convergence and "$\lim_{n\to\infty}$" for strong convergence. Also, a space X is always understood to be an infinite dimensional Banach space without Schur's property, i.e., the weak and strong convergence doesn't coincide for sequences.

A Banach space X is said to satisfy *Opial's condition* if for each sequence $\{x_n\}$ in X, the condition $w - \lim_{n\to\infty} x_n = x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$ ([25]). We can easily show that each "$\limsup_{n\to\infty}$" can be replaced by "$\liminf_{n\to\infty}$". Spaces possessing that property include the Hilbert space and the l^p spaces for $1 \le p < \infty$. However, $L^p (p \neq 2)$ do not satisfy that property ([6]). A sequence $\{x_n\}_{n\ge 0}$ in C is called *an almost-orbit of* T if

$$\lim_{n\to\infty} \left[\sup_{m\ge 0} \|x_{n+m} - T^m x_n\| \right] = 0.$$

A sequence $\{x_n\}$ in X is said to be *weakly almost convergence to* $z \in X$ if

$$\frac{1}{n} \sum_{i=0}^{n-1} x_{i+k}$$

converges weakly as $n \to \infty$ to z uniformly in $k \ge 0$ ([4], [22]). The convex hull of a set $E (\subset X)$ is denoted by coE, the closed convex hull by $\overline{co}E$. Also, $\omega_w(\{x_n\})$ denotes the set of all weak subsequential limits of a sequence $\{x_n\}$ in X. Let $F(T)$ denote the set of fixed points of T, that is,

$$F(T) = \{x \in C : Tx = x\}.$$

In this paper, unless other specified, X will be denoted a uniformly convex Banach space with modulus of convexity δ. The modulus of convexity of X is the function $\delta : [0,2] \to [0,1]$ defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1,\ \|y\| \leq 1,\ \|x-y\| \geq \epsilon\right\}$$

for $0 \leq \epsilon \leq 2$. X is uniformly convex if and only if $\delta(\epsilon) \geq 0$ for $0 \leq \epsilon \leq 2$ ([6]). We can see that if X is uniformly convex, then δ is strictly increasing and continuous on $[0,2]$.

3. Lemmas

In this section, we will prove many kind of lemmas for our main theorem. Throughout this section, $\{x_n\}_{n\geq 0}$ is an almost-orbit of T. To simplify in the following, we set

$$S(n,m) = \frac{1}{n}\sum_{i=0}^{n-1} x_{i+m} \qquad (n \geq 1,\ m \geq 0).$$

Also, for $\epsilon > 0$, we define

$$a(\epsilon) = \frac{\epsilon^2}{10R}\,\delta(\frac{\epsilon}{R}),$$

and

$$N_\epsilon = \{n_\epsilon \in N : \alpha_{m+n_\epsilon} \leq a(\epsilon) \quad \text{for each} \quad m \geq 0\},$$

where $R = 8M+1$, $M = \sup\{\|x\| : x \in C\}$ and δ is the modulus of convexity of the norm. Note that T is an asymptotically nonexpansive type mapping on C, N_ϵ is nonempty for each $\epsilon > 0$, and if $n_\epsilon \in N_\epsilon$, then $m + n_\epsilon \in N_\epsilon$ for each $m \geq 0$.

The next lemma is well known ([7]). This is a simple consequence of the definition of the modulus of convexity.

LEMMA 3.1 ([7]). *Let X be a uniformly convex Banach space with modulus of convexity δ. If $\|x\| \leq r$, $\|y\| \leq r$, $r \leq R$, and $\|x-y\| \geq \epsilon(>0)$, then*

$$\|\lambda x + (1-\lambda)y\| \leq r\left\{1 - 2\lambda(1-\lambda)\delta(\frac{\epsilon}{R})\right\}$$

for all $0 \leq \lambda \leq 1$.

