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MODULES WITH PRIME ENDOMORPHISM RINGS

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ABSTRACT. Some discrimination of modules whose endomorphism rings are prime is introduced, by means of structures of submodules inducing prime ideals of the endomorphism ring $\mathrm{End}_R(M)$ of a left $R-$module ${}_RM$ over a ring R. Modules with non-prime endomorphism rings are contrapositively studied as well.

1. Introduction

For any associative ring R and any left $R-$module ${}_RM$, its endomorphism ring $\mathrm{End}_R(M)$ will act on the right side of ${}_RM$, in other words, ${}_RM_{\mathrm{End}_R(M)}$ will be studied mainly. Thus the composite of functions preserves the order such that the composite

$$fg : A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

of $f : A \to B$ and $g : B \to C$ defined by $afg = (af)g$ for every $a \in A$. Without conflict, for any mapping $f : M \to N, K \subseteq M, L \subseteq N$ we also frequently will use notations of the image $f(K) = Kf$ of K under f and the preimage $f^{-1}(L) = Lf^{-1}$ of L under f as usual.

For any left $R-$module ${}_RM$, the endomorphism ring $\mathrm{End}_R(M)$ is said to be a prime ring if $fg = 0$ implies that $f = 0$ or $g = 0$. If $fg = 0$ with an epimorphism f or a monomorphism g, then $f = 0$ or $g = 0$ follows. For instance, if every nonzero endomorphism $f : {}_RM \to {}_RM$ is a monomorphism(or an epimorphism), then it clearly follows that $\mathrm{End}_R(M)$ is a prime ring. However there are some modules satisfying none of these. In order to study these modules having prime endomorphism rings we need some definitions of submodules of modules.

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For any subset J of $\mathrm{End}_R(M)$, let $\mathrm{Im}J = MJ = \sum_{f\in J}\mathrm{Im}f$ and $\ker J = \cap_{f\in J}\ker f$ be the sum of images of endomorphisms in J and the intersection of kernels of endomorphisms in J, respectively. Also we call N an *open* submodule if $N = N^o$, where $N^o = \sum_{f\in S, \mathrm{Im}f\leq N}\mathrm{Im}f$ is the sum of all images of endomorphisms contained in N and call N a *closed* submodule if $N = \overline{N}$, where $\overline{N} = \cap_{f\in S, N\leq \ker f}\ker f$ is the intersection of all kernels of endomorphisms containing N, and where $S = \mathrm{End}_R(M)$.

Here are some simple and easy conditions for any module $_RM$ to have a prime endomorphism ring:

(1) If each nonzero open submodule A is isomorphic or equal to M, it clearly follows that the endomorphism ring $\mathrm{End}_R(M)$ is a prime ring.
(2) If each nonzero closed submodule is isomorphic or equal to M, then the endomorphism ring $\mathrm{End}_R(M)$ is a prime ring.

However these kinds of definitions would give non-enough informations of prime endomorphism rings. Here are other definitions of submodules inducing prime ideals of endomorphism rings which was studied in [6]. Some results from [6] are written in this section.

DEFINITION 1.1 ([6]). For a submodule $P \leq M$ of a left $R-$module $_RM$, P is said to be a *meet-prime* submodule of $_RM$ if it satisfies the following conditions; for any *open* submodules $A, B \leq M$ with $P^o + A \neq M$ or $P^o + B \neq M$,

(1) if $A \cap B \leq P$, then $A \leq P$ or $B \leq P$,
(2) if $(P \cap A \cap B)^o \neq 0$, then $A \leq P$ or $B \leq P$,
(3) if $P \cap A = 0$, then $A = 0$ or $P + A = M$.

A module $_RM$ is said to be *meet-prime* if the trivial submodule 0 of $_RM$ is *meet-prime.*

In particular, if the trivial submodule $0 \leq M$ of a module $_RM$ satisfies the item (1), then we will call the trivial 0 a *quasi-meet-prime* submodule(or meet-irreducible in terms of open submodules)of $_RM$, or will call $_RM$ a *quasi-meet-prime* module.

DEFINITION 1.2. For a left $R-$module $_RM$, $0 \leq M$ is said to be a $\cap-$*prime*(or *intersection-prime*, or *cap-prime*) submodule of $_RM$ if it satisfies the following conditions: for any *open* submodules $A, B \leq M$,

(1) if $A \cap B \leq 0$, then $A = 0$ or $B = 0$,
(2) $A = 0$, or A is isomorphic or equal to M(briefly, denoted by $A \simeq M$).

A module ${}_RM$ is said to be $\cap-$prime if the trivial submodule 0 of ${}_RM$ is $\cap-$prime.

Clearly in any module if 0 is meet-prime, then 0 is $\cap-$prime, in other words, every meet-prime module is a $\cap$-prime module. However the converse is not true in general, for example, the integer ring ${}_{\mathbb{Z}}\mathbb{Z}$ has the trivial $0 \le \mathbb{Z}$ is a $\cap-$prime submodule $0 \le \mathbb{Z}$ but not a meet-prime submodule of it.

Easily for any submodule $P \le M$, we have that P is meet-prime if and only if P^o is meet-prime and that every module isomorphism preserves the meet-primeness and the $\cap$-primeness between isomorphic modules.

Recall a module ${}_RM$ is said to be *simple* if all submodules of ${}_RM$ are only the trivial submodules 0 and M itself. Likewise, we define a module ${}_RM$ to be *openly simple* by all open submodules of ${}_RM$ are only the trivial submodules 0 and M itself.

REMARK 1.3. Any simple module is openly simple, however the converse is not true in general. For the integer ring $\mathbb{Z}$, a left $\mathbb{Z}-$module ${}_{\mathbb{Z}}\mathbb{Z}(p^\infty)$ for prime p is openly simple but not simple.

LEMMA 1.4. *For any left $R-$module ${}_RM$, we have that $0 \le M$ is meet-prime in ${}_RM$ if and only if ${}_RM$ is openly simple.*

Hereafter S denotes the endomorphism ring $\mathrm{End}_R(M)$ of a left R-module ${}_RM$.

LEMMA 1.5. *For any left $R-$module ${}_RM$, we have the following:*

(1) *If $P \le M$ is any fully invariant meet-prime submodule of ${}_RM$, then $I^P = \{ f \in S \mid \mathrm{Im} f \le P \} \trianglelefteq S$ is a prime ideal of S.*
(2) *If $0 \le M$ is a $\cap$-prime submodule of ${}_RM$, then $0 \trianglelefteq S$ is a prime ideal of S, that is, S is a prime ring.*

PROPOSITION 1.6. *For any left $R-$module ${}_RM$, if at least one of the following is satisfied:*

(1) *${}_RM$ is an openly simple module.*
(2) *For each nonzero endomorphism $f : {}_RM \to {}_RM$, $(\ker f)^o = 0$.*
(3) *Every nonzero open submodule is isomorphic or equal to M.*

(4) *Every open submodule of* $_RM$ *is fully invariant essential(or large) and* $0 \leq M$ *is quasi-meet-prime.*
(5) S *is commutative and* $0 \leq M$ *is quasi-meet-prime.*
(6) *The zero submodule* $0 \leq M$ *is* $\cap$*-prime.*

Then the endomorphism ring S *is a prime ring.*

A left $R-$module $_RM$ is said to be *self-generated* if each submodule of $_RM$ is open ([4]). It is clear that for any self generated module $_RM$, 0 is meet-prime if and only if $_RM$ is simple.

DEFINITION 1.7 ([6]). For a submodule $P \leq M$ of a left $R-$module $_RM$, we will say that P is a sum-prime submodule of $_RM$ if it satisfies the following conditions: for any *closed* submodules $A, B \leq M$ with $\overline{P} \cap A \neq 0$ or $\overline{P} \cap B \neq 0$,

(1) if $P \leq A + B$, then $P \leq A$ or $P \leq B$,
(2) if $\overline{P + A + B} \neq M$, then $P \leq A$ or $P \leq B$,
(3) if $P + A = M$, then $A = M$ or $P \cap A = 0$.

A module $_RM$ is said to be *sum-prime* if M is a sum-prime submodule of $_RM$. In particular, if the trivial submodule M of a module $_RM$ satisfies the item (1), then we will call $_RM$ *quasi-sum-prime*(or sum-irreducible in terms of closed submodules).

DEFINITION 1.8. For a left $R-$module $_RM$, we will say that M is a +prime submodule of $_RM$ if it satisfies the following conditions: for any *closed* submodules $A, B \leq M$,

(1) if $M \leq A + B$, then $M = A$ or $M = B$,
(2) $A = 0$ or $A \simeq M$ is isomorphic or equal to M.

A module $_RM$ is said to be +prime if M is a +prime submodule of $_RM$.

Clearly for any submodule $P \leq M$, we have that P is a sum-prime submodule of $_RM$ if and only if $\overline{P}$ is a sum-prime submodule of $_RM$ and that every module isomorphism preserves the sum-primeness and the +primeness between isomorphic modules. We also have that every sum-prime module is a +prime module.

We also define a module $_RM$ to be *closedly simple* by all the closed submodules of $_RM$ are the trivial submodules 0 and M only.

REMARK 1.9. Any simple module is also closedly simple, however the converse is not true in general. For the integer ring $\mathbb{Z}$, a left $\mathbb{Z}-$module ${}_{\mathbb{Z}}\mathbb{Z}$ is closedly simple but not simple.

LEMMA 1.10. *For any left $R-$module ${}_RM$, we have that M is sum-prime in ${}_RM$ if and only if ${}_RM$ is closedly simple.*

LEMMA 1.11 ([6]). *For any left $R-$module ${}_RM$, we have the following.*

1. *If $P \leq M$ is any fully invariant sum-prime submodule of ${}_RM$, then $I_P = \{ f \in S \mid P \leq \ker f \}$ is a prime ideal of S.*
2. *If M is a $+$-prime submodule of ${}_RM$, then $0 \trianglelefteq S$ is a prime ideal of S, that is, S is a prime ring.*

PROPOSITION 1.12. *For any left $R-$module ${}_RM$, if at least one of the following is satisfied:*

1. *${}_RM$ is a closedly simple module.*
2. *For each nonzero endomorphism $f : {}_RM \to {}_RM$, $\overline{\mathrm{Im} f} = \overline{Mf}$ is improper, i.e., $\overline{\mathrm{Im} f} = \overline{Mf} = M$.*
3. *Every nonzero closed submodule is isomorphic or equal to M.*
4. *Every closed submodule of ${}_RM$ is fully invariant superfluous(or small) and $M \leq {}_RM$ is quasi-sum-prime.*
5. *S is commutative and $M \leq {}_RM$ is quasi-sum-prime.*
6. *The trivial submodule $M \leq {}_RM$ is $+$-prime.*

Then the endomorphism ring S is a prime ring.

