GEOMETRY OF ORTHONORMAL FRAME BUNDLES

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ABSTRACT. On $\mathbb{R}^{p+q+1}$ with the non-degenerate symmetric bilinear form $F$ of type $(p, q + 1)$, the pseudo-sphere is defined by $S^{p, q} = \{ x \in \mathbb{R}^{p+q+1} : F(x, x) = 1 \} \approx \mathbb{R}^p \times S^q \subset \mathbb{R}^{p+q+1}$. In this paper, we shall study the geometries modeled on the orthonormal frame bundle $SO(S^{p, q})$ of the pseudo-sphere $S^{p, q}$. This is a principal $G = SO_0(p, q)$-bundle over $S^{p, q}$. With respect to a natural pseudo-Riemannian metric, the group of weakly $G$-equivariant isometries of $SO(S^{p, q})$ turns out to be $\text{Isom}_0^0(SO(S^{p, q})) = \text{SO}_0(p, q + 1) \times \text{SO}_0(p, q)$.

Let $W$ be a connected, pseudo-Riemannian manifold of type $(p, q)$, and let $SO(W)$ be time-space orientation-preserving orthonormal frame bundle of $W$. That is, $SO(W)$ is a sub-bundle of the bundle of frames $L(W)$ with structural group $G = \text{SO}_0(p, q) = O^{++}(p, q)$. We shall equip $SO(W)$ with a pseudo-Riemannian metric using the Levi-Civita connection. More precisely, the horizontal spaces have the same metric as the base space $W$, and are orthogonal to the fiber which has the $-\frac{1}{2(p+q-1)}$ times the Killing-Cartan form. The space $SO(W)$ with such a metric appears naturally. For example, if $W$ is a 2-dimensional manifold, then $SO(W)$ is its unit tangent bundle. Thus,

$$SO(\mathbb{H}^2) = \text{PSL}(2, \mathbb{R}), \quad SO(S^2) = SO(3), \quad SO(\mathbb{R}^2) = SO(2) \times \mathbb{R}^2.$$  

Note that $\widetilde{SO}(W)$ for these are model spaces for 3-dimensional geometries.

The main concern of this paper is to study the group $\text{Isom}_C(SO(W))$ of fiber-preserving isometries of $SO(W)$. To describe the results, let $\text{Isom}(W)$ denote the group of isometries of $W$, $\Phi(u)$ the homogeneous holonomy of $W$ through a frame $u \in SO(W)$, and let $C(u)$ be its centralizer in $SO_0(p, q)$. Let $\text{Isom}^{++}(W)$ be the group of time-space orientation-preserving isometries of $W$. For a group $G$, $\ell(G)$ and $r(G)$ denote the left and right translations, respectively. The preimage of $\text{Isom}^{++}(W)$ under

Received October 16, 2000.
2000 Mathematics Subject Classification: Primary 53C50; Secondary 53C50.
Key words and phrases: frame bundle, indefinite metric, isometry group.
the natural homomorphism $\text{Isom}_G(\text{SO}(W)) \to \text{Isom}(W)$ is denoted by $\text{Isom}_G^{++}(\text{SO}(W))$. The main results are

A. There is an isometric action of $C(u)$ on $\text{SO}(W)$ as “left translations” on each fiber.

B. $\text{Isom}_G^{++}(\text{SO}(W))$ is naturally isomorphic to $(C(u) \cdot \text{SO}_0(p,q)) \times \text{Isom}^{++}(W)$.

C. If $W$ is a pseudo-Riemannian homogeneous manifold, then $\text{SO}(W)$ contains a sub-bundle which is isometric to the Lie group $\text{Isom}^{++}(W)$ with a natural left invariant metric.

D. When $W = S^{p,q} \simeq \mathbb{R}^p \times S^q$, the pseudo-sphere in $\mathbb{R}^{p,q+1}$, $\text{SO}(S^{p,q})$ is a principal $\text{SO}_0(p,q)$ bundle over $S^{p,q}$, and $\text{Isom}_G^{++}(\text{SO}(W)) = \text{SO}_0(p,q) \times \text{SO}_0(p,q+1)$.

E. The pseudo-sphere $S^{p,q}$ does not have a compact space form if $1 \leq p \leq q$.

Some of the results for Riemannian case can be found in [1].

1. Pseudo-Riemannian metric

1.1. In $\mathbb{R}^n$ with the natural basis, we consider a non-degenerate symmetric bilinear form

$$F(x,y) = -\sum_{i=1}^{p} x_i y_i + \sum_{j=p+1}^{p+q} x_j y_j$$

for $(x,y) \in \mathbb{R}^n$ and $n = p + q, n \geq 2$. We say the bilinear form $F$ has type $(p,q)$. The space $\mathbb{R}^n$ with the bilinear form $F$ is written as $\mathbb{R}^{p,q}$.

Let

$$I_{p,q} = \begin{bmatrix} -I_p & O \\ O & I_q \end{bmatrix} \in \text{GL}(p,\mathbb{R}) \times \text{GL}(q,\mathbb{R}),$$

and let

$$O(p,q) = \{ A \in \text{GL}(n,\mathbb{R}) : ^tA I_{p,q} A = I_{p,q} \}.$$

Then $O(p,q)$ is the orthogonal group of the bilinear form $F$. Note that $O(p,q)$ has 4 components for $p,q \geq 1$, denoted by $O^{++}(p,q), O^{--}(p,q), O^+(p,q), O^-(p,q)$; $O(p,q)$ has two components for $p = 0$ or $q = 0$. We shall denote $O^{++}(p,q)$ by $\text{SO}_0(p,q)$. This is the connected component containing the identity. An automorphism of $\mathbb{R}^{p,q}$ in $\text{SO}_0(p,q)$ is called a time-space orientation-preserving isometry. For a matrix $A = (a_{ij})$, $A \in \text{SO}_0(p,q)$ if and only if the column vector $a_i$'s satisfy

$$F(a_i, a_i) = -1, \quad F(a_j, a_j) = +1, \quad F(a_i, a_j) = 0,$$
for all \(1 \leq i \leq p\) and \(p+1 \leq j \leq p+q\). From \(\text{tr} A I_{p,q} A = I_{p,q}\), \(\det(A) = \pm 1\). If \(A \in SO_0(p,q)\), then \(\det(A) = 1\) but note that same is true for elements of \(O^{--}(p,q)\).

