J. Korean Math. Soc. **38** (2001), No. 5, pp. 971–986

GEOMETRY OF ORTHONORMAL FRAME BUNDLES

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ABSTRACT. On $\mathbb{R}^{p+q+1}$ with the non-degenerate symmetric bilinear form F of type $(p, q+1)$, the pseudo-sphere is defined by $S^{p,q} = \{x \in \mathbb{R}^{p+q+1} : F(x,x) = 1\} \approx \mathbb{R}^p \times S^q \subset \mathbb{R}^{p,q+1}$. In this paper, we shall study the geometries modelled on the orthonormal frame bundle $\mathrm{SO}(S^{p,q})$ of the pseudo-sphere $S^{p,q}$. This is a principal $G = \mathrm{SO}_0(p,q)$-bundle over $S^{p,q}$. With respect to a natural pseudo-Riemannian metric, the group of weakly G-equivariant isometries of $\mathrm{SO}(S^{p,q})$ turns out to be $\mathrm{Isom}^0_G(\mathrm{SO}(S^{p,q})) = \mathrm{SO}_0(p, q+1) \times \mathrm{SO}_0(p,q)$.

Let W be a connected, pseudo-Riemannian manifold of type (p,q), and let $\mathrm{SO}(W)$ be time-space orientation-preserving orthonormal frame bundle of W. That is, $\mathrm{SO}(W)$ is a sub-bundle of the bundle of frames $L(W)$ with structural group $G = \mathrm{SO}_0(p,q) = O^{++}(p,q)$. We shall equip $\mathrm{SO}(W)$ with a pseudo-Riemannian metric using the Levi-Civita connection. More precisely, the horizontal spaces have the same metric as the base space W, and are orthogonal to the fiber which has the $-\frac{1}{2(p+q-1)}$ times the Killing-Cartan form. The space $\mathrm{SO}(W)$ with such a metric appears naturally. For example, if W is a 2-dimensional manifold, then $\mathrm{SO}(W)$ is its unit tangent bundle. Thus,

$$\mathrm{SO}(\mathbf{H}^2) = \mathrm{PSL}(2,\mathbb{R}), \quad \mathrm{SO}(S^2) = \mathrm{SO}(3), \quad \mathrm{SO}(\mathbb{R}^2) = \mathrm{SO}(2) \times \mathbb{R}^2.$$

Note that $\widetilde{\mathrm{SO}}(W)$ for these are model spaces for 3-dimensional geometries.

The main concern of this paper is to study the group $\mathrm{Isom}_G(\mathrm{SO}(W))$ of fiber-preserving isometries of $\mathrm{SO}(W)$. To describe the results, let $\mathrm{Isom}(W)$ denote the group of isometries of W, $\Phi(u)$ the homogeneous holonomy of W through a frame $u \in \mathrm{SO}(W)$, and let $C(u)$ be its centralizer in $\mathrm{SO}_0(p,q)$. Let $\mathrm{Isom}^{++}(W)$ be the group of *time-space orientation-preserving* isometries of W. For a group G, $\ell(G)$ and $r(G)$ denote the left and right translations, respectively. The preimage of $\mathrm{Isom}^{++}(W)$ under

Received October 16, 2000.
2000 Mathematics Subject Classification: Primary 53C50; Secondary 53C50.
Key words and phrases: frame bundle, indefinite metric, isometry group.

the natural homomorphism $\mathrm{Isom}_G(\mathrm{SO}(W)) \to \mathrm{Isom}(W)$ is denoted by $\mathrm{Isom}_G^{++}(\mathrm{SO}(W))$. The main results are

A. There is an isometric action of $C(u)$ on $\mathrm{SO}(W)$ as "left translations" on each fiber.
B. $\mathrm{Isom}_G^{++}(\mathrm{SO}(W))$ is naturally isomorphic to $(C(u) \cdot \mathrm{SO}_0(p,q)) \rtimes \mathrm{Isom}^{++}(W)$.
C. If W is a pseudo-Riemannian homogeneous manifold, then $\mathrm{SO}(W)$ contains a sub-bundle which is isometric to the Lie group $\mathrm{Isom}^{++}(W)$ with a natural left invariant metric.
D. When $W = S^{p,q} \approx \mathbb{R}^p \times S^q$, the pseudo-sphere in $\mathbb{R}^{p,q+1}$, $\mathrm{SO}(S^{p,q})$ is a principal $\mathrm{SO}_0(p,q)$ bundle over $S^{p,q}$, and $\mathrm{Isom}_G^{++}(\mathrm{SO}(W)) = \mathrm{SO}_0(p,q) \times \mathrm{SO}_0(p,q+1)$.
E. The pseudo-sphere $S^{p,q}$ does not have a compact space form if $1 \leq p \leq q$.

Some of the results for Riemannian case can be found in [1].

1. Pseudo-Riemannian metric

1.1. In $\mathbb{R}^n$ with the natural basis, we consider a non-degenerate symmetric bilinear form

$$F(x,y) = -\sum_{i=1}^{p} x_i y_i + \sum_{j=p+1}^{p+q} x_j y_j$$

for $(x,y) \in \mathbb{R}^n$ and $n = p+q, n \geq 2$. We say the bilinear form F has type (p,q). The space $\mathbb{R}^n$ with the bilinear form F is written as $\mathbb{R}^{p,q}$. Let

$$I_{p,q} = \begin{bmatrix} -I_p & O \\ O & I_q \end{bmatrix} \in \mathrm{GL}(p,\mathbb{R}) \times \mathrm{GL}(q,\mathbb{R}),$$

and let

$$O(p,q) = \{A \in \mathrm{GL}(n,\mathbb{R}) : {}^tA I_{p,q} A = I_{p,q}\}.$$

Then $O(p,q)$ is the *orthogonal group* of the bilinear form F. Note that $O(p,q)$ has 4 components for $p,q \geq 1$, denoted by $O^{++}(p,q)$, $O^{--}(p,q)$, $O^{+-}(p,q)$, $O^{-+}(p,q)$; $O(p,q)$ has two components for $p=0$ or $q=0$. We shall denote $O^{++}(p,q)$ by $\mathrm{SO}_0(p,q)$. This is the connected component containing the identity. An automorphism of $\mathbb{R}^{p,q}$ in $\mathrm{SO}_0(p,q)$ is called a *time-space orientation-preserving* isometry. For a matrix $A = (a_{ij})$, $A \in \mathrm{SO}_0(p,q)$ if and only if the column vector $\mathbf{a}_i$'s satisfy

$$F(\mathbf{a}_i, \mathbf{a}_i) = -1, \quad F(\mathbf{a}_j, \mathbf{a}_j) = +1, \quad F(\mathbf{a}_i, \mathbf{a}_j) = 0,$$

for all $1 \leq i \leq p$ and $p+1 \leq j \leq p+q$. From ${}^tAI_{p,q}A = I_{p,q}$, $\det(A) = \pm 1$. If $A \in \mathrm{SO}_0(p,q)$, then $\det(A) = 1$ but note that same is true for elements of $O^{--}(p,q)$.