LEMMA 3.2. *Let $n \in N$, $\epsilon_1, \epsilon_2 \in (0,1)$. Then there exists $n_{\epsilon_2} \in N$, where n_{ϵ_2} is independent of ϵ_1, such that*

$$\|T^k S(n,m) - S(n,m+k)\| < \epsilon_1 + \epsilon_2$$

for all $m \geq n_{\epsilon_2}$ and *$k \in N_{\epsilon_1}$.*

Proof. Put

$$\varphi_m = \sup_{k \geq 0} \|x_{m+k} - T^k x_m\|.$$

We shall prove the Lemma by mathematical induction. If $n = 1$, then the assertion follows from the definition of almost-orbit. Now suppose that the assertion holds for $n-1$. We first claim that

$$\lim_{m \to \infty} \|S(n-1,m) - x_{n-1+m}\| \quad \text{exists.} \tag{3.1}$$

Let $\epsilon > 0$. From the induction assumption, one can choose $m_1 \in N$ such that

$$\varphi_m < \frac{1}{3}\epsilon,$$
$$\alpha_m < \frac{1}{3}\epsilon,$$

and

$$\|T^k S(n-1,m) - S(n-1,m+k)\| < \frac{1}{3}\epsilon$$

for all $k, m \geq m_1$. It follows that

$$\begin{aligned} &\|S(n-1,m+k) - x_{n-1+m+k}\| \\ \leq\ &\|S(n-1,m+k) - T^k S(n-1,m)\| \\ &+ \|T^k S(n-1,m) - T^k x_{n-1+m}\| \\ &+ \|T^k x_{n-1+m} - x_{n-1+m+k}\| \\ \leq\ &\|S(n-1,m) - x_{n-1+m}\| + \epsilon \end{aligned}$$

for all $m, k \geq m_1$. So, for fixed $m \geq m_1$, letting $k \to \infty$, we get

$$\limsup_{k \to \infty} \|S(n-1,k) - x_{n-1+k}\| \leq \|S(n-1,m) - x_{n-1+m}\| + \epsilon,$$

and hence

$$\begin{aligned}&\limsup_{k\to\infty}\|S(n-1,k)-x_{n-1+k}\|\\ &\leq \liminf_{m\to\infty}\|S(n-1,m)-x_{n-1+m}\|+\epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this implies (3.1) is true.

Put

$$r=\lim_{m\to\infty}\|S(n-1,m)-x_{n-1+m}\|.$$

By the induction assumption again, there is $m_2(=m_2(n,\epsilon_2))\in N$ such that

$$\Big|\|S(n-1,m)-x_{n-1+m}\|-r\Big|<\frac{1}{2}a(\epsilon_2), \tag{3.2}$$

$$\|T^jS(n-1,m)-S(n-1,m+j)\|<\frac{1}{2}a(\epsilon_2), \tag{3.3}$$

and

$$\varphi_m<\frac{1}{2}a(\epsilon_2) \tag{3.4}$$

for all $j, m \geq m_2$. Now, we put $n_{\epsilon_2}=m_2+m_2=2m_2\in N$. Since for $m\geq n_{\epsilon_2}$,

$$\begin{aligned}&\|T^kS(n-1,m)-S(n-1,m+k)\|\\ &\leq \|T^kS(n-1,m)-T^{k+m_2}S(n-1,m-m_2)\|\\ &\quad+\|T^{k+m_2}S(n-1,m-m_2)-S(n-1,m+k)\|\\ &\leq \alpha_k+\|S(n-1,m)-T^{m_2}S(n-1,m-m_2)\|\\ &\quad+\|T^{k+m_2}S(n-1,m-m_2)-S(n-1,m+k)\|,\end{aligned}$$

it then follows from (3.3) and (3.4) that

$$\|T^kS(n-1,m)-S(n-1,m+k)\|<a(\epsilon_1)+a(\epsilon_2) \tag{3.5}$$

for all $m\geq n_{\epsilon_2}$ and $k\in N_{\epsilon_1}$.