A left ${}_RM$ is said to be *self-cogenerated* if each submodule of ${}_RM$ is closed ([4]).

It is clear that any self cogenerated sum-prime module is simple.

2. Meet-prime or $\cap$-prime submodules under homomorphisms

For any function $f : {}_RM \to {}_RN$ the preimage assignment of f, conveniently denoted by f^{-1} or $f^{\leftarrow} : \mathcal{P}(N) \to \mathcal{P}(M)$ from the power set $\mathcal{P}(N)$ of ${}_RN$ into the power set $\mathcal{P}(M)$ of ${}_RM$ is a function always.

An $R-$homomorphism $f : {}_RM \to {}_RN$ is said to be open if the image assignment $f : \mathcal{P}(M) \to \mathcal{P}(N)$ preserves open submodules, in other words, $f(A) \leq N$ is an open submodule of ${}_RN$, for any open submodule $A \leq M$.

THEOREM 2.1. *For any open monomorphism $f : {}_RM \to {}_RN$, we have the following.*

(1) *If P is a meet-prime submodule of ${}_RN$, then $f^{-1}(P)$ is also a meet-prime submodule of ${}_RM$.*
(2) *If ${}_RN$ is a $\cap$-prime module, then ${}_RM$ is also a $\cap$-prime module.*

Proof. (1) For any open submodules $A, B \leq M$ such that $f^{-1}(P) + A \neq M$ or $f^{-1}(P) + B \neq M$, (i) if $A \cap B \leq f^{-1}(P)$, then since f is a monomorphism $f(A \cap B) = f(A) \cap f(B) \leq P$. Since f is open and P is meet(resp. $\cap$)-prime in ${}_RN$ it follows that $f(A) \leq P$ or $f(B) \leq P$. Therefore $A \leq f^{-1}(P)$ or $B \leq f^{-1}(P)$.

(ii) If $[A \cap B \cap f^{-1}(P)]^o \neq 0$, then $A \cap B \cap f^{-1}(P) \neq 0$ follows immediately. From the openness of the monomorphism f it follows easily that $0 \neq f(A) \cap f(B) \cap f(f^{-1}(P)) \leq f(A) \cap f(B) \cap P, f(A), f(B)$ are open submodules of ${}_RN$ such that $P^o + f(A) \neq N$ or $P^o + f(B) \neq N$. From the meet(resp. $\cap$)-primeness of P it follows that $f(A) \leq P$ or $f(B) \leq P$ and hence $A \leq f^{-1}(P)$ or $B \leq f^{-1}(P)$.

(iii) If $A \cap f^{-1}(P) = 0$(resp. with $f^{-1}(P) \neq 0$), then from a monomorphism f it follows that $f(A) \cap P = 0$.

Thus $f(A) = 0$ or $P + f(A) = N$ follows from the meet (resp. $\cap$)-primeness of P. Hence we have clearly that $A = 0$ or $f^{-1}(P) + A = M$.

(2): If $f^{-1}(P) = 0$, then $P \cap f(M) = 0$. For the case of $P \neq 0$ we have that $P + f(A) = P \oplus f(A) = N$ and hence $A = f^{-1}(N) = M$. For the case of $P = 0$ we have that $f(A) = 0$ or $f(A) \simeq N$. Since $f(M) \leq N$ is an open submodule of ${}_RN$ we have that $f(M) = 0$ or $f(M) \simeq N$. Therefore $f(A) = 0$ or $f(A) \simeq f(M)$ and hence $A = 0$ or $A \simeq M$. □

COROLLARY 2.2. *For any monomorphism $f : {}_RM \to {}_RN$ with a self-generated module ${}_RN$, we have the following.*

(1) *If P is a meet-prime submodule of ${}_RN$, then $f^{-1}(P)$ is also a meet-prime submodule of ${}_RM$.*
(2) *If ${}_RN$ is a $\cap$-prime module, then ${}_RM$ is also a $\cap$-prime module.*

Proof. Since for any self-generated module ${}_RN$ any homomorphism $f : {}_RM \to {}_RN$ is an open mapping. Thus the proof is completed by the same proof of Theorem 2.1. $\square$

REMARK 2.3. It is careful to apply the above Theorem 2.1 to the inclusion mapping $\iota : {}_RK \hookrightarrow {}_RM$. Since for any submodule $K \leq M$, the open submodule $A = \sum_{g \in \mathrm{End}_R(K), Kg \leq A} Kg \neq \sum_{f \in \mathrm{End}_R(M), Mf \leq A} Mf$, in general. In other words, it is not necessary for all open submodules in any submodule ${}_RK \leq {}_RM$ to be open submodules of ${}_RM$. For example, for any prime number p, a module ${}_{\mathbb{Z}}\mathbb{Z}(p^\infty)$ having a submodule $K = \{\, \overline{0}, \overline{1/p}, \overline{2/p}, \cdots, \overline{(p-1)/p}, \overline{1/p^2}, \overline{2/p^2}, \cdots, \overline{(p-1)/p^2}\,\} \leq \mathbb{Z}(p^\infty)$ is such a module that the inclusion mapping $\iota : {}_{\mathbb{Z}}K \hookrightarrow {}_{\mathbb{Z}}\mathbb{Z}(p^\infty)$ is not an open mapping.

COROLLARY 2.4. *For any module ${}_RM$ and for a submodule $K \leq M$ such that each open submodule $A \leq K$ of ${}_RK$ is open in ${}_RM$, that is,* $A = \sum\limits_{g \in \mathrm{End}_R(K), Kg \leq A} Kg = \sum\limits_{f \in \mathrm{End}_R(M), Mf \leq A} Mf$, *we have the following.*

(1) *If P is a meet-prime submodule of ${}_RM$, then $P \cap K$ is meet-prime in ${}_RK$.*
(2) *If ${}_RN$ is a $\cap$-prime module, then ${}_RM$ is also a $\cap$-prime module.*

Proof. Since the inclusion $\iota : {}_RK \to {}_RM$ is an open monomorphism by Theorem 2.1, we have that $P \cap K$ is a meet(resp. $0 \leq K$ is $\cap$)-prime submodule of ${}_RK$. $\square$

COROLLARY 2.5. *For any $\cap$-prime module ${}_RM$ and for a submodule $K \leq M$ such that each open submodule*

$$A = \sum_{g \in \mathrm{End}_R(K), Kg \leq A} Kg = \sum_{f \in \mathrm{End}_R(M), Mf \leq A} Mf,$$

we have a $\cap$-prime module ${}_RK$ and furthermore $\mathrm{End}_R(K)$ is a prime endomorphism ring.

Proof. Considering the inclusion mapping $\iota : {}_RK \hookrightarrow {}_RM$, then we have a monomorphism ι such that $\ker\iota = 0$ is also $\cap$-prime in ${}_RK$ by the $\cap$-primeness of 0 in ${}_RM$. Hence the endomorphism ring $\mathrm{End}_R(K)$ is prime. $\square$

COROLLARY 2.6. *For a self-generated $\cap$-prime module ${}_RM$ and for any submodule $K \leq M$, we have a $\cap$-prime module ${}_RK$ and furthermore $\mathrm{End}_R(K)$ is a prime endomorphism ring.*

Proof. Since the inclusion mapping $\iota : {}_RK \hookrightarrow {}_RM$ with a self-generated module ${}_RM$ is an open monomorphism always. From Corollary 2.5 it follows that ${}_RK$ is also a $\cap$-prime module, i.e., $0 \leq K$ is $\cap$-prime and hence the endomorphism ring $\mathrm{End}_R(K)$ is a prime ring. □

THEOREM 2.7. *For any R-epimorphism $f : {}_RM \to {}_RN$ with the open preimage assignment and for a submodule $Q \leq N$ of ${}_RN$, we have the following.*

(1) *If $f^{-1}(Q) \leq M$ is meet-prime, then Q is a meet-prime submodule of ${}_RN$.*
(2) *If $\mathrm{ker} f \leq M$ is a meet-prime submodule of ${}_RM$, then ${}_RN$ is a meet-prime module, and furthermore we have a meet-prime quotient module ${}_RM/\mathrm{ker} f$.*

COROLLARY 2.8. *For any R-epimorphism $f : {}_RM \to {}_RN$ with a self generated module ${}_RM$ and for a submodule $Q \leq N$ of ${}_RN$, we have the following.*

(1) *If $f^{-1}(Q) \leq M$ is meet-prime, then Q is a meet-prime submodule of ${}_RN$.*
(2) *If $\mathrm{ker} f \leq M$ is a meet-prime submodule of ${}_RM$, then ${}_RN$ is a meet-prime module. Furthermore we have a meet-prime quotient module ${}_RM/\mathrm{ker} f$.*

Proof. Since ${}_RM$ is self-generated module any homomorphism ${}_RM \to {}_RN$ has the open preimage assignment. By Theorem 2.7 the proof is established easily. □

For any module ${}_RM$ and for any submodule $K \leq M$ of ${}_RM$, considering the quotient module ${}_RM/K$ and the projection $\pi : {}_RM \to {}_RM/K$, additionally if K is open and fully invariant, then the projection $\pi : {}_RM \to {}_RM/K$ has an open image assignment, i.e. we have an open submodule $\pi(A) \leq M/K$ for any open submodule $A \leq M$ such that $A \supseteq K$.

REMARK 2.9. However the projection π doesn't have an open preimage assignment in general. For example, let ${}_{\mathbb{Z}}\mathbb{Q}$ be the $\mathbb{Z}$–module of rational numbers over the integer ring $\mathbb{Z}$. Then $\pi : {}_{\mathbb{Z}}\mathbb{Q} \to {}_{\mathbb{Z}}\mathbb{Q}/\mathbb{Z}$ doesn't have an open preimage assignment.

We have an immediate consequence of the above Theorem 2.7 that the meet-primeness is cohereditary in a kind of the factor modules.