2. Orthonormal frame bundle of a pseudo-Riemannian manifold

2.1. Let M be a differentiable manifold. A *pseudo-Riemannian metric* on M is a differentiable field $g = \{g_x\}_{x \in M}$ of non-degenerate symmetric bilinear forms g_x on the tangent spaces M_x of M. The g_x are the inner products on the tangent spaces. The metric is *Riemannian metric* if each g_x is positive definite. If the bilinear forms g_x have type (p,q), M is called a pseudo-Riemannian manifold of *type* (p,q).

We define a metric on $\mathrm{SO}_0(p,q)$. For $p+q \leq 2$, $\mathrm{SO}_0(p,q)$ is abelian and has an obvious bi-invariant metric. For $p+q > 2$, $\mathrm{SO}_0(p,q)$ is semi-simple. We define a symmetric bilinear, $\mathrm{ad}(G)$ invariant form on the tangent space of $\mathrm{SO}_0(p,q)$ at the identity:

$$\langle A, B\rangle = -\frac{1}{2(p+q-1)}\phi(A,B)$$

where $\phi(A,B) = (p+q-1)\mathrm{trace}(\mathrm{ad}(A)\mathrm{ad}(B))$ is the Killing-Cartan form on the Lie algebra $\mathfrak{o}(p,q)$ of $\mathrm{SO}_0(p,q)$. This is a bi-invariant pseudo-Riemannian metric of type $(pq, \frac{1}{2}(p^2+q^2-p-q))$ and is unique up to scale factor.

Example 2.2 (*Killing-Cartan form of* $\mathrm{SO}_0(1,2)$). Choose a basis for $\mathfrak{o}(1,2)$ as follows:

$$e_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Their exp are

$$\exp(te_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix}, \quad \exp(te_2) = \begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\exp(te_3) = \begin{bmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix}.$$

Then, with respect to the basis $\{e_1, e_2, e_3\}$,

$$\mathrm{ad}(e_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \ \mathrm{ad}(e_2) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ \mathrm{ad}(e_3) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus the Killing-Cartan form is defined by

$$[\phi(e_i, e_j)] = 2[\mathrm{trace}(\mathrm{ad}(e_i))(\mathrm{ad}(e_j))] = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Note that $\mathrm{SO}_0(1,2)$ is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$.

2.3. Let W be a pseudo-Riemannian manifold of type (p,q). Let $\mathrm{SO}(W)$ be the *time-space orientation-preserving orthonormal frame bundle* of W. That is, $\mathrm{SO}(W)$ is a sub-bundle of the bundle of frames $L(W)$ with fiber (and structural group) $\mathrm{SO}_0(p,q) = O^{++}(p,q)$. We shall equip $\mathrm{SO}(W)$ with a pseudo-Riemannian metric using the Levi-Civita connection. More precisely, the horizontal spaces have the same metric as the base space W, and are orthogonal to the fiber. Such a metric is of type $(p+pq, \frac{1}{2}(p^2+q^2-p+q))$. The universal covering space $\widetilde{\mathrm{SO}}(W)$ has a pseudo-Riemannian metric induced from $\mathrm{SO}(W)$.

2.4. Let $\mathrm{Isom}^{++}(W)$ be the group of *time-space orientation-preserving* isometries of W. Therefore, $f \in \mathrm{Isom}^{++}(W)$ if and only if the induced map $f_* : L(W) \to L(W)$ (on the frame bundle) preserves the sub-bundle $\mathrm{SO}(W)$.

Each point $u \in \mathrm{SO}(W)$ is considered as an isomorphism

$$u : \mathbb{R}^{p,q} \longrightarrow T_{\pi(u)}(W)$$

where $\pi : \mathrm{SO}(W) \to W$ is the projection. The group $\mathrm{SO}_0(p,q)$ acts on $\mathrm{SO}(W)$ from the right naturally, by composition.

$$\mathrm{SO}(W) \times \mathrm{SO}_0(p,q) \to \mathrm{SO}(W), \qquad (u,a) \mapsto u \circ a;$$

namely, for $a \in \mathrm{SO}_0(p,q)$ and $u \in \mathrm{SO}(W)$,

$$u \circ a : \ \mathbb{R}^{p,q} \xrightarrow{\ a\ } \mathbb{R}^{p,q} \xrightarrow{\ u\ } T_{\pi(u)}(W)$$

is a new isomorphism so that $u \circ a \in \mathrm{SO}(W)$. *With this right* $\mathrm{SO}_0(p,q)$*-action,* $\mathrm{SO}(W)$ *is a principal* $\mathrm{SO}_0(p,q)$*-bundle.*

LEMMA 2.5. *The principal* $\mathrm{SO}_0(p,q)$*-action on* $\mathrm{SO}(W)$ *is isometries.*

Proof. Let $a \in \mathrm{SO}_0(p,q)$. Let $X \in T_v(\mathrm{SO}(W))$ be a horizontal vector, and let $\gamma(t)$ be a horizontal curve in $\mathrm{SO}(W)$ fitting X. Then,

$$(r_a)_*(X) = \frac{d}{dt}\Big|_{t=0}(\gamma(t)\circ a) = X \circ a.$$

Since $a \in \mathrm{SO}_0(p,q)$ is an isometry of W, the homomorphism

$$\begin{array}{cccc} (r_a)_*: & T_v(\mathrm{SO}(W))_H & \longrightarrow & T_{v\circ a}(\mathrm{SO}(W))_H \quad \text{(horizontal spaces)} \\ & X & \mapsto & X\circ a \end{array}$$

is an isometry.