2. Orthonormal frame bundle of a pseudo-Riemannian manifold

2.1. Let \(M\) be a differentiable manifold. A \textit{pseudo-Riemannian metric} on \(M\) is a differentiable field \(g = \{g_x\}_{x \in M}\) of non-degenerate symmetric bilinear forms \(g_x\) on the tangent spaces \(M_x\) of \(M\). The \(g_x\) are the inner products on the tangent spaces. The metric is \textit{Riemannian metric} if each \(g_x\) is positive definite. If the bilinear forms \(g_x\) have type \((p,q)\), \(M\) is called a pseudo-Riemannian manifold of type \((p,q)\).

We define a metric on \(SO_0(p,q)\). For \(p + q \leq 2\), \(SO_0(p,q)\) is abelian and has an obvious bi-invariant metric. For \(p+q > 2\), \(SO_0(p,q)\) is semisimple. We define a symmetric bilinear, \(\text{ad}(G)\) invariant form on the tangent space of \(SO_0(p,q)\) at the identity:

\[
\langle A, B \rangle = -\frac{1}{2(p+q-1)} \phi(A, B)
\]

where \(\phi(A, B) = (p+q-1)\text{trace}((\text{ad}(A)\text{ad}(B)))\) is the Killing-Cartan form on the Lie algebra \(\mathfrak{o}(p,q)\) of \(SO_0(p,q)\). This is a bi-invariant pseudo-Riemannian metric of type \((pq, \frac{1}{2}(p^2 + q^2 - p - q))\) and is unique up to scale factor.

\textbf{Example 2.2 (Killing-Cartan form of SO}_0(1, 2))\textbf{).} Choose a basis for \(\mathfrak{o}(1,2)\) as follows:

\[
e_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
\]

Their exp are

\[
\exp(te_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix}, \quad \exp(te_2) = \begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

\[
\exp(te_3) = \begin{bmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix}.
\]
Then, with respect to the basis \( \{e_1, e_2, e_3\} \), 
\[
\text{ad}(e_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{ad}(e_2) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \text{ad}(e_3) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]
Thus the Killing-Cartan form is defined by
\[
[\phi(e_i, e_j)] = 2[\text{trace}(\text{ad}(e_i))(\text{ad}(e_j))] = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.
\]
Note that \( \text{SO}_0(1, 2) \) is isomorphic to \( \text{PSL}(2, \mathbb{R}) \).

2.3. Let \( W \) be a pseudo-Riemannian manifold of type \((p, q)\). Let \( \text{SO}(W) \) be the time-space orientation-preserving orthonormal frame bundle of \( W \). That is, \( \text{SO}(W) \) is a sub-bundle of the bundle of frames \( L(W) \) with fiber (and structural group) \( \text{SO}_0(p, q) = O^{++}(p, q) \). We shall equip \( \text{SO}(W) \) with a pseudo-Riemannian metric using the Levi-Civita connection. More precisely, the horizontal spaces have the same metric as the base space \( W \), and are orthogonal to the fiber. Such a metric is of type \((p + pq, \frac{1}{2}(p^2 + q^2 - p + q))\). The universal covering space \( \widetilde{\text{SO}}(W) \) has a pseudo-Riemannian metric induced from \( \text{SO}(W) \).

2.4. Let \( \text{Isom}^{++}(W) \) be the group of time-space orientation-preserving isometries of \( W \). Therefore, \( f \in \text{Isom}^{++}(W) \) if and only if the induced map \( f_* : L(W) \to L(W) \) (on the frame bundle) preserves the sub-bundle \( \text{SO}(W) \).

Each point \( u \in \text{SO}(W) \) is considered as an isomorphism 
\[
u : \mathbb{R}^{p,q} \to T_{\pi(u)}(W)
\]
where \( \pi : \text{SO}(W) \to W \) is the projection. The group \( \text{SO}_0(p, q) \) acts on \( \text{SO}(W) \) from the right naturally, by composition.
\[
\text{SO}(W) \times \text{SO}_0(p, q) \to \text{SO}(W), \quad (u, a) \mapsto u \circ a;
\]
namely, for \( a \in \text{SO}_0(p, q) \) and \( u \in \text{SO}(W) \),
\[
u \circ a : \mathbb{R}^{p,q} \xrightarrow{a} \mathbb{R}^{p,q} \xrightarrow{u} T_{\pi(u)}(W)
\]
is a new isomorphism so that \( u \circ a \in \text{SO}(W) \). With this right \( \text{SO}_0(p, q) \)-action, \( \text{SO}(W) \) is a principal \( \text{SO}_0(p, q) \)-bundle.

**Lemma 2.5.** The principal \( \text{SO}_0(p, q) \)-action on \( \text{SO}(W) \) is isometries.
Proof. Let \( a \in \text{SO}_0(p, q) \). Let \( X \in T_v(\text{SO}(W)) \) be a horizontal vector, and let \( \gamma(t) \) be a horizontal curve in \( \text{SO}(W) \) fitting \( X \). Then,

\[
(r_a)_*(X) = \left. \frac{d}{dt} \right|_{t=0} (\gamma(t) \circ a) = X \circ a.
\]

Since \( a \in \text{SO}_0(p, q) \) is an isometry of \( W \), the homomorphism

\[
(r_a)_*: T_v(\text{SO}(W))_H \rightarrow T_{v \circ a}(\text{SO}(W))_H \quad \text{(horizontal spaces)}
\]

is an isometry.