COROLLARY 2.10. *For any module ${}_RM$ and for any fully invariant open submodule $K \leq M$ of ${}_RM$, if $P \leq M$ such that $K \leq P$ and if $\pi(P) \leq M/K$ is meet-prime, then P is a meet-prime submodule of ${}_RM$.*

COROLLARY 2.11. *For any module ${}_RM$ and for any open fully invariant submodule K of ${}_RM$, if the quotient module ${}_RM/K$ is meet-prime, then $K \leq M$ is meet-prime.*

Proof. Since the projection mapping $\pi : {}_RM \to {}_RM/K$ has an open image assignment for each open fully invariant submodule $K \leq M$. Additionally if $0 = K \leq M/K$ is meet-prime in ${}_RM/K$, then we have immediately that $K \leq M$ is a meet-prime submodule of ${}_RM$. □

THEOREM 2.12. *For a self-generated module ${}_RN$ and for any R-epimorphism $f : {}_RM \to {}_RN$, if $P \leq N$ is meet-prime in ${}_RN$, then $f^{-1}(P) \leq M$ is a meet-prime submodule of ${}_RM$.*

Proof. For any R–homomorphism $f : {}_RM \to {}_RN$ with a self-generated module ${}_RN$, we have the induced isomorphism $\overline{f} : {}_RM/\ker f \to {}_RN$ of $f : {}_RM \to {}_RN$. From the self-generatedness of ${}_RN$ it follows that ${}_RM/\ker f$ is also a self-generated module. Thus the projection $\pi : {}_RM \to {}_RM/\ker f$ is an open epimorphism.

Now that $P \leq N$ is meet-prime if and only if $\overline{f}^{-1}(P) \leq M/\ker f$ is meet-prime it remains to show that $f^{-1}(P) \leq M$ is meet-prime for any given meet-prime submodule $P \leq N$.

By the Corollary 2.10 it immediately concludes that $f^{-1}(P) \leq M$ is meet-prime if $P \leq N$ is a meet-prime submodule of ${}_RN$. □

COROLLARY 2.13. *For self-generated modules ${}_RM, {}_RN$, for any submodule $P \leq N$ of ${}_RN$, and for any $R-$epimorphism $f : {}_RM \to {}_RN$, the following are equivalent:*

(1) *$P \leq N$ is a meet-prime submodule of ${}_RN$;*
(2) *$f^{-1}(P) \leq M$ is a meet-prime submodule of ${}_RM$.*

Proof. By the Corollary 2.10 and by the Theorem 2.12 the proof is completed at once. □

3. Sum-prime or +prime submodules under homomorphisms

An $R-$homomrophism $f : {}_RM \to {}_RN$ is said to be *closed* if the image assignment $f : \mathcal{P}(M) \to \mathcal{P}(N)$ preserves closed submodules, in other words, $f(A) \leq N$ is a closed submodule of ${}_RN$, for any closed submodule $A \leq M$. For example, any inclusion mapping $\iota : n\mathbb{Z} \hookrightarrow \mathbb{Z}$(for any $n \in \mathbb{N}$) is a closed monomorphism. We have some results for sum-prime submodules.

THEOREM 3.1. *For any closed monomorphism $f : {}_RM \to {}_RN$ and for a submodule $Q \leq M$ of ${}_RM$, we have the following.*

(1) *If $f(\overline{Q})$ is sum-prime in ${}_RN$, then Q is sum-prime in ${}_RM$.*
(2) *If ${}_RN$ is +(or sum−)prime, then ${}_RM$ is +(or sum−)prime, respectively.*

Proof. (1): It is elementary.

(2): (ii) It first is going to show that for any closed submodule $A \leq M, A = 0$ or $A \simeq M$. Since the improper submodule $M = \ker 0$ is a closed submodule of ${}_RM$ we have a closed submodule $f(M) \leq N$. From the +primeness of $N \leq N$ it follows that $f(A) = 0$ or $f(A) \simeq N$ for any closed submodule $A \leq M$ and also we have $f(M) \simeq N$. Hence we have that $A = 0$ or $A \simeq M$ by the monomorphism f. (i) if $M \leq A + B$ with closed submodules A, B such that $A \neq 0$ or $B \neq 0$, then $f(M) \leq f(A+B) = f(A) + f(B)$ with all cosed submodules $f(M), f(A) + f(B), f(A), f(B) \leq N$ and $f(M) \simeq N, f(A) + f(B) \simeq N, f(A) \simeq N$ or $f(B) \simeq N$. Since N is $\cap$-prime in ${}_RN$ we have that $f(M) \leq f(A)$ or $f(M) \leq f(B)$. Then it follows that $M \leq A$ or $M \leq B$. Therefore ${}_RM$ is a +prime module. For the case of a sum-prime module ${}_RN$, the similar method by replacing $\simeq$ by $=$ completes the proof. □

COROLLARY 3.2. *For any monomorphism* $f : {}_RM \to {}_RN$ *with a self-cogenerated module* ${}_RN$ *and for a submodule* $Q \leq M$ *of* ${}_RM$, *we have the following.*

(1) *If* $f(\overline{Q})$ *is sum-prime in* ${}_RN$, *then* Q *is sum-prime in* ${}_RM$.
(2) *If* ${}_RN$ *is* $+$(*or sum*$-$)*prime, then* ${}_RM$ *is* $+$(*or sum*$-$)*prime, respectively.*

Proof. Since any homomorphism $f : {}_RM \to {}_RN$ with a self-cogenerated module ${}_RN$ is a closed mapping, especially for any closed submodule $A \leq M$ we have that $f(A) \leq N$ is a closed submodule of a self-cogenerated module ${}_RN$. Thus the proof is completed by Theorem 3.1. □

REMARK 3.3. It is careful to apply the above Theorem 3.1 to the inclusion mapping $\iota : {}_RK \hookrightarrow {}_RM$. Since any closed submodule $A = \bigcap_{g \in \mathrm{End}_R(K); A \leq \ker g} \ker g \neq \bigcap_{f \in \mathrm{End}_R(M); A \leq \ker f} \ker f$ of a submodule ${}_RK$ $(\leq {}_RM)$ need not to be a closed submodule of ${}_RM$, in general. In other words, it is not necessary for all closed submodules in ${}_RK$(for $K \leq M$) to be closed submodules of ${}_RM$. For example, a module ${}_{\mathbb{Z}}\mathbb{Q}$ having a submodule ${}_{\mathbb{Z}}\mathbb{Z} \leq {}_{\mathbb{Z}}\mathbb{Q}$ is such a module that the inclusion mapping $\iota : {}_{\mathbb{Z}}\mathbb{Z} \hookrightarrow {}_{\mathbb{Z}}\mathbb{Q}$ is not a closed mapping.

COROLLARY 3.4. *For any module* ${}_RM$ *and for a submodule* $K \leq M$, *if the inclusion mapping* $\iota : {}_RK \hookrightarrow {}_RM$ *is a closed monomorphism, then we have the following.*

(1) *If* $f(\overline{Q})$ *is sum-prime in* ${}_RM$, *then* Q *is a sum-prime submodule of* ${}_RK$.
(2) *If* ${}_RN$ *is* $+$(*or sum*$-$)*prime, then* ${}_RM$ *is* $+$(*or sum*$-$)*prime, respectively.*

Proof. It is an immediate consequence of Theorem 3.1. □

COROLLARY 3.5. *For any module* ${}_RM$ *and for a submodule* $K \leq M$, *if the inclusion mapping* $\iota : {}_RK \hookrightarrow {}_RM$ *is a closed monomorphism, then we have the following.*

(1) *If* K *is sum-prime in* ${}_RM$, *then* K *is sum-prime in* ${}_RK$ *and hence* $\mathrm{End}_R(K)$ *is prime.*
(2) *If* ${}_RM$ *is* $+$(*or sum*$-$)*prime, then* ${}_RK$ *is* $+$(*or sum*$-$)*prime, respectively.*

Proof. Since a submodule $K \leq M$ is sum-prime if and only if $\overline{K} \leq M$ is sum-prime and since the inclusion mapping $\iota : {}_RK \to {}_RM$ is a closed monomorphism it follows quickly from Theorem 3.1 that K is sum-prime in ${}_RK$. Furthermore we have a prime endomorphism ring $\mathrm{End}_R(K)$. □

COROLLARY 3.6. *For any self-cogenerated module ${}_RM$ and any submodule $K \leq M$, we have the following.*

(1) *If K is sum-prime in ${}_RM$, then K is sum-prime in ${}_RK$ and hence the endomorphism ring $\mathrm{End}_R(K)$ is prime.*
(2) *Additionally if ${}_RM$ is $+$(or sum$-$)prime, then every submodule ${}_RK$ is $+$(or sum$-$)prime, respectively. And hence we have a prime endomorphism ring $\mathrm{End}_R(K)$.*

Proof. Since every submodule $K \leq M$ is a closed submodule of ${}_RM$ every closed submodule of ${}_RK$ is also a closed submodule of a self-cogenerated module ${}_RM$ and thus we have that the inclusion $\iota : {}_RK \hookrightarrow {}_RM$ is a closed monomorphism. By Theorem 3.1 we have that K is sum-prime in ${}_RK$. Therefore the prime endomorphism $\mathrm{End}_R(K)$ is obtained automatically. □

THEOREM 3.7. *For any epimorphism $f : {}_RM \to {}_RN$ with the closed preimage assignment and for a submodule $P \leq N$ of ${}_RN$, we have the following.*

(1) *If $f^{-1}(P)$ is a sum-prime submodule of ${}_RM$, then P is also a sum-prime submodule of ${}_RN$.*
(2) *If ${}_RM$ is a $+$(or sum$-$)prime module, then ${}_RN$ is $+$(or sum$-$) prime, respectively.*

Proof. (1): It is sufficient to show that $\overline{P}$ is a sum-prime submodule of ${}_RN$. For any closed submodule $C \leq N$ with $\overline{P} \cap C \neq 0$, we have that $f^{-1}(\overline{P}) \cap f^{-1}(C) \neq 0$. And $f^{-1}(\overline{P}) \cap f^{-1}(C) = \overline{f^{-1}(\overline{P})} \cap f^{-1}(C) \neq 0$ follows from the closed preimage assignment of f.

For any closed submodules $A, B \leq N$ with $\overline{P} \cap A \neq 0$ or $\overline{P} \cap B \neq 0$, we also have that $f^{-1}(\overline{P}) \cap f^{-1}(A) \neq 0$ or $f^{-1}(\overline{P}) \cap f^{-1}(B) \neq 0$.

(i) If $P \leq A+B$, then $\overline{P} \leq A+B$ since $A+B = \ker(I_A \cap I_B)$ is a closed submodule of ${}_RN$. Thus $f^{-1}(\overline{P}) \leq f^{-1}(A+B) = f^{-1}(A) + f^{-1}(B) = \overline{f^{-1}(A)} + \overline{f^{-1}(B)}$ implies that $f^{-1}(\overline{P}) \leq f^{-1}(A)$ or $f^{-1}(\overline{P}) \leq f^{-1}(B)$

by the sum-primeness of $f^{-1}(\overline{P})$. Thus it follows from an epimorphism f that $\overline{P} \leq A$ or $\overline{P} \leq B$.