Now let $X \in T_v(\mathrm{SO}(W))$ be a vertical vector, and let $\gamma(t)$ be a curve in $\mathrm{SO}_0(p,q)$ starting at the identity so that $v \circ \gamma(t)$ fits X. Then,

$$(r_a)_*(X) = \frac{d}{dt}\Big|_{t=0}(v\circ\gamma(t)\circ a) = \left(\frac{d}{dt}\Big|_{t=0}(v\circ\gamma(t))\right)\circ a = X\circ a.$$

Since $a \in \mathrm{SO}_0(p,q)$ and the metric on the fiber $\mathrm{SO}_0(p,q)$ is bi-invariant, the homomorphism

$$\begin{array}{cccc} (r_a)_*: & T_v(\mathrm{SO}(W))_V & \longrightarrow & T_{v\circ a}(\mathrm{SO}(W))_V \quad \text{(vertical spaces)} \\ & X & \mapsto & X\circ a \end{array}$$

is an isometry. □

2.6. Let us write $\mathrm{SO}_0(p,q)$ by G. We denote the *weakly G-equivariant* (with respect to the principal right G-action) isometries of $\mathrm{SO}(W)$ by $\mathrm{Isom}_G(\mathrm{SO}(W))$. More precisely, $h \in \mathrm{Isom}_G(\mathrm{SO}(W))$ if there is an automorphism α of $\mathrm{SO}_0(p,q)$ such that $h(u\circ x) = h(u)\circ\alpha(x)$ for all $u \in \mathrm{SO}(W)$ and $x \in \mathrm{SO}_0(p,q)$.

A weakly G-equivariant isometry of $\mathrm{SO}(W)$ induces an isometry of W, giving rise to a natural homomorphism

$$\widetilde{\pi} : \mathrm{Isom}_G(\mathrm{SO}(W)) \to \mathrm{Isom}(W).$$

Let

$$\begin{aligned} \mathcal{K} &= \ker(\widetilde{\pi}), \\ \mathrm{Isom}_G^{++}(\mathrm{SO}(W)) &= \widetilde{\pi}^{-1}(\mathrm{Isom}^{++}(W)). \end{aligned}$$

Then

$$1 \to \mathcal{K} \to \mathrm{Isom}_G^{++}(\mathrm{SO}(W)) \to \mathrm{Isom}^{++}(W) \to 1$$

is exact.

Lemma 2.7. *The homomorphism* $\mathrm{Isom}_G^{++}(\mathrm{SO}(W)) \to \mathrm{Isom}^{++}(W)$ *naturally splits. Furthermore,* $\mathrm{Isom}^{++}(W)$ *commutes with the right action of* $\mathrm{SO}_0(p,q)$ *on* $\mathrm{SO}(W)$.

Proof. Let $f \in \operatorname{Isom}^{++}(W)$. Then f naturally induces a map $\widetilde{f}$: $\mathrm{SO}(W) \to \mathrm{SO}(W)$ defined by

$$\widetilde{f}(u) = f_* \circ u : \ \mathbb{R}^{p,q} \xrightarrow{\ u\ } T_{\pi(u)}(W) \xrightarrow{\ f_*\ } T_{f(\pi(u))}(W).$$

Since f is an isometry of W, the induced map $\widetilde{f}$ of $\mathrm{SO}(W)$ leaves the connection form invariant [3, p. 226, Theorem 1.3]. This means that, for each $u \in \mathrm{SO}(W)$, $\widetilde{f}$ maps the horizontal subspace of $T_u(\mathrm{SO}(W))$ to the horizontal subspace of $T_{\widetilde{f}(u)}(\mathrm{SO}(W))$. However, the metric on $\mathrm{SO}(W)$ is defined so that the horizontal subspaces are isometric to the base space. Since f was an isometry of the base space, $\widetilde{f}$ induces an isometry on the horizontal spaces.

The map $\widetilde{f}$ commutes with the right action of $\mathrm{SO}_0(p,q)$ on $\mathrm{SO}(W)$. That is, $\widetilde{f}(u \circ a) = f_* \circ (u \circ a) = (f_* \circ u) \circ a = \widetilde{f}(u) \circ a$ for $u \in \mathrm{SO}(W)$ and $a \in \mathrm{SO}_0(p,q)$. Let $\sigma : \mathfrak{o}(p,q) \to \mathfrak{X}(\mathrm{SO}(W))$ be the fundamental vector field defined by $\sigma(A)_u = \frac{d}{dt}|_{t=0}(u \cdot \exp tA)$. Then $(\widetilde{f})_*(\sigma(A)_u) = \frac{d}{dt}|_{t=0} f_* \circ (u \cdot \exp tA) = \frac{d}{dt}|_{t=0}(f_* \circ u) \cdot \exp tA = \sigma(A)_{\widetilde{f}(u)}$ for each $A \in \mathfrak{o}(p,q)$. This shows that $(\widetilde{f})_*$ preserves the metric on the vertical subspaces as well. Consequently, $\widetilde{f}$ is an isometry of $\mathrm{SO}(W)$. Clearly $f \mapsto \widetilde{f}$ is a group homomorphism. □

2.8. Fix a point $u \in \mathrm{SO}(W)$. Let $\Phi(u) \subset G$ be the holonomy group of the metric connection with reference point u. See [3, p. 72]. Let $C_G(\Phi(u))$ be the centralizer of $\Phi(u)$ in G and let $\mathcal{Z}(G)$, the center of G.

LEMMA 2.9. *Let $g \in \mathcal{K}$. Then the restriction of g to each holonomy bundle $P(u)$ is a right translation by an element of* $\mathrm{SO}_0(p,q)$.

Proof. Define a map $\lambda : \mathrm{SO}(W) \to \mathrm{SO}_0(p,q)$ by

$$g(w) = w \circ \lambda(w).$$

We claim that λ is constant on each $P(u)$. Let γ be a horizontal curve in $\mathrm{SO}(W)$. Then by Leibniz rule,

$$\begin{aligned} g_*(\gamma'(t_0)) &= \tfrac{d}{dt}(\gamma(t) \circ \lambda(\gamma(t)))|_{t_0} \\ &= \tfrac{d}{dt}(\gamma(t_0) \circ \lambda(\gamma(t)))|_{t_0} + \gamma_{\lambda(\gamma(t_0))*}(\gamma'(t_0)). \end{aligned}$$

Notice that the first summand is vertical and the second is horizontal. Since g is an isometry, g_* maps the horizontal vector $\gamma'(t_0)$ to a horizontal vector. Thus

$$\frac{d}{dt}(\gamma(t_0) \circ \lambda(\gamma(t)))|_{t_0} = 0 \text{ for all } t_0.$$

This shows that λ is constant along γ. Since any two points in $P(u)$ can be connected via a horizontal path, the conclusion of the lemma follows. □

The right action of $G = \mathrm{SO}_0(p,q)$ on $\mathrm{SO}(W)$, $r(G)$ belongs to $\mathcal{K}$ by Lemma 2.8, and is transitive on the fiber. Therefore, to understand the isometry group of $\mathrm{SO}(W)$, it is enough to understand the stabilizer of the action of $\mathcal{K}$.