Put

$$x=\left(1-\frac{1}{n}\right)\Big(T^kS(n,m)-S(n-1,m+k)\Big)$$

and

$$y = \frac{1}{n}\left(x_{n-1+m+k} - T^k S(n,m)\right).$$

From (3.2), (3.4) and (3.5), then we get, for $m \geq n_{\epsilon_2}$ and $k \in N_{\epsilon_1}$,

$$\begin{aligned}
\|x\| &\leq \left(1 - \frac{1}{n}\right)\Big(\|T^k S(n,m) - T^k S(n-1,m)\| \\
&\quad + \|T^k S(n-1,m) - S(n-1,m+k)\|\Big) \\
&\leq \left(1 - \frac{1}{n}\right)\Big(\alpha_k + \|S(n,m) - S(n-1,m)\| + a(\epsilon_1) + a(\epsilon_2)\Big) \\
&\leq \frac{n-1}{n^2} r + 2a(\epsilon_1) + 2a(\epsilon_2) \quad (\leq R),
\end{aligned}$$

$$\|y\| \leq \frac{n-1}{n^2} r + a(\epsilon_1) + a(\epsilon_2) \quad (\leq R),$$

and

$$\|x - y\| = \|T^k S(n,m) - S(n,m+k)\|.$$

Suppose that

$$\|x - y\| \geq \epsilon_1 + \epsilon_2$$

for some $m \geq n_{\epsilon_2}$ and $k \in N_{\epsilon_1}$. We shall separately obtain contradictions in following two cases.

(I) If $\dfrac{4(n-1)}{n^2} r \leq \max\{\epsilon_1, \epsilon_2\}$, then

$$\begin{aligned}
\|x - y\| \leq \|x\| + \|y\| &\leq \frac{2(n-1)}{n^2} r + 3a(\epsilon_1) + 3a(\epsilon_2) \\
&< \epsilon_1 + \epsilon_2.
\end{aligned}$$

This is a contradiction.

(II) If $\dfrac{4(n-1)r}{n^2} > \max\{\epsilon_1, \epsilon_2\}$, using Lemma 3.1, we get

$$\begin{aligned}
&\left\|\frac{1}{n}x + (1 - \frac{1}{n})y\right\| \\
&\leq \left(\frac{n-1}{n^2} r + 2a(\epsilon_1) + 2a(\epsilon_2)\right)\left\{1 - \frac{2(n-1)}{n^2}\,\delta\left(\frac{\epsilon_1 + \epsilon_2}{R}\right)\right\}.
\end{aligned}$$

And hence

$$\frac{n-1}{n^2}\left\| S(n-1,m+k) - x_{n-1+m+k} \right\|$$
$$\leq \frac{n-1}{n^2} r + 2a(\epsilon_1) + 2a(\epsilon_2) - \frac{2(n-1)r}{n^2}\ \delta(\frac{\epsilon_1+\epsilon_2}{R}),$$

it then follows (3.4) that

$$0 < 2a(\epsilon_1) + 3a(\epsilon_2) - \frac{2(n-1)r}{n^2}\ \delta\left(\frac{\epsilon_1+\epsilon_2}{R}\right).$$

If $\epsilon_1 \geq \epsilon_2$, then $a(\epsilon_1) \geq a(\epsilon_2)$, $\frac{4(n-1)r}{n^2} > \epsilon_1$, and $\frac{n-1}{n^2} > \frac{\epsilon_1}{R}$. Therefore, we have

$$0 < 5a(\epsilon_1) - \frac{{\epsilon_1}^2}{2R}\ \delta(\frac{\epsilon_1}{R}),$$

this contradicts the definition of $a(\epsilon_1)$. If $\epsilon_1 < \epsilon_2$, then we also obtain a contradiction in the same way. The proof is completed. □

By Lemma 3.2, for $n \in N$, we can choose $k_n \in N$ such that

$$k_n \in N_{\frac{1}{n}}, \tag{3.6}$$

and for each $\epsilon > 0$,

$$\|T^k S(n,m) - S(n,m+k)\| < \epsilon + \frac{1}{n} \tag{3.7}$$

for all $k \in N_\epsilon$ and $m \geq k_n$.