(ii) If $\overline{\overline{P} + A + B} \neq N$, then the closed submodule $f^{-1}(\overline{P}) + f^{-1}(A) + f^{-1}(B) \neq M$ follows. $f^{-1}(\overline{P}) + f^{-1}(A) + f^{-1}(B) \neq M$ by the sum-primeness of $f^{-1}(\overline{P})$ implies that $f^{-1}(\overline{P}) \leq f^{-1}(A)$ or $f^{-1}(\overline{P}) \leq f^{-1}(B)$. Thus $\overline{P} \leq A$ or $\overline{P} \leq B$ follows immediately.

(iii) If $\overline{P} + A = N$, then $f^{-1}(\overline{P}) + f^{-1}(A) = \overline{f^{-1}(\overline{P}) + f^{-1}(A)} = M$. By the sum-primeness of $f^{-1}(\overline{P})$ it follows that $f^{-1}(A) = M$ or $f^{-1}(\overline{P}) \cap f^{-1}(A) = 0$. Thus $A = N$ or $\overline{P} \cap A = 0$ follows. Therefore $\overline{P}$ is sum-prime and hence P is sum-prime in ${}_RN$.

(2): For any nonzero closed submodules $A, B \leq N$, we have closed submodules $f^{-1}(A) \simeq M$ or $f^{-1}(B) \simeq M$ since M is +prime. Hence it follows clearly that $A \simeq f(M) = N$ or $B \simeq f(M) = N$. The rest of the proof are completed by the same methods done in the proof of (2) of Theorem 3.1. □

COROLLARY 3.8. *For any epimorphism $f : {}_RM \to {}_RN$ with a self-cogenerated module ${}_RM$ and for a submodule $P \leq N$ of ${}_RN$, we have the following.*

1. *If $f^{-1}(P)$ is a sum-prime submodule of ${}_RM$, then P is also a sum-prime submodule of ${}_RN$.*
2. *If ${}_RM$ is +(or sum−)prime, then ${}_RN$ is +(or sum−)prime, respectively.*

Proof. Since the preimage assignment of $f : {}_RM \to {}_RN$ for any self-cogenerated module ${}_RM$ is closed by Theorem 3.8 the proof is completed. □

REMARK 3.9. The preimage assignment $A + K \mapsto \pi^{-1}(A + K) = A$ of the projection $\pi : {}_RM \to {}_RM/K$ for each submodule $A \leq M$ is not necessary to be closed, in general.

However if $A \leq M$ is a closed submodule of ${}_RM$, then it follows easily that $A + K$ is also a closed submodule of ${}_RM$ which doesn't guarantee that $A + K$ is a closed submodule of ${}_RM/K$ for any submodule $K \leq M$. For example, for the Abelian group $\mathbb{Q}$ of rational numbers, considering a module ${}_{\mathbb{Z}}\mathbb{Q}$(forget the multiplication in $\mathbb{Q}$) with that $\tilde{h}|_{f(M)} = h$ as below.

$$\begin{array}{ccccc}
{}_RM & \xrightarrow[g']{} & {}_RM & \xrightarrow[h']{} & {}_RM \\
\big\downarrow f & & \big\downarrow f & & \big\downarrow f \\
{}_Rf(M) & \xrightarrow{g} & {}_Rf(M) & \xrightarrow{h} & {}_Rf(M) \\
 & & \big\downarrow & & \big\downarrow \\
 & & {}_RN & \xrightarrow[\exists \tilde{h}]{} & {}_RN
\end{array}$$

Since f is an open monomorphism we also have a nonzero open submodule $f((\ker h')^o) \leq N$ and hence

$$f((\ker h')^o) = \sum_{q \in \mathrm{End}_R(N); \mathrm{Im} q \leq f((\ker h')^o)} Nq = [f((\ker h')^o)]^o.$$

Therefore

$$0 \neq f((\ker h')^o) = \sum_{q \in \mathrm{End}_R(N); \mathrm{Im} q \leq f((\ker h')^o)} Nq \leq \ker \tilde{h}^o \lneq N$$

for some nonzero endomorphism $\tilde{h} \in \mathrm{End}_R(N)$. Therefore $\mathrm{End}_R(N)$ is not prime.

(2): This is the contraposition of (1). □

COROLLARY 4.1.3. *For an (quasi−)injective module ${}_RM$ and a submodule $K \leq M$, if the inclusion mapping $\iota : {}_RK \hookrightarrow {}_RM$ is open, then we have the following.*

(1) *If $\mathrm{End}_R(K)$ is not prime, then neither $\mathrm{End}_R(M)$ is.*
(2) *If $\mathrm{End}_R(M)$ is prime, then so $\mathrm{End}_R(K)$ is.*

Proof. It is easy to complete the proof by Theorem 4.1.2. □

COROLLARY 4.1.4. *For an (quasi−)injective self-generated module ${}_RM$ and any submodule $K \leq M$, we have the following.*

(1) *If $\mathrm{End}_R(K)$ is not prime, then neither $\mathrm{End}_R(M)$ is.*
(2) *If $\mathrm{End}_R(M)$ is prime, then so $\mathrm{End}_R(K)$ is.*

Proof. Since for a self-generated module ${}_RM$ the inclusion mapping $\iota : {}_RK \hookrightarrow {}_RM$ is always an open monomorphism. Thus the proof is completed by Corollary 4.1.3. □

EXAMPLES 4.1.5. For an injective module ${}_{\mathbb{Z}}\mathbb{Z}$ we also have an injective module ${}_{\mathbb{Z}}\mathbb{Z} \oplus \mathbb{Z} \leq {}_{\mathbb{Z}} \oplus^{\infty} \mathbb{Z} = \mathbb{Z}^{(\infty)}$ has a nonprime endomorphism ring. This fact says that ${}_{\mathbb{Z}}\mathbb{Z}^{(\infty)}$ has also a nonprime endomorphism ring.

And an injective non-self-generated module ${}_{\mathbb{Z}[x]}\mathbb{Z}[x]$ with a prime endomorphism ring has a submodule $k\mathbb{Z} + x\mathbb{Z}[x] \leq \mathbb{Z}[x]$(for $k \in \mathbb{N}$) has an open inclusion $\iota : {}_{\mathbb{Z}[x]}k\mathbb{Z} + x\mathbb{Z}[x] \hookrightarrow {}_{\mathbb{Z}[x]}\mathbb{Z}[x]$. It follows from the Corollary 2.3 that the endomorphism ring $\mathrm{End}_{\mathbb{Z}[x]}(k\mathbb{Z}+x\mathbb{Z}[x])$ is prime, on the other hand, a submodule $x\mathbb{Z}[x] \leq \mathbb{Z}[x]$ has a non-open inclusion $\iota : {}_{\mathbb{Z}[x]}x\mathbb{Z}[x] \hookrightarrow {}_{\mathbb{Z}[x]}\mathbb{Z}[x]$ and the Corollary 4.1.3 can't be applied to a submodule ${}_{\mathbb{Z}[x]}x\mathbb{Z}[x]$.

A left $R-$module ${}_RP$ is said to be *projective*([2], [3], [5]) if for any epimorphism $p : {}_RM \to {}_RN$ and for any homomorphism $g : {}_RP \to {}_RN$, there is a homomorphism $\tilde{g} : {}_RP \to {}_RM$ such that $\tilde{g}p = g$.

$$\begin{array}{ccccc} & & {}_RP & & \\ & \exists\tilde{g} \swarrow & \downarrow g & & \\ {}_RM & \xrightarrow{p} & {}_RN & \longrightarrow & 0 \end{array}$$

In the above definition of a projective module replacing ${}_RP$ with ${}_RM$ we have a definition of a *quasi-projective* module. Thus it is clear that any projective module is quasi-projective. Therefore the next results are for both quasi-projective modules and projective modules.

For any self-generated module ${}_RM$ and for an open fully invariant submodule $Q \leq M$ of ${}_RM$, the projection $\pi : {}_RM \to {}_RM/Q$ is an open epimorphism with the open preimage assignment of π.

THEOREM 4.1.6. *For a (quasi$-$)projective module ${}_RN$, if there is an R-epimorphism $f : {}_RM \to {}_RN$ with the open preimage assignment of f and with an open fully invariant submodule* $\ker f$, *then we have the following.*

(1) *If* $\mathrm{End}_R(N)$ *is not prime, then neither* $\mathrm{End}_R(M)$ *nor* End_R $(M/\ker f)$ *is.*
(2) *If* $\mathrm{End}_R(M)$ *is prime, then so* $\mathrm{End}_R(N)$ *and* End_R $(M/\ker f)$ *are.*

Proof. (1): Suppose that $\mathrm{End}_R(N)$ is not a prime ring. Then there is an endomorphism $g : {}_RN \to {}_RN$ such that $0 \neq \ker g^o \lneq N$. It is established immediately from the isomorphism theorem that $\mathrm{End}_R(M/\ker f)$ is not a prime ring.

So it remains to show that $\mathrm{End}_R(M)$ is not prime. Since the preimage assignment of f is open we have an open submodule $f^{-1}(\ker g^o) \leq M$ such that $0 \neq f^{-1}(\ker g^o) = \sum_{q \in \mathrm{End}(M); Mq \leq f^{-1}(\ker g^o)} Mq \lneq M$. On the other hand there is the induced isomorphism $\tilde{f} : {}_RM/\ker f \to {}_RN$ since $f : {}_RM \to {}_RN$ is an epimorphism.