PROPOSITION 2.10. *For $u \in \mathrm{SO}(W)$, let $\mathcal{K}_u$ be the stabilizer of the $\mathcal{K}$-action on* $\mathrm{SO}(W)$. *Then*

1. *$\mathcal{K}_u$ fixes the whole sub-bundle $P(u)$.*
2. *$\mathcal{K}_u$ is naturally isomorphic to $C_G(\Phi(u))/\mathcal{Z}(G) \subset \mathrm{Inn}(G)$, where $C_G(\Phi(u))$ is the centralizer of $\Phi(u)$ in G.*

Proof. Statement (1) follows directly from the preceding Lemma.

(2) We identify the fiber $\pi^{-1}(\pi(u))$ with G via $u \longleftrightarrow e$. Since elements of $\mathrm{Isom}_G(\mathrm{SO}(W))$ are weakly G-equivariant, $r(G)$ is normal in $\mathrm{Isom}_G(\mathrm{SO}(W))$, and hence in $\mathcal{K}$.

Let $\mathrm{M}_G(\mathrm{SO}(W), G)$ denote the group of all G-equivariant maps from $\mathrm{SO}(W)$ to G. That is, $\lambda : \mathrm{SO}(W) \to G$ is an element of $\mathrm{M}_G(\mathrm{SO}(W), G)$ if and only if

$$\lambda(v \circ a) = a^{-1} \circ \lambda(v) \circ a$$

for all $a \in G$. Such a map λ induces a map $\widetilde{\lambda} : \mathrm{SO}(W) \to \mathrm{SO}(W)$ by

$$\widetilde{\lambda}(v) = v \circ \lambda(v).$$

At a general point,

$$\widetilde{\lambda}(v \circ a) = (v \circ a) \circ \lambda(v \circ a) = (v \circ a) \circ (a^{-1} \lambda(v) \circ a) = v \circ \lambda(v) \circ a = \widetilde{\lambda}(v) \circ a,$$

showing that $\widetilde{\lambda}$ is G-equivariant. It is well known [5] that

$$\mathrm{M}_G(\mathrm{SO}(W), G) \cdot r(G) = r(G) \times_{Z(G)} \mathrm{M}_G(\mathrm{SO}(W), G)$$

is the full group of weakly G-equivariant maps of $\mathrm{SO}(W)$ inducing the identity on W. Thus,

$$\mathcal{K} \subset \mathrm{M}_G(\mathrm{SO}(W), G) \cdot r(G).$$

Let $g \in \mathcal{K}_u$. Then g is of the form $g = \widetilde{\lambda} \circ r(c^{-1})$. Now $g(u) = u$ yields

$$u \circ \lambda(u) \circ c^{-1} = u$$

so that $\lambda(u) = c$. Then, for any $k \in G$,

$$g(u \cdot k) = (\widetilde{\lambda} \circ r(c^{-1}))(u \cdot k) = \widetilde{\lambda}(u \cdot kc^{-1}) = \widetilde{\lambda}(u) \cdot kc^{-1} = u \cdot ckc^{-1}.$$

Thus g gives rise to a unique element of $\mathrm{Inn}(G)$, defining an injective homomorphism $\mathcal{K}_u \to \mathrm{Inn}(G)$.

By (1), g fixes $P(u)$, hence it also fixes $u \cdot \Phi(u)$. Therefore, g is an inner automorphism of G leaving the subgroup $\Phi(u)$ fixed. Thus $\mathcal{K}_u \subset C_G(\Phi(u))/\mathcal{Z}(G)$.

Conversely, let $c \in C_G(\Phi(u))$. Choose any "cross section" to the holonomy bundle $P(u) \to W$; that is, a map $s : W \to P(u)$ which is not necessarily continuous, satisfying $\pi \circ s = \mathrm{id}$. Then every point of $\mathrm{SO}(W)$ is of the form $s(x) \cdot k$ for some $x \in W$ and $k \in G$ Now define a map $\hat{c} : \mathrm{SO}(W) \to \mathrm{SO}(W)$ by

$$\hat{c}(s(x) \cdot k) = s(x) \cdot ckc^{-1}.$$

This is well defined because, if $t : W \to \mathrm{SO}(W)$ is another cross section, then s and t are related by elements of $P(u)$. More precisely, let $s(x) \cdot k = t(x) \cdot k'$. Then $t(x) = s(x) \cdot h$ for some $h \in \Phi(u)$, and hence $k' = h^{-1}k$. Therefore,

$$t(x) \cdot ck'c^{-1} = s(x) \cdot hch^{-1}kc^{-1} = s(x) \cdot ckc^{-1},$$

since c commutes with $\Phi(u)$. We could have used a local smooth section for s. This implies that $\hat{c}$ is smooth. We claim that $\hat{c}$ is in fact an isometry. Let $s(x) \cdot k \in \mathrm{SO}(W)$ be an arbitrary point. On a horizontal neighborhood (i.e., k fixed, and x varying),

$$\hat{c}(s(x) \cdot k) = s(x) \cdot ckc^{-1} = (s(x) \cdot k) \cdot (k^{-1}ckc^{-1})$$

so that $\hat{c} = r_{k^{-1}ckc^{-1}}$. Thus, for a horizontal vectors X, $\hat{c}_*(X) = r_{k^{-1}ckc^{-1}*}(X)$. On the vertical vectors, recall that G has a bi-invariant metric and so the conjugation by c is already an isometry. Consequently, $\hat{c}$ is an isometry of $\mathrm{SO}(W)$ inducing the identity on W. Thus we have shown that the homomorphism $\mathcal{K}_u \to C_G(\Phi(u))/\mathcal{Z}(G)$ is an isomorphism. □

THEOREM 2.11. *There is a left action of* $C_G(\Phi(u))$ *as left translations on each fiber, and* $\mathrm{Isom}_G^{++}(\mathrm{SO}(W)) = \Big(\ell(C_G(\Phi(u))) \times_{\mathcal{Z}(G)} r(G)\Big) \rtimes \mathrm{Isom}^{++}(W)$.