Now let \( X \in T_v(\text{SO}(W)) \) be a vertical vector, and let \( \gamma(t) \) be a curve in \( \text{SO}_0(p, q) \) starting at the identity so that \( v \circ \gamma(t) \) fits \( X \). Then,

\[
(r_a)_*(X) = \left. \frac{d}{dt} \right|_{t=0} (v \circ \gamma(t) \circ a) = \left( \left. \frac{d}{dt} \right|_{t=0} (v \circ \gamma(t)) \right) \circ a = X \circ a.
\]

Since \( a \in \text{SO}_0(p, q) \) and the metric on the fiber \( \text{SO}_0(p, q) \) is bi-invariant, the homomorphism

\[
(r_a)_*: T_v(\text{SO}(W))_V \rightarrow T_{v \circ a}(\text{SO}(W))_V \quad \text{(vertical spaces)}
\]

is an isometry.

2.6. Let us write \( \text{SO}_0(p, q) \) by \( G \). We denote the weakly \( G \)-equivariant (with respect to the principal right \( G \)-action) isometries of \( \text{SO}(W) \) by \( \text{Isom}_G(\text{SO}(W)) \). More precisely, \( h \in \text{Isom}_G(\text{SO}(W)) \) if there is an automorphism \( \alpha \) of \( \text{SO}_0(p, q) \) such that \( h(u \circ x) = h(u) \circ \alpha(x) \) for all \( u \in \text{SO}(W) \) and \( x \in \text{SO}_0(p, q) \).

A weakly \( G \)-equivariant isometry of \( \text{SO}(W) \) induces an isometry of \( W \), giving rise to a natural homomorphism

\[
\tilde{\pi} : \text{Isom}_G(\text{SO}(W)) \rightarrow \text{Isom}(W).
\]

Let

\[
\mathcal{K} = \ker(\tilde{\pi}),
\]

\[
\text{Isom}_G^{++}(\text{SO}(W)) = \tilde{\pi}^{-1}(\text{Isom}^{++}(W)).
\]

Then

\[
1 \rightarrow \mathcal{K} \rightarrow \text{Isom}_G^{++}(\text{SO}(W)) \rightarrow \text{Isom}^{++}(W) \rightarrow 1
\]

is exact.

Lemma 2.7. The homomorphism \( \text{Isom}_G^{++}(\text{SO}(W)) \rightarrow \text{Isom}^{++}(W) \) naturally splits. Furthermore, \( \text{Isom}^{++}(W) \) commutes with the right action of \( \text{SO}_0(p, q) \) on \( \text{SO}(W) \).
Proof. Let \( f \in \text{Isom}^+(W) \). Then \( f \) naturally induces a map \( \tilde{f} : \text{SO}(W) \rightarrow \text{SO}(W) \) defined by
\[
\tilde{f}(u) = f_* \circ u : \mathbb{R}^{p,q} \xrightarrow{u} T_{\pi(u)}(W) \xrightarrow{f_*} T_{f(\pi(u))}(W).
\]
Since \( f \) is an isometry of \( W \), the induced map \( \tilde{f} \) of \( \text{SO}(W) \) leaves the connection form invariant [3, p. 226, Theorem 1.3]. This means that, for each \( u \in \text{SO}(W) \), \( \tilde{f} \) maps the horizontal subspace of \( T_u(\text{SO}(W)) \) to the horizontal subspace of \( T_{\tilde{f}(u)}(\text{SO}(W)) \). However, the metric on \( \text{SO}(W) \) is defined so that the horizontal subspaces are isometric to the base space. Since \( f \) was an isometry of the base space, \( \tilde{f} \) induces an isometry on the horizontal spaces.

The map \( \tilde{f} \) commutes with the right action of \( \text{SO}_0(p,q) \) on \( \text{SO}(W) \). That is, \( \tilde{f}(u \circ a) = f_* \circ (u \circ a) = (f_* \circ u) \circ a = \tilde{f}(u) \circ a \) for \( u \in \text{SO}(W) \) and \( a \in \text{SO}_0(p,q) \). Let \( \sigma : \mathfrak{o}(p,q) \rightarrow \mathfrak{X}(\text{SO}(W)) \) be the fundamental vector field defined by \( \sigma(A)_u = \frac{d}{dt}|_{t=0}(u \cdot \exp tA) \). Then \( (\tilde{f})_*(\sigma(A)_u) = \frac{d}{dt}|_{t=0}f_* \circ (u \cdot \exp tA) = \frac{d}{dt}|_{t=0}(f_* \circ u) \cdot \exp tA = \sigma(A)_{\tilde{f}(u)} \) for each \( A \in \mathfrak{o}(p,q) \). This shows that \( (\tilde{f})_* \) preserves the metric on the vertical subspaces as well. Consequently, \( \tilde{f} \) is an isometry of \( \text{SO}(W) \). Clearly \( f \mapsto \tilde{f} \) is a group homomorphism. \( \square \)

2.8. Fix a point \( u \in \text{SO}(W) \). Let \( \Phi(u) \subset G \) be the holonomy group of the metric connection with reference point \( u \). See [3, p.72]. Let \( C_G(\Phi(u)) \) be the centralizer of \( \Phi(u) \) in \( G \) and let \( Z(G) \), the center of \( G \).

**Lemma 2.9.** Let \( g \in \mathcal{K} \). Then the restriction of \( g \) to each holonomy bundle \( P(u) \) is a right translation by an element of \( \text{SO}_0(p,q) \).