For an endomorphism $\tilde{g} = \tilde{f} g \tilde{f}^{-1} : {}_RM/\ker f \simeq {}_RN \to {}_RM/\ker f$ since ${}_RN$ is (quasi-)projective there is an endomorphism $g' : {}_RM/\ker f \to {}_RM$ such that $g'\pi = \tilde{g}$ as in the diagram:

$$\begin{array}{ccccc} & & {}_RM/\ker f \simeq {}_RN & & \\ & \exists g' \swarrow & \tilde{g} \downarrow & & \\ {}_RM & \xrightarrow{\ \pi\ } & {}_RM/\ker f & \longrightarrow & 0 \end{array}$$

Hence we have found an endomorphism $\pi g' : {}_RM \to {}_RM/\ker f \to {}_RM$ such that $0 \neq [\ker(\pi g')]^o \lneq M$ followed easily from the following commutative diagram:

$$\begin{array}{cccccccc} & & & & {}_RN & \xrightarrow{\ g\ } & {}_RN & 0 \\ & & & f \nearrow & \tilde{f} \uparrow & & \tilde{f}^{-1} \downarrow & \nearrow \\ {}_RM & \xrightarrow{\ q\ } & {}_RM & \xrightarrow{\ \pi\ } & {}_RM/\ker f & \xrightarrow{\ \tilde{g}\ } & {}_RM/\ker f & \\ & & & & g' \downarrow & \nearrow \pi & & \\ & & & & {}_RM & & & \end{array}$$

Since $\pi g' \pi = \pi \tilde{g}$ and since the preimage assignment of π is open it follows that $0 \neq Mq \leq \pi^{-1}(\ker(g'\pi)^o) = \pi^{-1}(\ker \tilde{g}^o) = \ker(\pi g')^o \lneq M$, for some $0 \neq q \in \mathrm{End}_R(M)$ which implies that $0 \neq \ker(\pi g')^o = \pi^{-1}((\ker g')^o) \lneq M$. Therefore $\mathrm{End}_R(M)$ is not a prime ring.

(2): This is the contraposition of (1). □

COROLLARY 4.1.7. *For a (quasi-)projective module $_RN$ and for a self-generated module $_RM$, if there is an R-epimorphism $f : {_RM} \to {_RN}$ with a fully invariant kernel* $\ker f$, *then we have the following.*

(1) *If* $\mathrm{End}_R(N)$ *is not prime, then neither* $\mathrm{End}_R(M/\ker f)$ *is.*
(2) *If* $\mathrm{End}_R(M)$ *is prime, then so* $\mathrm{End}_R(N)$ *and* $\mathrm{End}_R(M/\ker f)$ *are.*

Proof. Since each homomorphism $f : {_RM} \to {_RN}$ with a self-generated module $_RM$ has the open preimage assignment and $\ker f \leq M$ is an open submodule of $_RM$ Theorem 4.1.6 completes the proof. □

EXAMPLES 4.1.8. It is easy to find an epimorphism $f : {_\mathbb{Z}\mathbb{Z}^{(\infty)}} \to {_\mathbb{Z}\mathbb{Z}^{(2)}}$ with a fully invariant kernel $\ker f$ from a self-generated module $_\mathbb{Z}\mathbb{Z}^{(\infty)}$ onto a projective module $_\mathbb{Z}\mathbb{Z}^{(2)}$, where $_\mathbb{Z}\mathbb{Z}^{(\infty)}$ and $_\mathbb{Z}\mathbb{Z}^{(2)}$ are direct sums of infinite and 2-copies of $\mathbb{Z}$, respectively. It follows immediately from Corollary 4.1.7 that $\mathrm{End}_\mathbb{Z}(\mathbb{Z}^{(\infty)})$ is not prime.

4.2. Using kernels of images of endomorphisms

If we have a nonprime endomorphism ring $S = \mathrm{End}_R(M)$ of a module $_RM$, then there is some nonzero endomorphism $f \in S$ such that $0 \neq \overline{\mathrm{Im} f} \lneq M$, vice versa. More precisely, if S is not prime, then there are nonzero endomorphisms $f, g \in S$ such that $fg = 0$. Thus the fact of $fg = 0$ implies that $0 \neq \mathrm{Im} f = Mf \leq \ker g \lneq M$. Hence $0 \neq \overline{\mathrm{Im} f} \lneq M$.

REMARK 4.2.1. For a module $_RM$, the endomorphism ring $\mathrm{End}_R(M)$ is not prime if and only if there is a nonzero endomorphism $f \in \mathrm{End}_R(M)$ such that $0 \neq \overline{Mf} \lneq M$.

THEOREM 4.2.2. *For an (quasi–)injective module $_RN$, if there is a closed monomorphism $f : {_RM} \to {_RN}$, then we have the following.*

(1) *If* $\mathrm{End}_R(M)$ *is not prime, then neither* $\mathrm{End}_R(f(M))$ *nor* $\mathrm{End}_R(N)$ *is.*
(2) *If* $\mathrm{End}_R(N)$ *is prime, then* $\mathrm{End}_R(M)$ *is prime.*

Proof. (1): If $\mathrm{End}_R(M)$ is not a prime ring, then by the isomorphism between $_RM$ and $_Rf(M)$ it is clearly obtained that $\mathrm{End}_R(f(M))$ is not a prime ring. Thus there is some endomorphism $g \in \mathrm{End}_R(M)$ such that

$0 \neq \overline{Mg} \neq M$. Since f is closed monomorphism we have a closed submodule $f(\overline{Mg}) \leq N$ and $f(\overline{Mg}) = \bigcap_{q \in \mathrm{End}_R(N); f(\overline{Mg}) \leq \ker q} \ker q \leq N$. Since ${}_RN$ is (quasi-)injective there is an extension $\tilde{g} : {}_RN \to {}_RN$ such that $\tilde{g}|_{f(M)} = f^{-1}gf : {}_Rf(M) \to {}_Rf(M)$ and $0 \neq f(\overline{Mg}) = \bigcap_{q \in \mathrm{End}_R(N); f(\overline{Mg}) \leq \ker q} \ker q \leq \overline{N\tilde{g}} \lneq N$, showing that $\mathrm{End}_R(N)$ is not a prime ring.

(2): This is the contraposition of (1). □

Corollary 4.2.3. *For any (quasi–)injective self-cogenerated module ${}_RN$, if there is a monomorphism $f : {}_RM \to {}_RN$, then we have the following.*

(1) *If $\mathrm{End}_R(M)$ is not a prime ring. Then neither $\mathrm{End}_R(N)$ nor $\mathrm{End}_R(f(M))$ is prime.*
(2) *If $\mathrm{End}_R(N)$ is a prime ring. Then so $\mathrm{End}_R(M)$ and $\mathrm{End}_R(f(M))$ are prime.*

Proof. Since ${}_RN$ is self-cogenerated any homomorphism $f : {}_RM \to {}_RN$ is a closed mapping. Theorem 4.2.2 completes the proof. □

Corollary 4.2.4. *For any (quasi–)injective self-cogenerated module ${}_RN$ and for any submodule $K \leq {}_RN$, we have the following.*

(1) *If $\mathrm{End}_R(K)$ is not prime, then neither $\mathrm{End}_R(N)$ is.*
(2) *If $\mathrm{End}_R(N)$ is prime, then so $\mathrm{End}_R(K)$ is.*

Proof. Since ${}_RN$ is self-cogenerated the inclusion mapping $\iota : {}_RK \hookrightarrow {}_RN$ is a closed monomorphism. It follows immediately from Theorem 4.2.2. □

Examples 4.2.5. Clearly there is a closed monomorphism $f : {}_{\mathbb{Z}}\,\mathbb{Q}^{(2)} \to {}_{\mathbb{Z}}\,\mathbb{Q}^{(\infty)}$ from a module ${}_{\mathbb{Z}}\mathbb{Q}^{(2)}$ into an injective module ${}_{\mathbb{Z}}\mathbb{Q}^{(\infty)}$, where $\mathbb{Z}$ is the integer ring and where ${}_{\mathbb{Z}}\mathbb{Q}^{(\infty)}$ and ${}_{\mathbb{Z}}\mathbb{Q}^{(2)}$ are direct sums of infinite copies and 2-copies of the rational field $\mathbb{Q}$, respectively. Thus it follows that the endomorphism ring $\mathrm{End}_{\mathbb{Z}}(\mathbb{Q}^{(\infty)})$ is not prime from the nonprimeness of $\mathrm{End}_{\mathbb{Z}}(\mathbb{Q}^{(2)})$.

For any module ${}_RM$ and for a closed fully invariant submodule Q of ${}_RM$, the projection $\pi : {}_RM \to {}_RM/Q$ is a closed epimorphism with the closed preimage assignment of π.

THEOREM 4.2.6. *For a (quasi–)projective module ${}_RN$, if there is a closed epimorphism $f : {}_RM \to {}_RN$ with the closed preimage assignment and with a closed fully invariant submodule* $\ker f \leq M$, *then we have the following.*

(1) *If* $\mathrm{End}_R(N)$ *is not prime,*
then neither $\mathrm{End}_R(M)$ *nor* $\mathrm{End}_R(M/\ker f)$ *is.*
(2) *If* $\mathrm{End}_R(M)$ *is prime,*
then so $\mathrm{End}_R(N)$ *and* $\mathrm{End}_R(M/\ker f)$ *are.*

Proof. (1): From the nonprime endomorphism ring $\mathrm{End}_R(N)$ it follows that there is a nonzero endomorphism $g : {}_RN \to {}_RN$ such that $0 \neq \overline{\mathrm{Im} g} = \overline{Ng} \lneq N$ and $\mathrm{End}_R(M/\ker f)$ is not a prime ring. In other words, there are endomorphisms $g, \phi : {}_RN \to {}_RN$ such that $0 \neq \mathrm{Im} g \leq \ker k \lneq N$, i.e., $g\phi = 0_{RN}$.

Let $\tilde{f} : {}_RM/\ker f \to {}_RN$ be the induced isomorphism by f. Then we have endomorphisms $\tilde{g} = \tilde{f} g \tilde{f}^{-1}$ and $\tilde{\phi} = \tilde{f}\phi\tilde{f}^{-1} : {}_RM/\ker f \to {}_RM/\ker f$ such that $0 \neq \overline{\mathrm{Im}\tilde{g}} = \overline{(M/\ker f)\tilde{g}} \leq \ker\tilde{\phi} \lneq M/\ker f$.