Proof. By the previous lemma, we have

$$\begin{aligned}\mathrm{Isom}_G^{++}(\mathrm{SO}(W)) &= \mathcal{K}\rtimes\mathrm{Isom}^{++}(W)\\ &= (r(G)\rtimes\mathcal{K}_u)\rtimes\mathrm{Isom}^{++}(W)\\ &= (r(G)\rtimes(C_G(\Phi(u))/\mathcal{Z}(G))\rtimes\mathrm{Isom}^{++}(W)\\ &= \Big(\ell(C_G(\Phi(u)))\times_{\mathcal{Z}(G)} r(G)\Big)\rtimes \mathrm{Isom}^{++}(W).\end{aligned}$$

So, it only remains to understand how the inner automorphism group $C_G(\Phi(u))/\mathcal{Z}(G)$ gives rise to the left multiplication action $\ell(C_G(\Phi(u)))$. For any $c\in C_G(\Phi(u))$, define $\ell_c:\mathrm{SO}(W)\to\mathrm{SO}(W)$ by

$$\ell_c(s(x)\cdot k)=s(x)\cdot ck$$

so that $\ell_c=\hat{c}\circ r_{c^{-1}}$. (The map $\hat{c}$ was defined in the proof of Lemma 2.10). This finishes the proof of theorem. □

COROLLARY 2.12. *For the two extreme cases, we have the following:*

1. *If* $\Phi(u)=\mathrm{SO}_0(p,q)$, *then* $\mathrm{Isom}_G^{++}(\mathrm{SO}(W))=r(G)\times\mathrm{Isom}^{++}(W)$.
2. *If* $\Phi(u)$ *is trivial, then*

$$\mathrm{Isom}_G^{++}(\mathrm{SO}(W))=(\ell(G)\times_{\mathcal{Z}(G)} r(G))\rtimes\mathrm{Isom}^{++}(W).$$

EXAMPLE 2.13. If $W=\mathbb{R}^{p,q}$ with the indefinite metric of type (p,q), then the bundle $\mathrm{SO}(W)$ has trivial holonomy so that $C_G(\Phi(u))=G$. Thus $\mathrm{Isom}_G^{++}(\mathrm{SO}(W)) = (G\times_{\mathcal{Z}(G)} G)\rtimes\{\mathbb{R}^{p,q}\rtimes O^{++}(p,q)\}$ since $\mathrm{Isom}^{++}(W)=\mathbb{R}^{p,q}\rtimes O^{++}(p,q)$.

A pseudo-Riemannian manifold is *homogeneous* if $\mathrm{Isom}(W)$ acts on W transitively.

THEOREM 2.14. *If W is a pseudo-Riemannian homogeneous manifold, then $\mathrm{SO}(W)$ contains a sub-bundle which is isometric to the Lie group $\mathrm{Isom}^{++}(W)$ with a left invariant metric.*

Proof. Fix $u\in\mathrm{SO}(W)$, and let $x=\pi(u)\in W$, where $\pi:\mathrm{SO}(W)\to W$ is the projection. Let $H=\mathrm{Isom}^{++}(W)_x$, the stabilizer at x.

We have two bundles over W:

$$\begin{array}{ccc} H & & \mathrm{SO}_0(p,q) \\ \downarrow & & \downarrow \\ \mathrm{Isom}^{++}(W) & \longrightarrow & \mathrm{SO}(W) \\ \downarrow & & \downarrow \\ W & \xrightarrow{=} & W \end{array}$$

Both bundles are principal (with right actions). Define $\phi : H \to \mathrm{SO}_0(p,q)$ by $\phi(h) = u^{-1} \circ h_* \circ u$.

$$\begin{array}{ccc} \mathbb{R}^{p+q} & \xrightarrow{u} & T_x(W) \\ {\scriptstyle \phi(h)}\downarrow & & {\scriptstyle h_*}\downarrow \\ \mathbb{R}^{p+q} & \xleftarrow{u^{-1}} & T_x(W) \end{array}$$

Clearly ϕ is 1-1. Define $\psi : \mathrm{Isom}^{++}(W) \to \mathrm{SO}(W)$ by $\psi(f) = f_* \circ u$. Then $(\mathrm{Isom}^{++}(W), H) \to (\mathrm{SO}(W), \mathrm{SO}_0(p,q))$ *is weakly equivariant*. That is, the diagram

$$\begin{array}{ccccccc} \mathrm{Isom}^{++}(W) \times H & \longrightarrow & \mathrm{Isom}^{++}(W) & & (f,h) & \longrightarrow & f \circ h \\ {\scriptstyle \psi\times\phi}\downarrow & & {\scriptstyle \psi}\downarrow & & \downarrow & & \downarrow \\ \mathrm{SO}(W) \times \mathrm{SO}_0(p,q) & \longrightarrow & \mathrm{SO}(W) & & (\psi(f), \phi(h)) & \longrightarrow & \psi(f) \circ \phi(h) \end{array}$$

is commutative, because $\psi(f \circ h) = (f \circ h)_* \circ u = (f_* \circ u) \circ (u^{-1} \circ h_* \circ u) = \psi(f) \circ (u^{-1} \circ h_* \circ u)$. Therefore, $\mathrm{SO}(W)$ contains $\mathrm{Isom}^{++}(W)$ with a natural left invariant metric. $\square$

COROLLARY 2.15. *Assume W is a pseudo-Riemannian homogeneous manifold with $\mathrm{Isom}^{++}(W)_x \cong \mathrm{SO}_0(p,q)$. Then $\mathrm{Isom}^{++}(W) \hookrightarrow \mathrm{SO}(W)$ is a bundle isomorphism.*

3. Orthonormal frame bundle of a pseudo-sphere

3.1. Now we specialize to the case when W is a pseudo-sphere. The quadric

$$S^{p,q} = \{x \in \mathbb{R}^{p,q+1} | F(x,x) = 1\}$$

is called a *pseudo-Riemannian sphere.* It is diffeomorphic to $\mathbb{R}^p \times S^q$ via $(x, y) \mapsto (x, \frac{y}{|y|})$. It is connected for $q > 0$, and has 2 components if $q = 0$.