Denote by

$$\Lambda = \left\{ \{j_n\}_{n\geq 0} : j_n \geq k_n \quad \text{for all} \quad n \in N \right\}.$$

LEMMA 3.3. *Let $\{j_n\}_{n\geq 0} \in \Lambda$ and $f \in F(T)$. Then*

$$\lim_{n\to\infty} \|S(n,j_n) - f\|$$

exists.

Proof. Since

$$
\begin{aligned}
&\left\| S(n, j_n) - \frac{1}{n} \sum_{i=0}^{n-1} S(m,\ i + j_n + j_m) \right\| \\
&\le \frac{1}{mn} \sum_{k=0}^{m-1} \left\| \sum_{i=0}^{n-1} (x_{i+j_n} - x_{k+i+j_n+j_m}) \right\| \\
&\le \frac{1}{mn} \sum_{k=0}^{m-1} \sum_{l=1}^{k+j_m} \left\| \sum_{i=0}^{n-1} (x_{i+j_n+l} - x_{i+j_n+l-1}) \right\| \\
&\le \frac{2(m+j_m)M}{n},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\| \frac{1}{n} \sum_{i=0}^{n-1} S(m,\ i + j_n + j_m) - f \right\| \\
\le\ &\left\| \frac{1}{n} \sum_{i=0}^{n-1} S(m,\ i + j_n + j_m) - \frac{1}{n} \sum_{i=0}^{n-1} T^{i+j_n} S(m, j_m) \right\| \\
&+ \left\| \frac{1}{n} \sum_{i=0}^{n-1} T^{i+j_n} S(m, j_m) - f \right\| \\
\le\ &\frac{1}{n} \sum_{i=0}^{n-1} \left\| S(m,\ i + j_n + j_m) - T^{i+j_n} S(m, j_m) \right\| \\
&+ \frac{1}{n} \sum_{i=0}^{n-1} \left(\alpha_{i+j_n} + \left\| S(m, j_m) - f \right\| \right) \\
\le\ &\frac{1}{m} + \frac{2}{n} + \| S(m, j_m) - f \|,
\end{aligned}
$$

we have

$$
\| S(n, j_n) - f \| \le \| S(m, j_m) - f \| + \frac{2}{n} + \frac{1}{m} + \frac{2(m+j_m)M}{n}.
$$

For fixed $m \in N$, letting $n \to \infty$, we have

$$
\limsup_{n \to \infty} \| S(n, j_n) - f \| \le \| S(m, j_m) - f \| + \frac{1}{m},
$$

and hence

$$\limsup_{n\to\infty} \|S(n, j_n) - f\| \leq \liminf_{m\to\infty} \|S(m, j_m) - f\|.$$

This completes the proof. □

LEMMA 3.4. *Suppose that X satisfies Opial's condition. Then we have the followings.*

(a) *$F(T) \neq \emptyset$, and for each $\{j_n\}_{n\geq 0} \in \Lambda$, there exists $z \in F(T)$ such that*

$$w - \lim_{n\to\infty} S(n, j_n) = z.$$

(b) *$F(T)$ is a closed convex subset of X.*

Proof. Let W be the set of weak limit points of subsequence of the sequence $\{S(n, j_n) : n \in N\}$. Clearly, W is nonempty, since $\{S(n, j_n) : n \in N\}$ is bounded. Let $v \in W$ and $v = w - \lim_{m\to\infty} S(n_m, j_{n_m})$. Then for each $\epsilon > 0$ and $k \in N_\epsilon$,

$$\begin{aligned} \|S(n_m, j_{n_m}) - T^k v\| &\leq \|S(n_m, j_{n_m}) - S(n_m, j_{n_m} + k)\| \\ &\quad + \|S(n_m, j_{n_m} + k) - T^k S(n_m, j_{n_m})\| \\ &\quad + \|T^k S(n_m, j_{n_m}) - T^k v\| \\ &\leq \frac{2Mk}{n_m} + \frac{1}{n_m} + 2\epsilon + \|S(n_m, j_{n_m}) - v\|, \end{aligned}$$

it follows that

$$\limsup_{k\to\infty} \limsup_{m\to\infty} \|S(n_m, j_{n_m}) - T^k v\| \leq \limsup_{m\to\infty} \|S(n_m, j_{n_m}) - v\|.$$