Since ${}_RN \simeq {}_RM/\ker f$ is (quasi-)projective there are homomorphisms $g', \phi' : {}_RM/\ker f \to {}_RM$ and hence there are endomorphisms $k = \pi g', l = \pi\tilde{\phi}' : {}_RM \to {}_RM$ such that $g'\pi = \tilde{g}$ and $\phi'\pi = \tilde{\phi}$.

$$
\begin{array}{ccccc}
 & & {}_RM/\ker f & & \\
 & \exists g',\ \phi' \swarrow & \Big\downarrow{\scriptstyle \tilde{g},\,\tilde{\phi}} & & \\
{}_RM & \xrightarrow{\ \pi\ } & {}_RM/\ker f & \longrightarrow & 0
\end{array}
$$

Hence we have found endomorphisms $\pi g', \pi\phi' : {}_RM \xrightarrow{\pi} {}_RM/\ker f \to {}_RM$ such that $0 \neq \overline{\mathrm{Im}(\pi g')} \leq \ker\pi\phi' \lneq M$ followed easily from the following commutative diagram:

$$
\begin{array}{ccccccc}
 & & {}_RN & \xrightarrow{\ g\ } & {}_RN & \xrightarrow{\ \phi\ } & {}_RN \\
 & & {\scriptstyle \tilde{f}}\Big\uparrow & & {\scriptstyle \tilde{f}^{-1}}\Big\downarrow & & {\scriptstyle \tilde{f}}\Big\uparrow \\
 & & {}_RM/\ker f & \xrightarrow{\ \tilde{g}\ } & {}_RM/\ker f & \xrightarrow{\ \tilde{\phi}\ } & {}_RM/\ker f \\
 & \nearrow \pi & {\scriptstyle g'}\Big\downarrow & \nearrow \pi & \Big\downarrow{\scriptstyle \phi'} & \nearrow \pi & \\
{}_RM & & {}_RM & & {}_RM & &
\end{array}
$$

Thus $\mathrm{End}_R(M)$ is not a prime ring.

(2): This is the contraposition of (1). □

COROLLARY 4.2.7. *For a (quasi–)projective module $_RN$ and for a self-cogenerated module $_RM$ if there is an epimorphism $f : {_RM} \to {_RN}$ with a fully invariant kernel* $\ker f$, *then we have the following.*

(1) *If* $\mathrm{End}_R(N)$ *is not prime,*
then neither $\mathrm{End}_R(M)$ *nor* $\mathrm{End}_R(M/\ker f)$ *is.*
(2) *If* $\mathrm{End}_R(M)$ *is prime,*
then $\mathrm{End}_R(N)$ *is prime, and thus* $\mathrm{End}_R(M/\ker f)$ *is prime.*

Proof. Since any homomorphism $f : {_RM} \to {_RN}$ with a self-cogenerated module $_RM$ is a closed mapping and since the projection $\pi : {_RM} \to {_RM}/\ker f$ has the closed image assignment and the closed preimage assignment the proof is established by Theorem 4.2.6. □

EXAMPLES 4.2.8. For a self-cogenerated module $_{\mathbb{Z}}\mathbb{Z}_k \times (\prod_{n \in \mathbb{N} \setminus k\mathbb{N}} \mathbb{Z}_n)$ with any composite number k and for a projective module $_{\mathbb{Z}}\mathbb{Z}_k$, we have an epimorphism $f : {_{\mathbb{Z}}\mathbb{Z}_k} \times (\prod_{n \in \mathbb{N} \setminus k\mathbb{N}} \mathbb{Z}_n) \to {_{\mathbb{Z}}\mathbb{Z}_k}$ such that $\ker f$ is closed fully invariant. From the nonprimeness of the endomorphism ring $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_k)$ it follows that the endomorphism ring $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_k \times (\prod_{n \in \mathbb{N} \setminus k\mathbb{N}} \mathbb{Z}_n))$ is non-prime.

5. Open meet-prime or closed sum-prime submodules of modules

For fully invariant submodules $A, B \leq M$, we have that

$$I^A I^B,\ I^B I^A \subseteq I^A \cap I^B = I^{A \cap B}$$

and

$$I_A I_B,\ I_B I_A \subseteq I_A \cap I_B = I_{A+B}$$

hold.

LEMMA 5.1. *For any open $A \leq M$, open fully invariant $A_1, A_2, \cdots, A_n \leq M$, and any fully invariant meet-prime submodules $P, P_1, P_2, \cdots, P_n \leq M$ of a left $R-$module ${}_RM$ we have the following.*

(1) *If $A \subseteq \cup_1^n P_i$, then $A \leq P_i$ for some i.*
(2) *If $\cap_1^n A_i \leq P$, then $A_i \leq P$ for some i.*
(3) *If $\cap_1^n A_i = P$, then $A_i = P$ for some i.*

The following proof is just as the same as the proof of Proposition 1.11 [p. 8, 1].

Proof. For fully invariant meet-prime submodules $P, P_1, P_2, \cdots, P_n \leq M$ we have prime ideals $I^P, I^{P_1}, I^{P_2}, \cdots, I^{P_n} \trianglelefteq \mathrm{End}_R(M)$ of the endomorphism ring $\mathrm{End}_R(M)$ of ${}_RM$.

(1): By the induction on n in the form;

$$A \not\leq P_i \ (1 \leq i \leq n) \ \text{impliy that} \ A \not\leq \cup_1^n P_i \, .$$

For $n = 1$, it clearly holds.

For $n \gneq 1$ we assume that the item (1) is true for $n-1$. Then for each i, there is an endomorphism $f_i \in I^A$ such that $f_i \notin I^{P_j}$ for all $j \neq i$.

If for some i, there is an isomorphism $f_i \in I^A$ such that $f_i \notin I^{P_i}$. Then it is proved. If not, there is an isomorphism $f_i \in I^A$ such that $f_i \notin I^{P_i}$ for all i. Considering an endomorphism $g = \sum_{i=1}^n f_1 f_2 \cdots f_{i-1} f_{i+1} \cdots f_n \notin I^{\cup_1^n P_i}$. Then we have that $Mg \leq A$ but $Mg \nsubseteq \cup_1^n P_i$. From the openness of A it follows that $A \leq P_i$ for some i. Therefore the item (1) is true.

(2): Suppose that $P \not\leq A_i$ for every $i (1 \leq i \leq n)$. Then there is some endomorphism $f_i \in I^{A_i}$ such that $f_i \notin I^P$ for every i. And hence $g = \prod_1^n f_i \in \prod_1^n I^{A_i} \subseteq \cap_1^n I^{A_i} \setminus I^P = I^{\cap_1^n A_i} \setminus I^P$ since I^P is prime. Then it concludes that $P \ngeq \cap_1^n A_i$.

(3): If $P = \cap_1^n A_i$, then from the above (2) it follows immediately that $P = A_i$ for some i. □

LEMMA 5.2. *For any closed submodule $B \leq M$, any closed fully invariant submodules $B_1, B_2, \cdots, B_n \leq M$, and any fully invariant sum-prime submodules $Q, Q_1, Q_2, \cdots, Q_n \leq M$ of any $R-$module ${}_RM$, we have the following.*

(1) *If $B \supseteq \cup_1^n Q_i$, then $B \geq Q_i$ for some i.*
(2) *If $Q \leq \sum_1^n B_i$, then $Q \leq B_i$ for some i.*
(3) *If $Q = \sum_1^n B_i$, then $B_i = Q$ for some i.*

Proof. For fully invariant meet-prime submodules $Q_1, Q_2, \cdots, Q_n \leq M$ we have prime ideals $I_{Q_1}, I_{Q_2}, \cdots, I_{Q_n} \trianglelefteq \mathrm{End}_R(M)$ of the endomorphism ring $\mathrm{End}_R(M)$.

(1): By the induction on n in the form;

$$B \not\geq Q_i \ (1 \leq i \leq n) \text{ imply that } B \not\supseteq \cup_1^n Q_i \ .$$

For $n = 1$, it clearly holds.

For $n \gtrsim 1$ we assume that the item (1) is true for $n-1$. Then for each i, there is an endomorphism $f_i \in I_B$ such that $f_i \notin I_{Q_j}$ for all $j \neq i$.

If for some i, there is an isomorphism $f_i \in I_B$ such that $f_i \notin I_{Q_i}$. Then it is proved. If not, there is an isomorphism $f_i \in I_B$ such that $f_i \notin I_{Q_i}$ for all i. Considering an endomorphism $g = \sum_{i=1}^n f_1 f_2 \cdots f_{i-1} f_{i+1} \cdots f_n \notin I_{\cup_1^n Q_i}$. Then we have that $\ker g \geq B$ but $\ker g \not\supseteq \cup_1^n Q_i$. From the closedness of B it follows that $B \geq Q_i$ for some i. Therefore the item (1) is true.

(2): Suppose that $Q \not\leq B_i$ for every $i (1 \leq i \leq n)$. Then there is some endomorphism $f_i \in I_{B_i}$ such that $f_i \notin I_Q$ for every i. And hence $g = \prod_1^n f_i \in \prod_1^n I_{B_i} \subseteq \cap_1^n I_{B_i} \setminus I_Q = I_{\sum_1^n B_i} \setminus I_Q$ since I_Q is prime. Then it concludes that $Q \not\leq \sum_1^n B_i$.

(3): If $Q = \sum_1^n B_i$, then from the above (2) it follows immediately that $Q = B_i$ for some i. □

REMARK 5.3. Any maximal submodule $N \leq M$ of a module ${}_RM$(if ${}_RM$ has any) is meet-prime and any minimal submodule(if ${}_RM$ has any) is sum-prime.

PROPOSITION 5.4. *For any module ${}_RM$, we have the following.*

(1) *There exists at least one proper maximal open submodule (that is, maximal submodule among the open submodules) of ${}_RM$.*
(2) *There exists at least one nonzero minimal closed submodule (that is, minimal submodule among the closed submodules) of ${}_RM$.*

Proof. (1): Let $\mathfrak{S} = \{A \lneq M | A \text{ is a proper open submodule of } {}_RM\}$ be the set of all proper open submodules of ${}_RM$. Then $\mathfrak{S} \neq \emptyset$ since the trivial submodule 0 is open. Let $\mathfrak{C}$ be any chain in $\mathfrak{S}$ of proper open submodules of ${}_RM$. Then $\mathfrak{C} : \cdots \leq A_1 \leq A_2 \leq \cdots \leq A_n \leq A_{n+1} \leq \cdots$ has an upper bound $\cup_i A_i$ which is an open submodule of ${}_RM$. By the

Zorn's lemma there exists a maximal element $\cup A_i = A \leq M$ in $\mathfrak{S}$, in fact, which is a maximal among proper open submodules of ${}_RM$.

Easily it follows from Definition 1.1 that such a maximal element A is a meet-prime submodule of ${}_RM$.

(2): Let $\mathfrak{T} = \{B(\neq 0) \leq M | B$ is a nonzero closed submodule of ${}_RM\}$ be the set of all nonzero closed submodules of ${}_RM$. Then $\mathfrak{T} \neq \emptyset$ since the trivial submodule M is closed. Let $\mathfrak{D}$ be any chain in $\mathfrak{T}$ of nonzero closed submodules of ${}_RM$. Then $\mathfrak{D} : \cdots \geq B_1 \geq B_2 \geq \cdots \geq B_n \geq B_{n+1} \geq \cdots$ has a lower bound $\cap B_i$ which is a closed submodule of ${}_RM$. By the Zorn's lemma with a reversing set inclusion order there exists a minimal element $\cap B_i = B \leq M$ in $\mathfrak{T}$.