**Proof.** Define a map \( \lambda : \text{SO}(W) \rightarrow \text{SO}_0(p,q) \) by
\[
g(w) = w \circ \lambda(w).
\]
We claim that \( \lambda \) is constant on each \( P(u) \). Let \( \gamma \) be a horizontal curve in \( \text{SO}(W) \). Then by Leibniz rule,
\[
g_*(\gamma'(t_0)) = \frac{d}{dt}(\gamma(t) \circ \lambda(\gamma(t)))|_{t_0} = \frac{d}{dt}(\gamma(t_0) \circ \lambda(\gamma(t)))|_{t_0} + \gamma_*(\gamma(t_0)) \lambda_*(\gamma'(t_0)).
\]
Notice that the first summand is vertical and the second is horizontal. Since \( g \) is an isometry, \( g_* \) maps the horizontal vector \( \gamma'(t_0) \) to a horizontal vector. Thus \( \frac{d}{dt}(\gamma(t_0) \circ \lambda(\gamma(t)))|_{t_0} = 0 \) for all \( t_0 \).
This shows that $\lambda$ is constant along $\gamma$. Since any two points in $P(u)$ can be connected via a horizontal path, the conclusion of the lemma follows. \qed

The right action of $G = \text{SO}_0(p, q)$ on $\text{SO}(W)$, $r(G)$ belongs to $\mathcal{K}$ by Lemma 2.8, and is transitive on the fiber. Therefore, to understand the isometry group of $\text{SO}(W)$, it is enough to understand the stabilizer of the action of $\mathcal{K}$.

**Proposition 2.10.** For $u \in \text{SO}(W)$, let $\mathcal{K}_u$ be the stabilizer of the $\mathcal{K}$-action on $\text{SO}(W)$. Then

1. $\mathcal{K}_u$ fixes the whole sub-bundle $P(u)$.
2. $\mathcal{K}_u$ is naturally isomorphic to $C_G(\Phi(u))/\mathcal{Z}(G) \subset \text{Inn}(G)$, where $C_G(\Phi(u))$ is the centralizer of $\Phi(u)$ in $G$.

*Proof.* Statement (1) follows directly from the preceding Lemma.

(2) We identify the fiber $\pi^{-1}(\pi(u))$ with $G$ via $u \mapsto e$. Since elements of $\text{Isom}_G(\text{SO}(W))$ are weakly $G$-equivariant, $r(G)$ is normal in $\text{Isom}_G(\text{SO}(W))$, and hence in $\mathcal{K}$.

Let $M_G(\text{SO}(W), G)$ denote the group of all $G$-equivariant maps from $\text{SO}(W)$ to $G$. That is, $\lambda : \text{SO}(W) \to G$ is an element of $M_G(\text{SO}(W), G)$ if and only if

$$\lambda(v \circ a) = a^{-1} \circ \lambda(v) \circ a$$

for all $a \in G$. Such a map $\lambda$ induces a map $\tilde{\lambda} : \text{SO}(W) \to \text{SO}(W)$ by

$$\tilde{\lambda}(v) = v \circ \lambda(v).$$

At a general point,

$$\tilde{\lambda}(v \circ a) = (v \circ a) \circ \lambda(v \circ a) = (v \circ a) \circ (a^{-1} \lambda(v) \circ a) = v \circ \lambda(v) \circ a = \tilde{\lambda}(v) \circ a,$n

showing that $\tilde{\lambda}$ is $G$-equivariant. It is well known [5] that

$$M_G(\text{SO}(W), G) \cdot r(G) = r(G) \times_{\mathcal{Z}(G)} M_G(\text{SO}(W), G)$$

is the full group of weakly $G$-equivariant maps of $\text{SO}(W)$ inducing the identity on $W$. Thus,

$$\mathcal{K} \subset M_G(\text{SO}(W), G) \cdot r(G).$$

Let $g \in \mathcal{K}_u$. Then $g$ is of the form $g = \tilde{\lambda} \circ r(c^{-1})$. Now $g(u) = u$ yields

$$u \circ \lambda(u) \circ c^{-1} = u.$$
so that $\lambda(u) = c$. Then, for any $k \in G$,
\[
g(u \cdot k) = (\tilde{\lambda} \circ r(c^{-1}))(u \cdot k) = \tilde{\lambda}(u \cdot kc^{-1}) = \tilde{\lambda}(u) \cdot kc^{-1} = u \cdot ckc^{-1}.
\]
Thus $g$ gives rise to a unique element of $\text{Inn}(G)$, defining an injective homomorphism $\mathcal{K}_u \to \text{Inn}(G)$.

By (1), $g$ fixes $P(u)$, hence it also fixes $u \cdot \Phi(u)$. Therefore, $g$ is an inner automorphism of $G$ leaving the subgroup $\Phi(u)$ fixed. Thus $\mathcal{K}_u \subset C_G(\Phi(u))/Z(G)$.

Conversely, let $c \in C_G(\Phi(u))$. Choose any “cross section” to the holonomy bundle $P(u) \to W$; that is, a map $s : W \to P(u)$ which is not necessarily continuous, satisfying $\pi \circ s = \text{id}$. Then every point of $\text{SO}(W)$ is of the form $s(x) \cdot k$ for some $x \in W$ and $k \in G$. Now define a map $\hat{c} : \text{SO}(W) \to \text{SO}(W)$ by
\[
\hat{c}(s(x) \cdot k) = s(x) \cdot ckc^{-1}.
\]
This is well defined because, if $t : W \to \text{SO}(W)$ is another cross section, then $s$ and $t$ are related by elements of $P(u)$. More precisely, let $s(x) \cdot k = t(x) \cdot k'$. Then $t(x) = s(x) \cdot h$ for some $h \in \Phi(u)$, and hence $k' = h^{-1}k$. Therefore,
\[
t(x) \cdot ckc^{-1} = s(x) \cdot hch^{-1}kc^{-1} = s(x) \cdot ckc^{-1},
\]
since $c$ commutes with $\Phi(u)$. We could have used a local smooth section for $s$. This implies that $\hat{c}$ is smooth. We claim that $\hat{c}$ is in fact an isometry. Let $s(x) \cdot k \in \text{SO}(W)$ be an arbitrary point. On a horizontal neighborhood (i.e., $k$ fixed, and $x$ varying),
\[
\hat{c}(s(x) \cdot k) = s(x) \cdot ckc^{-1} = (s(x) \cdot k) \cdot (k^{-1} ckc^{-1})
\]
so that $\hat{c} = r_{k^{-1} ckc^{-1}}$. Thus, for a horizontal vectors $X$, $\hat{c}_*(X) = r_{k^{-1} ckc^{-1}}(X)$. On the vertical vectors, recall that $G$ has a bi-invariant metric and so the conjugation by $c$ is already an isometry. Consequently, $\hat{c}$ is an isometry of $\text{SO}(W)$ inducing the identity on $W$. Thus we have shown that the homomorphism $\mathcal{K}_u \to C_G(\Phi(u))/Z(G)$ is an isomorphism. \(\square\)