With this pseudo-Riemannian metric, $S^{p,q}$ has constant sectional curvature $+1$, and $\mathrm{Isom}(S^{p,q}) = O(p, q+1)$ is the full group of isometries. The isotropy subgroup at $(0, 0, \cdots, 1)$ is exactly $O(p, q)$. The connected component is $\mathrm{Isom}^{++}(W) = O^{++}(p, q+1) = \mathrm{SO}_0(p, q+1)$. By Corollary 2.15, they are equal and we have

$$\mathrm{SO}(S^{p,q}) = \mathrm{Isom}^{++}(S^{p,q}) = \mathrm{SO}_0(p, q+1)$$

as bundles over $S^{p,q}$.

3.2. *Geometry of* $\mathrm{SO}(S^{p,q})$. We would like to understand the metric on $\mathrm{SO}_0(p, q+1)$ coming from the identification with $\mathrm{SO}(S^{p,q})$. $\mathrm{SO}_0(p, q+1)$ acts on $S^{p,q}$ transitively with isotropy subgroup is $\mathrm{SO}_0(p, q)$ so that

$$\mathrm{SO}_0(p, q) \to \mathrm{SO}_0(p, q+1) \to S^{p,q}$$

is a principal $\mathrm{SO}_0(p, q)$-bundle.

Moreover, $S^{p,q}$ is a symmetric space with a symmetric structure $(\mathrm{SO}_0(p, q+1), \mathrm{SO}_0(p, q), \sigma)$, where σ is conjugation by $\begin{bmatrix} I_{p+q} & O \\ O & -1 \end{bmatrix}$. Then $\mathrm{SO}_0(p, q)$ is the connected component of

$$(\mathrm{SO}_0(p, q+1))_\sigma = \mathrm{Fix}(\sigma, \mathrm{SO}_0(p, q+1)).$$

Thus $\mathrm{SO}_0(p, q+1)/\mathrm{SO}_0(p, q)$ is a symmetric space and the canonical decomposition of the Lie algebra is

$$\mathfrak{o}(p, q+1) = \mathfrak{o}(p, q) + \mathfrak{m}$$

where $\mathfrak{o}(p, q)$ is considered as a subalgebra of $\mathfrak{o}(p, q+1)$ and $\mathfrak{m}$ is the subspace of all matrices of the following form

$$\mathfrak{o}(p, q) = \left\{ \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}, \qquad \mathfrak{m} = \left\{ \begin{bmatrix} O & O & \xi \\ O & O & \eta \\ {}^t\xi & -{}^t\eta & 0 \end{bmatrix} : \xi \in \mathbb{R}^p, \eta \in \mathbb{R}^q \right\}$$

[These are $+1$ and -1 eigen spaces of σ, respectively]. The restriction of the Killing-Cartan form of $\mathrm{SO}_0(p, q+1)$ on $\mathfrak{g}$ to $\mathfrak{m}$ defines a $\mathrm{SO}_0(p, q+1)$-invariant (indefinite) Riemannian metric on the symmetric space $S^{p,q} = \mathrm{SO}_0(p, q+1)/\mathrm{SO}_0(p, q)$. Furthermore, under the Killing-Cartan form, the factors $\mathfrak{g}$ and $\mathfrak{m}$ are orthogonal.

Thus our metric on $\mathrm{SO}(W)$ and the metric on $\mathrm{SO}_0(p,q+1)$ have the following properties: With the bundle isomorphism,

$$\begin{array}{ccc} \mathrm{SO}_0(p,q) & & \mathrm{SO}_0(p,q) \\ \downarrow & & \downarrow \\ \mathrm{SO}_0(p,q+1) & \longrightarrow & \mathrm{SO}(W) \\ \downarrow & & \downarrow \\ W & = & W \end{array}$$

1. the metrics on W are the same
2. the metrics on $\mathrm{SO}_0(p,q)$ are the same
3. With the same holonomy bundle, the fiber is orthogonal to the base.

These are enough to conclude that the two spaces are isometric. See [3, p. 232, Theorem 3.4].

3.3 (Calculation of holonomy groups). Clearly $\mathfrak{o}(p,q+1) = \mathfrak{o}(p,q)+\mathfrak{m}$ as vector spaces. Furthermore, one can show that $[\mathfrak{m},\mathfrak{m}] = \mathfrak{o}(p,q)$. Now by [3, p.103, Theorem 11.1], the Lie algebra of the holonomy group is generated by $[\mathfrak{m},\mathfrak{m}]_{\mathfrak{o}(p,q)}$, the $\mathfrak{o}(p,q)$ component of $[\mathfrak{m},\mathfrak{m}]$. But since $[\mathfrak{m},\mathfrak{m}] = \mathfrak{o}(p,q)$, $[\mathfrak{m},\mathfrak{m}]_{\mathfrak{o}(p,q)} = \mathfrak{o}(p,q)$. Therefore, the holonomy group must be the whole group $\mathrm{SO}_0(p,q)$. Now by Lemma 2.12, we have

THEOREM 3.4. $$\begin{aligned}\mathrm{Isom}_G^{++}(\mathrm{SO}(W)) &= r(G) \times_{\mathcal{Z}(G)} \mathrm{Isom}^{++}(W) \\ &= \mathrm{SO}_0(p,q) \times_{\mathcal{Z}(G)} \mathrm{SO}_0(p,q+1).\end{aligned}$$

Note that in this expression, the group G acts on $\mathrm{SO}(W) = \mathrm{SO}_0(p,q+1)$ as right multiplication by inverse elements. The group $\mathrm{Isom}^{++}(W)$ acts on $\mathrm{SO}(W)$ as lift of the isometries of W. We have seen this can be identified as left multiplication of $\mathrm{Isom}^{++}(W) = \mathrm{SO}_0(p,q+1)$ on $\mathrm{SO}(W) = \mathrm{SO}_0(p,q+1)$. Thus $(b,a) \in G \times \mathrm{Isom}^{++}(W)$ acts on $x \in \mathrm{SO}_0(p,q)$ by $(b,a)\cdot x = axb^{-1}$. Therefore we write $\mathrm{Isom}^{++}(W)$ first so that $\mathrm{Isom}_G^{++}(\mathrm{SO}(W)) = \mathrm{Isom}^{++}(W)\times G$. Thus, $\mathrm{SO}_0(p,q+1)\times \mathrm{SO}_0(p,q)$ acts on $\mathrm{SO}(S^{p,q}) = \mathrm{SO}_0(p,q+1)$ by

$$\begin{array}{ccc} (\mathrm{SO}_0(p,q+1) \times_{\mathcal{Z}(G)} \mathrm{SO}_0(p,q)) \times \mathrm{SO}(S^{p,q}) & \longrightarrow & \mathrm{SO}(S^{p,q}) \\ ((a,b),x) & \longrightarrow & a\cdot x \cdot b^{-1} \end{array}$$

multiplication in the Lie group $\mathrm{SO}_0(p,q+1)$.