Easily it follows from Definition 1.7 that such a minimal element B is a sum-prime submodule of ${}_RM$. □

REMARK 5.5. In spite of the Proposition 5.4 it is not guaranteeded for the sets

$\{P \lneq M \mid P$ is a proper fully invariant meet-prime submodule of ${}_RM\}$

and

$\{P \neq 0 \mid P$ is a nonzero fully invariant sum-prime submodule of ${}_RM\}$

(which will be studied in the sections 7 and 8) to be nonempty sets, for any module ${}_RM$.

6. Zariski topologies for endomorphism rings

It is trivial that if an endomorphism ring S has no prime ideal of S, then S is not prime.

For any left module ${}_RM$ over a ring R, there exists a proper fully invariant meet-prime or proper fully invariant sum-prime submodule P, respectively, we have a prime ideal I^P or I_P in the endomorphism ring $S = \mathrm{End}_R(M)$. Unfortunately this does not guarantee the existence of a proper prime ideal of S.

We let Spec(S) be the set of all prime ideals of S(even though S need not to be a commutative ring), precisely

$$\mathrm{Spec}(S) = \{\ J \lhd S \mid J \text{ is a prime ideal of } S\ \}$$

which will be called the prime spectrum of the endomorphism ring S. Then we also have a topological space which will be named by Zariski topology on the spectrum Spec(S) as follows:

THEOREM 6.1. *For any module $_RM$, the prime spectrum* Spec(S) *of the endomorphism ring S is a topological space, if as closed sets we take all sets of form $v(E) = \{ I \in \text{Spec}(S) \mid E \subseteq I \}$, where E is any subset of S. Precisely, the sets $v(E)$ satisfy the axioms for closed sets in a topological space.*

(1) *For any subset $E \subseteq S$, if $\langle E \rangle$ is the ideal of S generated by E, then $v(E) = v(\langle E \rangle) = v(r(E))$, where $r(E) = \cap_{E \subseteq J_\alpha \in \text{Spec}(S)} J_\alpha$ is the prime radical of E.*
(2) $v(0) = \text{Spec}(S)$, $v(S) = \emptyset$.
(3) $v(\cup_{i \in I} E_i) = \cap_{i \in I} v(E_i)$, *for each $E_i \subseteq S$.*
(4) $v(AB) = v(A) \cup v(B)$ *for $A, B \subseteq S$.*

PROPOSITION 6.2. *For any left $R-$module $_RM$,* Spec(S) *is a topological space, if as open sets we take all sets of form*

$$\Gamma A = \{ J \in \text{Spec}(S) \mid A \not\subseteq J \},$$

where A is any subset of S.

Before a proof, it is convenient to note that

$$\Gamma A = \{ J \in \text{Spec}(S) \mid A \not\subseteq J \} = \{ J \in \text{Spec}(S) \mid \langle A \rangle \nleq J \},$$

for A is any subset of S, where $\langle A \rangle$ is the ideal generated by the set A.

Additionally notice that for any subset A of S

$$\begin{aligned} \Gamma A &= \Gamma(\sum_{a \in A} \langle a \rangle) = \cap_{a \in A} \Gamma a = \cap_{a \in A} \Gamma \langle a \rangle \\ &= \{J \in \text{Spec}(S) \mid A \not\subseteq J\} = \{J \in \text{Spec}(S) \mid \langle A \rangle \nleq J\} \\ &= \Gamma(\cap_{A \not\subseteq J_\beta} J_\beta), \quad J_\beta \text{ is a prime ideal of } S. \end{aligned}$$

The resulting topology is called the *Zariski topology* named after the Zariski topology on the prime spectrum of a commutative ring. The topological space Spec(S) is called the *prime spectrum* of the endomorphism ring S of a module $_RM$.

Remind that a topological space X is said to be *irreducible* if $X \neq \emptyset$ and if every nonempty two open sets intersect, or equivalently if every nonempty open set is dense in X(p. 13 in [1]).

THEOREM 6.3. *For any module $_RM$, the following are equivalent:*

(1) Spec(S) *is irreducible;*
(2) *The prime radical $rad(S) = \cap_{J \in \text{Spec}(S)} J$ is in* Spec(S), *i.e.,* rad(S) *is a prime ideal of S.*

7. Zariski image topologies for openly regular modules

A module ${}_RM$ is said to be *openly regular* if for any submodules $C, D \leq M$, the following properties are satisfied:

(1) $C^o \lneq D^o$ implies that $C \lneq D$,

(2) $C^o = D^o$ implies that $C \leq D$ or $D \lneq C$.

Clearly any self-generated module is openly regular. There are openly regular modules which are not self-generated, for instance, a module ${}_{\mathbb{Z}_p[x]}\mathbb{Z}_p[x]$ for the polynomial ring ${}_{\mathbb{Z}_p[x]}\mathbb{Z}_p[x]$ in an indeterminate x over the ring $\mathbb{Z}_p$ modulo p has nonopen submodules $x^n\mathbb{Z}_p[x] \leq {}_{\mathbb{Z}_p[x]}\mathbb{Z}_p[x]$ ($n \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers) having the trivial submodule $0 = (x^n\mathbb{Z}_p[x])^o \leq {}_{\mathbb{Z}_p[x]}\mathbb{Z}_p[x]$ ($n \in \mathbb{N}$). Clearly it is seen that $\{x^n\mathbb{Z}_p[x] \mid n \in \mathbb{N}\}$ is linearly ordered. We have the following results relative to meet-prime submodules of left R-modules: Let

$\Pi = \{P_\alpha \lneq M | P_\alpha$ is a proper fully invariant meet-prime submodule of ${}_RM\}$ be the set of all proper fully invariant meet-prime submodules of ${}_RM$. Then we have the following proposition.

PROPOSITION 7.1. *For any openly regular left $R-$module ${}_RM$, Π is a topological space, if as closed sets we take all sets of form $v(E) = \{ P \in \Pi \mid E \subseteq P \}$, where E is any subset of ${}_RM$. Precisely, the sets $v(E)$ satisfy the axioms for closed sets in a topological space:*

(1) *For any subset $E \subseteq M$, if $\langle E\rangle$ is the submodule of M generated by E, then $v(E) = v(\langle E\rangle) = v(r(E))$, where $r(E) = \cap_{E \subseteq P_\alpha \in \Pi} P_\alpha$ is the prime radical of E.*

(2) $v(0) = v(r(0)) = \Pi$, $v(M) = \emptyset$.

(3) $v(\cup_{i\in I} E_i) = \cap_{i \in I} v(E_i)$, *for each* $E_i \subseteq M$.

(4) $v(A \cap B) = v(A) \cup v(B)$ *for* $A, B \subseteq M$.

The prime radical $\mathrm{rad}(M) = r(0) = \cap_{P_\alpha \in \Pi} P_\alpha$ of any ${}_RM$ is an open fully invariant submodule of ${}_RM$.

Proof. (4): If $A \cap B \subseteq P$ for $P \in \Pi$, then $\langle A\rangle^o \cap \langle B\rangle^o \leq P^o$ implies that $\langle A\rangle^o \leq P^o$ or $\langle B\rangle^o \leq P^o$ since P is meet-prime if and only if P^o is meet-prime. Then it follows that $A \subseteq \langle A\rangle \leq P$ or $B \subseteq \langle B\rangle \leq P$ by letting $A = B$ in (*).

(*) If $A \cap B \subseteq P$, then $\langle A\rangle^o \cap \langle B\rangle^o \leq P^o \Longleftrightarrow \langle A\rangle \cap \langle B\rangle \leq P$ for any meet-prime $P \lneq M$ in any openly regular module ${}_RM$. In order to show (*), suppose that $\langle A\rangle \gneq P$ and $\langle B\rangle \gneq P$. Then $A^o \cap B^o = P^o$ follows and hence $\langle A\rangle^o = \langle B\rangle^o = (\langle A\rangle \cap \langle B\rangle)^o = P^o$ is fully invariant meet-prime. Hence $P^o \leq \langle A\rangle \cap \langle B\rangle$. Since ${}_RM$ is openly regular we have that $\langle A\rangle, \langle B\rangle, \langle A\rangle \cap \langle B\rangle$ and P are submodules of ${}_RM$ which are linearly ordered. Thus $P \subset \langle A\rangle \cap \langle B\rangle = \langle A \cap B\rangle \subset \langle A\rangle, \langle B\rangle$(which is contradicted to $A \cap B \subseteq P$) or $\langle A\rangle \cap \langle B\rangle \subseteq P \subset \langle A\rangle, \langle B\rangle$(which is the required) follows. Hence the only case of $\langle A\rangle \cap \langle B\rangle \subseteq P \subset \langle A\rangle, \langle B\rangle$ remains to be considered, and hence we have that $A \cap B \subseteq P$. Therefore if $A \cap B \subseteq P$, we have that $\langle A\rangle^o \cap \langle B\rangle^o \leq P^o \Longleftrightarrow \langle A\rangle \cap \langle B\rangle \leq P$ for any meet-prime $P \lneq M$ in any openly regular module ${}_RM$. Conversely, $v(A) \cup v(B) \subseteq v(A \cap B)$ is elementary. Therefore we have proved (4).□

PROPOSITION 7.2. *For any openly regular left R–module ${}_RM$, Π is a topological space, if as open sets we take all sets of form*

$$\Gamma A = \{\, P \in \Pi \mid A \nsubseteq P \,\},$$

where A is any subset of ${}_RM$.

It is convenient to note that

$$\Gamma A = \{\, P \in \Pi \mid A \nsubseteq P \,\} = \{\, P \in \Pi \mid \langle A\rangle \nleq P \,\},$$

for A is any subset of ${}_RM$, where $\langle A\rangle$ is the submodule generated by the set A.

Additionally notice that for any subset $A \subseteq M$ of ${}_RM$

$$\begin{aligned}\Gamma A &= \Gamma(\sum_{a\in A}\langle a\rangle) = \cap_{a\in A}\Gamma a = \cap_{a\in A}\Gamma\langle a\rangle \\ &= \{P \in \Pi \mid A \nsubseteq P\} = \{P \in \Pi \mid \langle A\rangle \nleq P\} \\ &= \Gamma(\cap_{A\nsubseteq P_\beta} P_\beta),\end{aligned}$$

where P_β is a *fully invariant meet-prime* submodule of ${}_RM$.