By Opial's condition, v is the asymptotic center of $\{S(n_m, j_{n_m})\}$. Therefore, we have $\lim_{k\to\infty} T^k v = v$ strongly ([16]), and hence $v \in F(T)$ from the continuity of T.

Now, let $v_i \in W$, $(i = 1, 2)$ and $v_i = w - \lim_{n(i)\to\infty} S(n(i), j_{n(i)})$, where $\{n(i)\}$, $(i = 1, 2)$ are subsequences of $\{n\}$. Suppose that $v_1 \neq v_2$. Then, by Opial's condition and Lemma 3.3, we have

$$\begin{aligned} \lim_{n\to\infty} \|S(n, j_n) - v_1\| &= \lim_{n(1)\to\infty} \|S(n(1), j_{n(1)}) - v_1\| \\ &< \lim_{n(1)\to\infty} \|S(n(1), j_{n(1)}) - v_2\| \\ &= \lim_{n\to\infty} \|S(n, j_n) - v_2\|. \end{aligned}$$

In the same way, we also have

$$\lim_{n\to\infty} \|S(n,j_n) - v_2\| < \lim_{n\to\infty} \|S(n,j_n) - v_1\|.$$

This is a contradiction. Consequently, $v_1 = v_2$ and hence W is a singleton. This proves (a).

Since the closedness of $F(T)$ is clear, it remains only to show the convexity of $F(T)$. Suppose $x, y \in F(T)$, $x \neq y$. Let $z = \frac{1}{2}(x+y)$. Then,

$$\begin{aligned} \|T^k z - x\| &= \|T^k z - T^k x\| \\ &\leq \alpha_k + \frac{1}{2}\|x-y\|, \end{aligned}$$

and

$$\|T^k z - y\| \leq \alpha_k + \frac{1}{2}\|x-y\|.$$

It follows that

$$\|T^k z - z\| \leq \left(\alpha_k + \frac{1}{2}\|x-y\|\right)\left\{1 - \delta\left(\frac{2\|x-y\|}{2\alpha_k + \|x-y\|}\right)\right\}$$

for all $k \in N$. Since $\lim_{k\to\infty} \alpha_k = 0$, this implies that $\lim_{k\to\infty} \|T^k z - z\| = 0$, and $z \in F(T)$ by the continuity of T. This completes the proof. $\square$

LEMMA 3.5. *Let $\{t_n\} \in \Lambda$. If $S(n, t_n + k)$ converges weakly as $n \to \infty$, uniformly in $k \in N$, to an element p of $F(T)$. Then $S(n,k)$ converges weakly to p as $n \to \infty$, uniformly in $k \in N$.*

Proof. Suppose that $w - \lim_{n\to\infty} S(n, t_n + k) = p$ uniformly in $k \in N$. Let $\epsilon > 0$ and $x^* \in X^*$ where X^* is dual space of X. Then there is $m \in N$ such that

$$\left|\left\langle S(m, t_m + k) - p,\ \ x^*\right\rangle\right| < \epsilon$$

for all $k \in N$. Since for $n \geq m + t_m$,

$$\begin{aligned}
&\left\| \frac{1}{m} \sum_{i=0}^{m-1} S(n,\ i + t_m + k) - S(n,k) \right\| \\
&\leq \frac{1}{m} \sum_{i=0}^{m-1} \left\| S(n,\ i + t_m + k) - S(n,k) \right\| \\
&\leq \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{l=1}^{i+t_m} \left\| \sum_{j=0}^{n-1} (x_{j+k+l} - x_{j+k+l-1}) \right\| \\
&\leq \frac{2(m+t_m)M}{n}
\end{aligned}$$