**Theorem 2.11.** There is a left action of $C_G(\Phi(u))$ as left translations on each fiber, and $\text{Isom}^+(\text{SO}(W)) = \left(\ell(C_G(\Phi(u))) \times Z(G) r(G)\right) \times \text{Isom}^+(W)$. 
Proof. By the previous lemma, we have
\[ \text{Isom}^\leftrightarrow_G(\text{SO}(W)) = \mathcal{K} \rtimes \text{Isom}^\leftrightarrow(W) \]
\[ = (r(G) \rtimes \mathcal{K}_u) \rtimes \text{Isom}^\leftrightarrow(W) \]
\[ = (r(G) \rtimes (C_G(\Phi(u))/Z(G))) \rtimes \text{Isom}^\leftrightarrow(W) \]
\[ = \left( \ell(G(\Phi(u))) \times Z(G), r(G) \right) \rtimes \text{Isom}^\leftrightarrow(W). \]

So, it only remains to understand how the inner automorphism group $C_G(\Phi(u))/Z(G)$ gives rise to the left multiplication action $\ell(C_G(\Phi(u)))$. For any $c \in C_G(\Phi(u))$, define $\ell_c : \text{SO}(W) \to \text{SO}(W)$ by
\[ \ell_c(s(x) \cdot k) = s(x) \cdot ck \]
so that $\ell_c = \hat{c} \circ r_{c^{-1}}$. (The map $\hat{c}$ was defined in the proof of Lemma 2.10). This finishes the proof of theorem. \qed

**Corollary 2.12.** For the two extreme cases, we have the following:

1. If $\Phi(u) = \text{SO}_0(p, q)$, then $\text{Isom}^\leftrightarrow_G(\text{SO}(W)) = r(G) \rtimes \text{Isom}^\leftrightarrow(W)$.

2. If $\Phi(u)$ is trivial, then
\[ \text{Isom}^\leftrightarrow_G(\text{SO}(W)) = (\ell(G) \times Z(G), r(G)) \rtimes \text{Isom}^\leftrightarrow(W). \]

**Example 2.13.** If $W = \mathbb{R}^{p,q}$ with the indefinite metric of type $(p, q)$, then the bundle $\text{SO}(W)$ has trivial holonomy so that $C_G(\Phi(u)) = G$. Thus $\text{Isom}^\leftrightarrow_G(\text{SO}(W)) = (G \times Z(G)) \rtimes \{\mathbb{R}^{p,q} \rtimes O^{\leftrightarrow}(p, q)\}$ since $\text{Isom}^{\leftrightarrow}(W) = \mathbb{R}^{p,q} \rtimes O^{\leftrightarrow}(p, q)$.

A pseudo-Riemannian manifold is **homogeneous** if $\text{Isom}(W)$ acts on $W$ transitively.

**Theorem 2.14.** If $W$ is a pseudo-Riemannian homogeneous manifold, then $\text{SO}(W)$ contains a sub-bundle which is isometric to the Lie group $\text{Isom}^{\leftrightarrow}(W)$ with a left invariant metric.

**Proof.** Fix $u \in \text{SO}(W)$, and let $x = \pi(u) \in W$, where $\pi : \text{SO}(W) \to W$ is the projection. Let $H = \text{Isom}^{\leftrightarrow}(W)_x$, the stabilizer at $x$. 
We have two bundles over $W$:
\[
\begin{array}{cccc}
H & \text{SO}_0(p, q) & \downarrow & \\
\downarrow & & \downarrow & \\
\text{Isom}^{++}(W) & \longrightarrow & \text{SO}(W) & \\
\downarrow & & \downarrow & \\
W & \longrightarrow & W & \\
\end{array}
\]
Both bundles are principal (with right actions). Define $\phi : H \to \text{SO}_0(p, q)$ by $\phi(h) = u^{-1} \circ h_* \circ u$.

\[
\begin{array}{cccc}
\mathbb{R}^{p+q} & \longrightarrow & T_x(W) & \\
\downarrow & & \downarrow & \\
\phi(h) & \downarrow & h_* & \\
\mathbb{R}^{p+q} & \leftarrow & T_x(W) & \\
\end{array}
\]
Clearly $\phi$ is 1-1. Define $\psi : \text{Isom}^{++}(W) \to \text{SO}(W)$ by $\psi(f) = f_* \circ u$. Then $(\text{Isom}^{++}(W), H) \to (\text{SO}(W), \text{SO}_0(p, q))$ is weakly equivariant. That is, the diagram
\[
\begin{array}{cccc}
\text{Isom}^{++}(W) \times H & \longrightarrow & \text{Isom}^{++}(W) & (f, h) \longrightarrow f \circ h \\
\psi \times \phi & \downarrow & \psi & \downarrow \\
\text{SO}(W) \times \text{SO}_0(p, q) & \longrightarrow & \text{SO}(W) & (\psi(f), \phi(h)) \longrightarrow \psi(f) \circ \phi(h) \\
\end{array}
\]
is commutative, because $\psi(f \circ h) = (f \circ h)_* \circ u = (f_* \circ u) \circ (u^{-1} \circ h_* \circ u) = \psi(f) \circ (u^{-1} \circ h_* \circ u)$. Therefore, $\text{SO}(W)$ contains $\text{Isom}^{++}(W)$ with a natural left invariant metric.