3.5. On the other hand, the group $\mathrm{SO}_0(p,q+1)$ is semi-simple, the Killing-Cartan form defines a bi-invariant pseudo-Riemannian metric on $\mathrm{SO}_0(p,q+1)$. Thus

$$r(G) \subset r(\mathrm{SO}_0(p,q+1)) \subset \mathrm{Isom}(\mathrm{SO}_0(p,q+1)).$$

One can prove that then weakly G-equivariant isometries of $\mathrm{SO}_0(p,q+1)$ is the normalizer of $r(G)$ in $\ell(\mathrm{SO}_0(p,q+1)) \cdot r(\mathrm{SO}_0(p,q+1))$, which becomes

$$r(G) \times_{\mathcal{Z}(G)} \ell(\mathrm{SO}_0(p,q+1)).$$

Consequently, we get

$$\mathrm{Isom}_G^{++}(\mathrm{SO}_0(p,q+1)) = \mathrm{SO}_0(p,q) \times_{\mathcal{Z}(G)} \mathrm{SO}_0(p,q+1).$$

3.6. Let $\widetilde{\mathrm{SO}}_0(p,q+1)$ be the universal covering group of $\mathrm{SO}_0(p,q+1)$; and let $\widehat{\mathrm{SO}}(p,q)$ be the covering group of $\mathrm{SO}_0(p,q)$ induced by the universal covering $\widetilde{\mathrm{SO}}_0(p,q+1) \to \mathrm{SO}_0(p,q+1)$; that is, $\widehat{\mathrm{SO}}(p,q)$ is the preimage of $\mathrm{SO}_0(p,q)$ under this map. The principal fibering is

$$\widehat{\mathrm{SO}}(p,q) \to \widetilde{\mathrm{SO}}_0(p,q+1) \to \mathrm{SO}_0(p,q+1).$$

The isometry group is then

$$(\mathrm{Isom}_G)_0(\widetilde{\mathrm{SO}}(S^{p,q})) = \widetilde{\mathrm{SO}}_0(p,q+1) \times_{\mathcal{Z}} \widehat{\mathrm{SO}}(p,q)$$

where $\mathcal{Z}$ is the diagonal subgroup which is the center of $\mathrm{SO}_0(p,q+1)$. Thus we have a short exact sequence

$$1 \to \widehat{\mathrm{SO}}(p,q) \to (\mathrm{Isom}_G)_0(\widetilde{\mathrm{SO}}(S^{p,q})) \to \mathrm{SO}_0(p,q+1) \to 1.$$

4. Spaces modelled on $\mathrm{SO}(S^{p,q})$

We shall describe the geometries of orthonormal frame bundles of some low dimensional pseudo-spheres $S^{p,q}$. The discreteness of a group Q in $\mathrm{SO}_0(p,q+1)$ does not necessarily imply that Q acts properly on $S^{p,q}$, and in fact, such proper actions $(Q, S^{p,q})$ are quite rare. We give two examples: $S^{2,0}$ and $S^{1,2}$. The pseudo-sphere $S^{2,0}$ admits compact forms but $S^{1,2}$ does not.

THEOREM 4.1. (cf. [9, Theorem 11.1.7]) *Suppose a discrete group Q acts on $S^{p,q}$ effectively and properly. If $p \leq q$, then Q is finite.*

Proof. Let V be the subspace of $\mathbb{R}^{p,q+1}$ given by $x_1 = x_2 = \cdots = x_p = 0$, so $V \cap S^{p,q}$ is the equatorial sphere S^q. If $g \in \mathrm{GL}(p+q+1, \mathbb{R})$, then $p \le q < q+1$ implies

$$\dim(V \cap g(V)) \ge 2\dim(V) - (p+q+1) > 0.$$

Thus, for every $\alpha \in Q$, $\alpha(S^q)$ meets S^q.

If Q were infinite, then compactness of S^q would give us a point $x \in S^q$, a sequence $\{\alpha_i\} \subset Q$ of distinct elements, and a sequence $\{x_i\} \subset S^q$, such that $\alpha_i(x_i) \in S^q$ and $\{\alpha_i(x_i)\} \to x$. Passing to a subsequence, $\{x_i\} \to x' \in S^q$. As Q has closed orbits, this says $\alpha(x') = x$ for some $\alpha \in Q$. Then $\{\alpha^{-1}\alpha_i(x_i)\} \to \alpha^{-1}(x) = x'$, contradicting discontinuity of Q at x'. Thus Q is finite. □

EXAMPLE 4.2 ($\mathrm{SO}(S^{2,0})$). For $W = S^{2,0}$, $\mathrm{SO}(S^{2,0}) = \mathrm{SO}_0(2,1)$ has a principal $\mathrm{SO}_0(2,0)$-fibering structure:

$$\begin{array}{ccccc} \mathrm{SO}_0(2,0) & \longrightarrow & \mathrm{SO}_0(2,1) & \longrightarrow & S^{2,0} \\ \| & & \| & & \| \\ \mathrm{SO}(2) & \longrightarrow & \mathrm{PSL}(2,\mathbb{R}) & \longrightarrow & \mathbf{H}^2 \end{array}$$

The pseudo-sphere $S^{2,0}$ has an indefinite metric of type $(2,0)$, which is the negative of the ordinary Poincaré metric. Thus the space $\mathrm{SO}_0(2,1)$ has an indefinite metric of type $(2,1)$, negative of the Lorenz metric, (cf. [4]). The group of weakly $\mathrm{SO}_0(2,0)$-equivariant isometries is $(\mathrm{Isom}_{\mathrm{SO}(2,0)})_0(\mathrm{SO}(2,1)) = \mathrm{SO}(2,1) \times \mathrm{SO}(2)$.