and

$$\begin{aligned}
&\left| \left\langle \frac{1}{m} \sum_{i=0}^{m-1} S(n, i + t_m + k) - p,\ x^* \right\rangle \right| \\
&= \left| \left\langle \frac{1}{n} \sum_{j=0}^{n-1} S(m, j + t_m + k) - p,\ x^* \right\rangle \right| \\
&\leq \frac{1}{n} \sum_{j=0}^{n-1} \left| \left\langle S(m, j + t_m + k) - p,\ x^* \right\rangle \right| \\
&< \epsilon,
\end{aligned}$$

we have

$$|\langle S(n,k) - p,\ x^* \rangle| \leq \frac{2(m+t_m)M}{n} \cdot \|x^*\| + \epsilon$$

for $n \geq m + t_m$ and $k \in N$. This shows that $w - \lim_{n \to \infty} S(n,k) = p$ uniformly in $k \in N$. This completes the proof. $\square$

4. Nonlinear mean ergodic theorems

In [19], Oka proved a nonlinear mean ergodic theorem for the almost-orbit of an asymptotically nonexpansive mapping in a uniformly convex Banach space with Opial's condition.

We are now in a position to prove the weak almost convergence theorem for almost-orbits of asymptotically nonexpansive type mappings in a uniformly convex Banach space with Opial's condition.

THEOREM 4.1. *Let C be a bounded closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition, T an asymptotically nonexpansive type mapping on C and continuous, and $\{x_n\}_{n\geq 0}$ an almost-orbit of T. Then $\{x_n\}$ is weakly almost convergent to an element of $F(T)$, that is, there exists an element p of $F(T)$ such that*

$$w-\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}x_{i+k}=p \quad \textit{uniformly in } k\in N.$$

Proof. By Lemma 3.3 and 3.4, $\lim_{n\to\infty}\|S(n,j_n)-f\|$ exists and $F(T)$ is nonempty. For each $f\in F(T)$ and $\{j_n\}\in\Lambda$, define $r(\{j_n\},f)$, $r(\{j_n\})$ and r by

$$r(\{j_n\},f)=\lim_{n\to\infty}\|S(n,j_n)-f\| \ \text{ for } \{j_n\}\in\Lambda \text{ and } f\in F(T),$$
$$r(\{j_n\})=\inf\{r(\{j_n\},f):f\in F(T)\} \ \text{ for } \{j_n\}\in\Lambda,$$

and

$$r=\inf\{r(\{j_n\}):\{j_n\}\in\Lambda\},$$

respectively. Now, choose $\{t_n^{(i)}\}_{n\geq 0}\in\Lambda$, $i=1,2,\cdots$, such that $\lim_{i\to\infty}r(\{t_n^{(i)}\})=r$ and let

$$t_n=\sum_{i=1}^{n}t_n^{(i)} \qquad \text{for } n\geq 0.$$

Clearly $\{t_n\}_{n\geq 0}\in\Lambda$. Moreover we obtain

$$r(\{t_n\})=r. \tag{4.1}$$

To show this, let $n\geq i\geq 1$ and $f\in F(T)$. Then $\{t_n-t_n^{(i)}\}_n\in\Lambda$ and hence we have

$$\begin{aligned}
&\|S(n,t_n)-f\|\\
&\leq\|S(n,t_n)-T^{t_n-t_n^{(i)}}S(n,t_n^{(i)})\|+\|T^{t_n-t_n^{(i)}}S(n,t_n^{(i)})-f\|\\
&\leq\frac{2}{n}+\alpha_{t_n-t_n^{(i)}}+\|S(n,t_n^{(i)})-f\|\\
&\leq\frac{3}{n}+\|S(n,t_n^{(i)})-f\|.
\end{aligned}\tag{4.2}$$