\textbf{Corollary 2.15.} Assume $W$ is a pseudo-Riemannian homogeneous manifold with $\text{Isom}^{++}(W)_x \cong \text{SO}_0(p, q)$. Then $\text{Isom}^{++}(W) \hookrightarrow \text{SO}(W)$ is a bundle isomorphism.

\section{Orthonormal frame bundle of a pseudo-sphere}

\subsection*{3.1.} Now we specialize to the case when $W$ is a pseudo-sphere. The quadric
\[S^{p,q} = \{x \in \mathbb{R}^{p+q+1} | F(x, x) = 1\}\]
is called a pseudo-Riemannian sphere. It is diffeomorphic to \( \mathbb{R}^p \times S^q \) via \( (x, y) \mapsto (x, \frac{y}{|y|}) \). It is connected for \( q > 0 \), and has 2 components if \( q = 0 \).

With this pseudo-Riemannian metric, \( S^{p,q} \) has constant sectional curvature \(+1\), and \( \text{Isom}(S^{p,q}) = O(p, q + 1) \) is the full group of isometries. The isotropy subgroup at \((0, 0, \cdots, 1)\) is exactly \( O(p, q) \). The connected component is \( \text{Isom}^+(W) = O^+(p, q + 1) = SO_0(p, q+1) \). By Corollary 2.15, they are equal and we have

\[
\text{SO}(S^{p,q}) = \text{Isom}^+(S^{p,q}) = SO_0(p, q + 1)
\]
as bundles over \( S^{p,q} \).

3.2. Geometry of \( SO(S^{p,q}) \). We would like to understand the metric on \( SO_0(p, q+1) \) coming from the identification with \( SO(S^{p,q}) \). \( SO_0(p, q+1) \) acts on \( S^{p,q} \) transitively with isotropy subgroup is \( SO_0(p, q) \) so that

\[
SO_0(p, q) \to SO_0(p, q + 1) \to S^{p,q}
\]
is a principal \( SO_0(p, q) \)-bundle.

Moreover, \( S^{p,q} \) is a symmetric space with a symmetric structure \((SO_0(p, q+1), SO_0(p, q), \sigma)\), where \( \sigma \) is conjugation by \[
\begin{bmatrix}
I_{p+q} & 0 \\
O & -1
\end{bmatrix}.
\]
Then \( SO_0(p, q) \) is the connected component of

\[
(SO_0(p, q + 1))_\sigma = \text{Fix}(\sigma, SO_0(p, q + 1)).
\]
Thus \( SO_0(p, q + 1)/SO_0(p, q) \) is a symmetric space and the canonical decomposition of the Lie algebra is

\[
\mathfrak{o}(p, q + 1) = \mathfrak{o}(p, q) + \mathfrak{m}
\]
where \( \mathfrak{o}(p, q) \) is considered as a subalgebra of \( \mathfrak{o}(p, q + 1) \) and \( \mathfrak{m} \) is the subspace of all matrices of the following form

\[
\mathfrak{o}(p, q) = \left\{ \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}, \quad \mathfrak{m} = \left\{ \begin{bmatrix} O & O & \xi \\ O & O & \eta \\ t\xi & -t\eta & 0 \end{bmatrix} : \xi \in \mathbb{R}^p, \eta \in \mathbb{R}^q \right\}
\]

[These are \(+1\) and \(-1\) eigen spaces of \( \sigma \), respectively]. The restriction of the Killing-Cartan form of \( SO_0(p, q+1) \) on \( \mathfrak{g} \) to \( \mathfrak{m} \) defines a \( SO_0(p, q+1) \)-invariant (indefinite) Riemannian metric on the symmetric space \( S^{p,q} = SO_0(p, q + 1)/SO_0(p, q) \). Furthermore, under the Killing-Cartan form, the factors \( \mathfrak{g} \) and \( \mathfrak{m} \) are orthogonal.
Thus our metric on \( SO(W) \) and the metric on \( SO_0(p, q + 1) \) have the following properties: With the bundle isomorphism,

\[
\begin{array}{ccc}
SO_0(p, q) & \longrightarrow & SO_0(p, q) \\
\downarrow & & \downarrow \\
SO_0(p, q + 1) & \longrightarrow & SO(W) \\
\downarrow & & \downarrow \\
W & = & W
\end{array}
\]

1. the metrics on \( W \) are the same
2. the metrics on \( SO_0(p, q) \) are the same
3. With the same holonomy bundle, the fiber is orthogonal to the base.

These are enough to conclude that the two spaces are isometric. See [3, p. 232, Theorem 3.4].

3.3 (Calculation of holonomy groups). Clearly \( \mathfrak{o}(p, q + 1) = \mathfrak{o}(p, q) + \mathfrak{m} \) as vector spaces. Furthermore, one can show that \([\mathfrak{m}, \mathfrak{m}] = \mathfrak{o}(p, q)\). Now by [3, p.103, Theorem 11.1], the Lie algebra of the holonomy group is generated by \([\mathfrak{m}, \mathfrak{m}]_{\mathfrak{o}(p, q)}\), the \( \mathfrak{o}(p, q) \) component of \([\mathfrak{m}, \mathfrak{m}]\). But since \([\mathfrak{m}, \mathfrak{m}] = \mathfrak{o}(p, q), [\mathfrak{m}, \mathfrak{m}]_{\mathfrak{o}(p, q)} = \mathfrak{o}(p, q)\). Therefore, the holonomy group must be the whole group \( SO_0(p, q) \). Now by Lemma 2.12, we have

**Theorem 3.4.** \( \text{Isom}^{++}_G(SO(W)) = \tau(G) \times_{Z(G)} \text{Isom}^{++}(W) = SO_0(p, q) \times_{Z(G)} SO_0(p, q + 1) \).