The universal covering of $\mathrm{SO}_0(2,1)$ has a fibration

$$\begin{array}{ccccc} \widetilde{\mathrm{SO}}_0(2,0) & \longrightarrow & \widetilde{\mathrm{SO}}_0(2,1) & \longrightarrow & S^{2,0} \\ \| & & \| & & \| \\ \mathbb{R} & \longrightarrow & \widetilde{\mathrm{PSL}}(2,\mathbb{R}) & \longrightarrow & \mathbf{H}^2 \end{array}$$

The group of weakly $\widetilde{\mathrm{SO}}_0(2,0)$-equivariant isometries is

$$(\mathrm{Isom}_{\widetilde{\mathrm{SO}}(2,0)})_0(\widetilde{\mathrm{SO}}(2,1)) = \mathbb{R} \times_{\mathbb{Z}} \widetilde{\mathrm{SO}}(2,1).$$

Note that this is also the group of isometries for the $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$-geometry.

Let $1 \to \mathbb{Z} \to \Pi \to Q \to 1$ be an central extension of $\mathbb{Z}$ by a Fuchsian group Q. If Q is cocompact, then $H^2(Q;\mathbb{Z}) \approx \mathbb{Z} \oplus \mathrm{Torsion}$; if Q is finitely generated and not cocompact, then $H^2(Q;\mathbb{Z})$ is a direct sum of finite cyclic groups corresponding to the conjugacy classes of the maximal finite subgroups of Q.

(Structure Theorem) [4]. *Let* $1 \to \mathbb{Z} \to \Pi \to Q \to 1$ *be an central extension of* $\mathbb{Z}$ *by a Fuchsian group* Q. *Then* Π *embeds into* $(\mathrm{Isom}_{\widetilde{\mathrm{SO}}(2,0)})_0(\widetilde{\mathrm{SO}}(2,1))$ *if and only if* $[\Pi] \in H^2(Q, \mathbb{Z})$ *has infinite order.*

This embedding yields $\Pi\backslash P$ as a Seifert orbifold with a complete Lorentz metric of constant curvature. The fibers are time-like geodesics. If $Q\backslash S^{2,0}$ is non-compact, then each $[\Pi]$ embeds. The same construction also yields embeddings of Π into $\mathrm{Isom}(\widetilde{\mathrm{PSL}}(2, \mathbb{R}))$ and $\Pi\backslash P$ is a Seifert orbifold with the 3-dimensional $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$-geometry. See [6] and [4] for the embeddings.

EXAMPLE 4.3 ($\mathrm{SO}(S^{1,2})$). For $W = S^{1,2}$, $\mathrm{SO}(S^{1,2}) = \mathrm{SO}_0(1,3)$ has a principal $\mathrm{SO}_0(1,2)$-fibering structure:

$$\mathrm{SO}_0(1,2) \longrightarrow \mathrm{SO}_0(1,3) \longrightarrow S^{1,2}.$$

Since $\pi_1(\mathrm{SO}_0(1,3)) = \mathbb{Z}_2$ and $\pi_1(\mathrm{SO}_0(1,2)) = \mathbb{Z}$, the bundle is nontrivial. Then $(\mathrm{Isom}_{\mathrm{SO}(1,2)})_0(\mathrm{SO}(S^{1,2})) = \mathrm{SO}_0(1,2) \times \mathrm{SO}_0(1,3)$ and the action on $\mathrm{SO}(S^{1,2}) \approx \mathrm{SO}_0(1,3)$ is given by

$$(a,b)x = b \cdot x \cdot a^{-1}$$

for $(a,b) \in \mathrm{SO}_0(1,2) \times \mathrm{SO}_0(1,3)$ and $x \in \mathrm{SO}(S^{1,2})$.

Suppose there exists a compact space form $\Pi\backslash\mathrm{SO}(S^{1,2})$. That is, Π is a discrete subgroup of $\mathrm{SO}_0(1,2) \times \mathrm{SO}_0(1,3)$ which acts on $\mathrm{SO}(S^{1,2})$ properly with $\Pi\backslash\mathrm{SO}(S^{1,2})$ compact. [Such a Π exists]. Then $\Gamma = \Pi \cap \mathrm{SO}_0(1,2)$ is never cocompact in $\mathrm{SO}_0(1,2)$. If it were, the quotient $Q = \Pi/\Gamma$ will act on $S^{1,2}$ properly. By Theorem 4.1, Q must be finite. This implies that $\Pi\backslash\mathrm{SO}(S^{1,2})$ is not compact, a contradiction.

Our model space for the geometry must be simply connected. Therefore, we take the universal covering space of $\mathrm{SO}(S^{1,2})$ as our model space. Since $\pi_1(\mathrm{SO}_0(1,3)) = \mathbb{Z}_2$, $\widetilde{\mathrm{SO}}_0(1,3)$ is a double covering of $\mathrm{SO}_0(1,3)$. Let $\widehat{\mathrm{SO}}_0(1,2)$ be the corresponding double covering of $\mathrm{SO}_0(1,2)$. Then $\widetilde{\mathrm{SO}}_0(1,3)$ is a principal $\widehat{\mathrm{SO}}_0(1,2)$-bundle,

$$\widehat{\mathrm{SO}}_0(1,2) \longrightarrow \widetilde{\mathrm{SO}}_0(1,3) \longrightarrow S^{1,2}.$$

Note that $\widehat{\mathrm{SO}}_0(1,2)$ is "level 2" group in [4], and is isomorphic to $\mathrm{SL}(2,\mathbb{R})$, since $\pi_1(\widehat{\mathrm{SO}}_0(1,2)) = \mathbb{Z}_2$. Therefore, $\widetilde{\mathrm{SO}}_0(1,3)$ is diffeomorphic to $S^3 \times \mathbb{R}^3$. From $(\mathrm{Isom}_{\mathrm{SO}(1,2)})_0(\mathrm{SO}(S^{1,2})) = \mathrm{SO}_0(1,2) \times \mathrm{SO}_0(1,3)$, we have

$$(\mathrm{Isom}_{\widehat{\mathrm{SO}}(1,2)})_0(\widetilde{\mathrm{SO}}(S^{1,2})) = \widehat{\mathrm{SO}}_0(1,2) \times_{\mathbb{Z}_2} \widetilde{\mathrm{SO}}_0(1,3),$$

where $\mathbb{Z}_2$ is the diagonal center.

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