Letting $n \to \infty$, we have $r(\{t_n\}, f) \leq r(\{t_n^{(i)}\}, f)$ for all $f \in F(T)$ and so $r(\{t_n\}) \leq \lim_{i\to\infty} r(\{t_n^{(i)}\}) = r$. But $r \leq r(\{t_n\})$ by the definition of r. Thus (4.1) holds. Since $F(T)$ is closed convex by Lemma 3.4, the reflexivity of X implies that there is an element p of $F(T)$ such that $r(\{t_n\}, p) = r(\{t_n\})(= r)$.
Set $t_n' = 2t_n$. Then we shall show

$$w - \lim_{n\to\infty} S(n, t_n' + k) = p \tag{4.3}$$

uniformly in $k \in N$. If this is shown, the conclusion follows from Lemma 3.5. To show (4.3), let $\{l_n\}$ be an arbitrary sequence such that $l_n \geq t_n'$ for all n. Then $\{l_n\} \in \Lambda$ and by Lemma 3.4, there is $z \in F(T)$ such that $w - \lim_{n\to\infty} S(n, l_n) = z$. Suppose $z \neq p$. Then Opial's condition implies that

$$\begin{aligned} r(\{l_n\}) &\leq \lim_{n\to\infty} \|S(n, l_n) - z\| \\ &< \lim_{n\to\infty} \|S(n, l_n) - p\| \\ &= r(\{l_n\}, p). \end{aligned}$$

But by the same way as in (4.2), we have

$$r(\{l_n\}, p) \leq r(\{t_n\}, p) = r(\{t_n\}) = r.$$

Thus $r(\{l_n\}) < r$ and this contradicts the definition of r. Hence $z = p$ and so $w - \lim_{n\to\infty} S(n, l_n) = p$. This implies (4.3). Thus, the theorem is completely proved. □

COROLLARY 4.2 ([19]). *Let C be a bounded closed convex subset of a uniformly convex Banach space X, T an asymptotically nonexpansive mapping on C, and $\{x_n\}_{n\geq 0}$ an almost-orbit of T. Then $\{x_n\}$ is weakly almost convergent to an element of $F(T)$.*

Theorem 4.1 is also an extension of [5, Corollary 2.1] and [9, Theorem 2.1].

Now, we shall prove the weak convergence theorem for an almost-orbit of asymptotically nonexpansive type mappings.

THEOREM 4.3. *Under the same assumptions as in Theorem 4.2, the following conditions are equivalent:*

(1) $w - \lim_{n\to\infty} x_n = p \in F(T)$.
(2) $\omega_w(\{x_n\}) \subset F(T)$.
(3) $w - \lim_{n\to\infty} (x_{n+1} - x_n) = 0$.

Proof. We only need to prove (3) $\Rightarrow$ (1). By Theorem 4.2, for each $\epsilon > 0$ and $x^* \in X^*$, there exists $n \in N$ such that

$$\left|\left\langle \frac{1}{n}\sum_{i=0}^{n-1} x_{i+k} - p,\ x^* \right\rangle\right| < \epsilon \tag{4.4}$$

for all $k \in N$. Let $z \in \omega_w(\{x_n\})$ and $z = w - \lim_{m\to\infty} x_{n_m}$. Then $w - \lim_{m\to\infty} x_{n_m+i} = z$ for each $1 \le i \le n$. Then (4.4) implies that

$$\left|\langle z - p,\ x^*\rangle\right| < \epsilon.$$

Since $\epsilon > 0$ and $x^* \in X^*$ are arbitrary, we have $z = p$. This completes the proof. □

REMARK. Recently, Kim-Li [13] proved a weak convergence theorem for an almost-orbit of right reversible semigroups of nonexpansive mappings in a general Banach space with Opial's condition. The corresponding theorem of this for the asymptotically nonexpansive type mappings is open.

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