Note that in this expression, the group \( G \) acts on \( SO(W) = SO_0(p, q + 1) \) as right multiplication by inverse elements. The group \( \text{Isom}^{++}(W) \) acts on \( SO(W) \) as lift of the isometries of \( W \). We have seen this can be identified as left multiplication of \( \text{Isom}^{++}(W) = SO_0(p, q + 1) \) on \( SO(W) = SO_0(p, q + 1) \). Thus \((b, a) \in G \times \text{Isom}^{++}(W)\) acts on \( x \in SO_0(p, q) \) by \((b, a) \cdot x = axb^{-1}\). Therefore we write \( \text{Isom}^{++}(W) \) first so that \( \text{Isom}^{++}_G(SO(W)) = \text{Isom}^{++}(W) \times G \). Thus, \( SO_0(p, q + 1) \times SO_0(p, q) \) acts on \( SO(S^{p,q}) = SO_0(p, q + 1) \) by

\[
\begin{array}{c}
(SO_0(p, q + 1) \times SO_0(p, q)) \times SO(S^{p,q}) \longrightarrow SO(S^{p,q}) \\
((a, b), x) \longrightarrow a \cdot x \cdot b^{-1}
\end{array}
\]

multiplication in the Lie group \( SO_0(p, q + 1) \).
3.5. On the other hand, the group $SO_0(p, q + 1)$ is semi-simple, the Killing-Cartan form defines a bi-invariant pseudo-Riemannian metric on $SO_0(p, q + 1)$. Thus

$$r(G) \subset r(SO_0(p, q + 1)) \subset Isom(SO_0(p, q + 1)).$$

One can prove that then weakly $G$-equivariant isometries of $SO_0(p, q + 1)$ is the normalizer of $r(G)$ in $\ell(SO_0(p, q + 1)) \cdot r(SO_0(p, q + 1))$, which becomes

$$r(G) \times Z(G) \ell(SO_0(p, q + 1)).$$

Consequently, we get

$$\text{Isom}_G^{++}(SO_0(p, q + 1)) = SO_0(p, q) \times Z(G) SO_0(p, q + 1).$$

3.6. Let $\widetilde{SO}_0(p, q + 1)$ be the universal covering group of $SO_0(p, q + 1)$; and let $\widetilde{SO}(p, q)$ be the covering group of $SO_0(p, q)$ induced by the universal covering $\widetilde{SO}_0(p, q + 1) \rightarrow SO_0(p, q + 1)$; that is, $\widetilde{SO}(p, q)$ is the preimage of $SO_0(p, q)$ under this map. The principal fibering is

$$\widetilde{SO}(p, q) \rightarrow \widetilde{SO}_0(p, q + 1) \rightarrow SO_0(p, q + 1).$$

The isometry group is then

$$(\text{Isom}_G)_0(\widetilde{SO}(S^{p,q})) = \widetilde{SO}_0(p, q + 1) \times_Z \widetilde{SO}(p, q)$$

where $Z$ is the diagonal subgroup which is the center of $SO_0(p, q + 1)$. Thus we have a short exact sequence

$$1 \rightarrow \widetilde{SO}(p, q) \rightarrow (\text{Isom}_G)_0(\widetilde{SO}(S^{p,q})) \rightarrow SO_0(p, q + 1) \rightarrow 1.$$

4. Spaces modelled on $SO(S^{p,q})$

We shall describe the geometries of orthonormal frame bundles of some low dimensional pseudo-spheres $S^{p,q}$. The discreteness of a group $Q$ in $SO_0(p, q + 1)$ does not necessarily imply that $Q$ acts properly on $S^{p,q}$, and in fact, such proper actions $(Q, S^{p,q})$ are quite rare. We give two examples: $S^{2,0}$ and $S^{1,2}$. The pseudo-sphere $S^{2,0}$ admits compact forms but $S^{1,2}$ does not.

**Theorem 4.1.** (cf. [9, Theorem 11.1.7]) Suppose a discrete group $Q$ acts on $S^{p,q}$ effectively and properly. If $p \leq q$, then $Q$ is finite.
Proof. Let $V$ be the subspace of $\mathbb{R}^{p,q+1}$ given by $x_1 = x_2 = \cdots = x_p = 0$, so $V \cap S^{p,q}$ is the equatorial sphere $S^q$. If $g \in \text{GL}(p + q + 1, \mathbb{R})$, then $p \leq q < q + 1$ implies

$$\dim(V \cap g(V)) \geq 2 \dim(V) - (p + q + 1) > 0.$$  

Thus, for every $\alpha \in Q$, $\alpha(S^q)$ meets $S^q$.

If $Q$ were infinite, then compactness of $S^q$ would give us a point $x \in S^q$, a sequence $\{\alpha_i\} \subset Q$ of distinct elements, and a sequence $\{x_i\} \subset S^q$, such that $\alpha_i(x_i) \in S^q$ and $\{\alpha_i(x_i)\} \to x$. Passing to a subsequence, $\{x_i\} \to x' \in S^q$. As $Q$ has closed orbits, this says $\alpha(x') = x$ for some $\alpha \in Q$. Then $\{\alpha^{-1}\alpha_i(x_i)\} \to \alpha^{-1}(x) = x'$, contradicting discontinuity of $Q$ at $x'$. Thus $Q$ is finite. \qed

Example 4.2 ($\text{SO}(S^{2,0})$). For $W = S^{2,0}$, $\text{SO}(S^{2,0}) = \text{SO}_0(2,1)$ has a principal $\text{SO}_0(2,0)$-fibering structure:

$$\begin{array}{ccc}
\text{SO}_0(2,0) & \longrightarrow & \text{SO}_0(2,1) \\
\| & & \| \\
\text{SO}(2) & \longrightarrow & \text{PSL}(2, \mathbb{R}) \\
\| & & \| \\
\mathbb{R} & \longrightarrow & \tilde{\text{PSL}}(2, \mathbb{R}) & \longrightarrow & \mathbb{H}^2
\end{array}$$

The pseudo-sphere $S^{2,0}$ has an indefinite metric of type $(2,0)$, which is the negative of the ordinary Poincaré metric. Thus the space $\text{SO}_0(2,1)$ has an indefinite metric of type $(2,1)$, negative of the Lorenz metric, (cf. [4]). The group of weakly $\text{SO}_0(2,0)$-equivariant isometries is $(\text{Isom}_{\text{SO}(2,0)})_0(\text{SO}(2,1)) = \text{SO}(2,1) \times \text{SO}(2)$.

The universal covering of $\text{SO}_0(2,1)$ has a fibration

$$\begin{array}{ccc}
\tilde{\text{SO}}_0(2,0) & \longrightarrow & \tilde{\text{SO}}_0(2,1) \\
\| & & \| \\
\mathbb{R} & \longrightarrow & \tilde{\text{PSL}}(2, \mathbb{R}) & \longrightarrow & \mathbb{H}^2
\end{array}$$

The group of weakly $\tilde{\text{SO}}_0(2,0)$-equivariant isometries is 

$$(\text{Isom}_{\tilde{\text{SO}}(2,0)})_0(\tilde{\text{SO}}(2,1)) = \mathbb{R} \times_{\mathbb{Z}} \tilde{\text{SO}}(2,1).$$

Note that this is also the group of isometries for the $\tilde{\text{PSL}}(2, \mathbb{R})$-geometry.

Let $1 \to \mathbb{Z} \to \Pi \to Q \to 1$ be an central extension of $\mathbb{Z}$ by a Fuchsian group $Q$. If $Q$ is cocompact, then $H^2(Q; \mathbb{Z}) \approx \mathbb{Z} \oplus \text{Torsion}$; if $Q$ is finitely generated and not cocompact, then $H^2(Q; \mathbb{Z})$ is a direct sum of finite cyclic groups corresponding to the conjugacy classes of the maximal finite subgroups of $Q$. 
(Structure Theorem) [4]. Let $1 \to \mathbb{Z} \to \Pi \to Q \to 1$ be an central extension of $\mathbb{Z}$ by a Fuchsian group $Q$. Then $\Pi$ embeds into $(\text{Isom}_{\widetilde{SO}(2,0)})_0(\widetilde{SO}(2,1))$ if and only if $[\Pi] \in H^2(Q, \mathbb{Z})$ has infinite order.

This embedding yields $\Pi \setminus P$ as a Seifert orbifold with a complete Lorentz metric of constant curvature. The fibers are time-like geodesics. If $Q \setminus S^{2,0}$ is non-compact, then each $[\Pi]$ embeds. The same construction also yields embeddings of $\Pi$ into $\text{Isom}(\widetilde{PSL}(2, \mathbb{R}))$ and $\Pi \setminus P$ is a Seifert orbifold with the 3-dimensional $\widetilde{PSL}(2, \mathbb{R})$-geometry. See [6] and [4] for the embeddings.

**Example 4.3 (SO($S^{1,2}$)).** For $W = S^{1,2}$, $SO(S^{1,2}) = SO_0(1,3)$ has a principal $SO_0(1,2)$-fibering structure:

$$SO_0(1,2) \longrightarrow SO_0(1,3) \longrightarrow S^{1,2}.$$ 

Since $\pi_1(SO_0(1,3)) = \mathbb{Z}_2$ and $\pi_1(SO_0(1,2)) = \mathbb{Z}$, the bundle is non-trivial. Then $(\text{Isom}_{SO_0(1,2)})_0(SO(S^{1,2})) = SO_0(1,2) \times SO_0(1,3)$ and the action on $SO(S^{1,2}) \cong SO_0(1,3)$ is given by

$$(a,b)x = b \cdot x \cdot a^{-1}$$

for $(a,b) \in SO_0(1,2) \times SO_0(1,3)$ and $x \in SO(S^{1,2})$.

Suppose there exists a compact space form $\Pi \setminus SO(S^{1,2})$. That is, $\Pi$ is a discrete subgroup of $SO_0(1,2) \times SO_0(1,3)$ which acts on $SO(S^{1,2})$ properly with $\Pi \setminus SO(S^{1,2})$ compact. [Such a $\Pi$ exists]. Then $\Gamma = \Pi \cap SO_0(1,2)$ is never cocompact in $SO_0(1,2)$. If it were, the quotient $Q = \Pi / \Gamma$ will act on $S^{1,2}$ properly. By Theorem 4.1, $Q$ must be finite. This implies that $\Pi \setminus SO(S^{1,2})$ is not compact, a contradiction.

Our model space for the geometry must be simply connected. Therefore, we take the universal covering space of $SO(S^{1,2})$ as our model space. Since $\pi_1(SO_0(1,3)) = \mathbb{Z}_2$, $\widetilde{SO}_0(1,3)$ is a double covering of $SO_0(1,3)$. Let $\widetilde{SO}_0(1,2)$ be the corresponding double covering of $SO_0(1,2)$. Then $\widetilde{SO}_0(1,3)$ is a principal $\widetilde{SO}_0(1,2)$-bundle,

$$\widetilde{SO}_0(1,2) \longrightarrow \widetilde{SO}_0(1,3) \longrightarrow S^{1,2}.$$ 

Note that $\widetilde{SO}_0(1,2)$ is "level 2" group in [4], and is isomorphic to $SL(2, \mathbb{R})$, since $\pi_1(\widetilde{SO}_0(1,2)) = \mathbb{Z}_2$. Therefore, $\widetilde{SO}_0(1,3)$ is diffeomorphic to $S^3 \times \mathbb{R}^3$. From $(\text{Isom}_{\widetilde{SO}_0(1,2)})_0(SO(S^{1,2})) = SO_0(1,2) \times SO_0(1,3)$, we have

$$(\text{Isom}_{\widetilde{SO}_0(1,2)})_0(SO(S^{1,2})) = \widetilde{SO}_0(1,2) \times \mathbb{Z}_2 \widetilde{SO}_0(1,3),$$
where $Z_2$ is the diagonal center.

References


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