The resulting topology is called the *Zariski image topology* for the openly regular ${}_RM$ named after the Zariski topology on the prime spectrum of a commutative ring. The topological space Π is called the *image spectrum* of ${}_RM$, denoted by $\mathrm{Spec}_I(M)$.

Also we define the *prime radical* $\mathrm{rad}(M)$ by the intersection of all meet-prime submodules of ${}_RM$, in other words, $\mathrm{rad}(M) = \cap_\alpha P_\alpha$ (cf. the Jacobson Radical $\mathrm{Rad}(M)$ the intersection of all maximal submodules of ${}_RM$).

Clearly in any openly regular module ${}_RM$ it is easily shown that $\mathrm{rad}(M) \leq \mathrm{Rad}(M)$(if $\mathrm{Rad}(M) \neq M$ i.e., if ${}_RM$ has any maximal submodule of ${}_RM$).

Let $\mathfrak{S}$ be the set of all open submodules of ${}_RM$, then by the Zorn's lemma there are maximal submodules among open submodules of ${}_RM$, being open fully invariant meet-prime submodules of ${}_RM$. This says that $\mathrm{Spec}_I(M)$ is a nonempty set.

If the prime radical $\mathrm{rad}(M)$ is a meet-prime submodule of ${}_RM$, then the image spectrum $\mathrm{Spec}_I(M) = \{ L \lneq M \mid \mathrm{rad}(M) \leq L \}$ contains $\mathrm{rad}(M)$ since the prime radical $\mathrm{rad}(M)$ is open and fully invariant in ${}_RM$.

THEOREM 7.3. *For any openly regular module* ${}_RM$, *if a submodule* $K \leq \mathrm{rad}(M)$ *of* ${}_RM$ *is in* $\mathrm{Spec}_I(M)$, *then we have that* $K = \mathrm{rad}(M)$ *and* $\mathrm{Spec}_I(M)$ *is irreducible.*

Proof. If $K \in \mathrm{Spec}_I(M)$, then K is fully invariant meet-prime, then the open submodule K^o is also fully invariant meet-prime in ${}_RM$. Thus $\mathrm{rad}(M) \leq K^o \in \mathrm{Spec}_I(M)$ implies that $\mathrm{rad}(M) = K \in \mathrm{Spec}_I(M)$.

And every basic open set in the image spectrum $\mathrm{Spec}_I(M)$ contains $\mathrm{rad}(M)$, in other words, $\mathrm{Spec}_I(M)$ is irreducible. And by the hypothesis of $K \leq \mathrm{rad}(M)$, we have an open submodule $\mathrm{rad}(M) = K^o$ which is in $\mathrm{Spec}(M)$. □

COROLLARY 7.4. *For any openly regular module* ${}_RM$, *we have that* $\mathrm{Spec}_I(M)$ *is irreducible if and only if* $\mathrm{rad}(M) \in \mathrm{Spec}_I(M)$.

For any module ${}_RM$, we have a surjective mapping from the image spectrum $\mathrm{Spec}_I(M)$ onto a subset $\{ I^P \mid P \in \mathrm{Spec}_I(M)\} \subseteq \mathrm{Spec}(S)$ of the prime spectrum $\mathrm{Spec}(S)$ of the endomorphism ring S of ${}_RM$. Let this subspace $\{I^P | P \in \mathrm{Spec}_I(M)\}$ be the topological subspace of the Zariski topology of the spectrum $\mathrm{Spec}(S)$ of the endomorphism ring. Then we have the next theorem.

LEMMA 7.5. *For any openly regular module* ${}_RM$, *let*

$$Y = \{I^P | P \in \mathrm{Spec}_I(M)\} \subseteq \mathrm{Spec}(S).$$

Then we have the following.

(1) *If* Y *is open in* $\mathrm{Spec}(S)$ *and if the prime spectrum* $\mathrm{Spec}(S)$ *is irreducible. then the image spectrum* $\mathrm{Spec}_I(M)$ *is irreducible.*

(2) *If Y is dense in* $\operatorname{Spec}(S)$ *and if the image spectrum* $\operatorname{Spec}_I(M)$ *is irreducible, then the prime spectrum* $\operatorname{Spec}(S)$ *is irreducible.*
(3) *If Y is open dense in* $\operatorname{Spec}(S)$, *then the prime spectrum* $\operatorname{Spec}(S)$ *is irreducible if and only if the image spectrum* $\operatorname{Spec}_I(M)$ *is irreducible.*

COROLLARY 7.6. *For any openly regular module* ${}_RM$, *if*

$$\{I^P | P \in \operatorname{Spec}_I(M)\}$$

is open dense in $\operatorname{Spec}(S)$, *then the following are equivalent:*
(1) *The prime spectrum* $\operatorname{Spec}(S)$ *is reducible;*
(2) *The image spectrum* $\operatorname{Spec}_I(M)$ *is reducible.*

REMARK 7.7. The openness and density of $\{I^P | P \in \operatorname{Spec}_I(M)\}$ in the hypotheses of the Proposition 7.5 and Corollary 7.6 is essential. Without the openness of the subspace Y, it is impossible for Y to contain the prime radical of S. For example, a module ${}_{\mathbb{Z}}\mathbb{Z}$ over the integer ring $\mathbb{Z}$ has a non-open prime image spectrum $\operatorname{Spec}_I(M)$ isomorphic to $\{ p\mathbb{Z} \mid p$ is a prime number$\}$ but its prime radical $\operatorname{rad}(\mathbb{Z}) = 0 \notin \operatorname{Spec}_I(\mathbb{Z})$, in other words, $Y = \{I^{p\mathbb{Z}} \mid p$ is a prime number$\}$ is not open in $\operatorname{Spec}(S)$. However it is well-known that the prime spectrum $\operatorname{Spec}(\operatorname{End}_{\mathbb{Z}}(\mathbb{Z}))$ is irreducible. And for a prime number p considering a left ${}_{\mathbb{Z}}\mathbb{Z}(p^\infty)$ having an empty set $Y = \{I^P \mid P$ is a meet-prime submodule of ${}_{\mathbb{Z}}\mathbb{Z}(p^\infty)\} = \emptyset \subseteq \operatorname{Spec}(\operatorname{End}_{\mathbb{Z}}(\mathbb{Z}(p^\infty)))$, then we have that Y is reducible and $\operatorname{Spec}(\operatorname{End}_{\mathbb{Z}}(\mathbb{Z}(p^\infty)))$ is a singleton being irreducible in the Zariski topology. This shows that the reducibility of Y does not imply that of $\operatorname{Spec}(S)$ without the density of Y.

Considering the quotient module ${}_RM/\operatorname{rad}(M)$ of any module ${}_RM$ over the prime radical $\operatorname{rad}(M)$ of module ${}_RM$, let $T = \operatorname{End}_R(M/\operatorname{rad}(M))$ denote the endomorphism ring of the quotient module ${}_RM/\operatorname{rad}(M)$ over the prime radical $\operatorname{rad}(M)$.

THEOREM 7.8. *For an openly regular module* ${}_RM$ *with the prime radical* $\operatorname{rad}(M)$, *if* $\{I^L | L \in \operatorname{Spec}_I(M/\operatorname{rad}(M))\}$ *is open dense in* $\operatorname{Spec}(T)$, *where* $T = \operatorname{End}_R(M/\operatorname{rad}(M))$ *is the endomorphism ring of the quotient module* ${}_RM/\operatorname{rad}(M)$, *the following are equivalent:*
(1) *The endomorphism ring* $\operatorname{End}_R(M/\operatorname{rad}(M))$ *is prime;*

(2) *The prime spectrum* $\mathrm{Spec}(T)$ *is irreducible;*
(3) *The image spectrum* $\mathrm{Spec}_I(M/\mathrm{rad}(M))$ *is irreducible;*
(4) *The prime radical* $\mathrm{rad}(M)$ *of* $_RM$ *is meet-prime.*

Proof. (1) $\Longrightarrow$ (2): It is trivial.

(2) $\Longrightarrow$ (3): Assume (1), then the prime spectrum $\mathrm{Spec}(T)$ is irreducible. Thus by the above Lemma 7.4 we have the irreducible image spectrum $\mathrm{Spec}_I(M/\mathrm{rad}(M))$.

(3) $\Longrightarrow$ (4): From Corollary 7.4 it follows immediately.

(4) $\Longrightarrow$ (1): Assume that the image spectrum $\mathrm{Spec}_I(M/\mathrm{rad}(M))$ is irreducible, then the prime radical $\mathrm{rad}(M/\mathrm{rad}(M)) = \mathrm{rad}(M) \in \mathrm{Spec}_I$ $(M/\mathrm{rad}\ (M))$ is a fully invariant meet-prime submodule of an openly regular module $_RM/\mathrm{rad}(M)$. Therefore we obtain a prime ideal $I^{\mathrm{rad}(M)}$ $= I^{\overline{0}} = 0 \trianglelefteq T$ of the endomorphism ring of $_RM/\mathrm{rad}(M)$. Therefore the endomorphism ring T is a prime ring. □

THEOREM 7.9. *For an openly regular module* $_RM$ *with the prime radical* $\mathrm{rad}(M)$, *if* $\{I^L | L \in \mathrm{Spec}_I(M/\mathrm{rad}(M))\}$ *is open dense in* $\mathrm{Spec}(T)$, *where* $T = \mathrm{End}_R(M/\mathrm{rad}(M))$ *is the endomorphism ring of the quotient module* $_RM/\mathrm{rad}(M)$, *the following are equivalent:*

(1) *The endomorphism ring* $\mathrm{End}_R(M/\mathrm{rad}(M))$ *is not prime;*
(2) *The prime spectrum* $\mathrm{Spec}(T)$ *is reducible;*
(3) *The Image spectrum* $\mathrm{Spec}_I(M/\mathrm{rad}(M))$ *is reducible;*
(4) *The prime radical* $\mathrm{rad}(M)$ *of* $_RM$ *is not meet-prime.*

THEOREM 7.10. *For an openly regular module* $_RM$ *with* $\mathrm{rad}(M) = 0$, *if* $\{I^P | P \in \mathrm{Spec}_I(M)\}$ *is open dense in* $\mathrm{Spec}(S)$, *then the following are equivalent:*

(1) *The endomorphism ring* S *is prime;*
(2) *The prime spectrum* $\mathrm{Spec}(S)$ *is irreducible;*
(3) *The image spectrum* $\mathrm{Spec}_I(M)$ *is irreducible;*
(4) 0 *is meet-prime.*

Proof. Replacing $\mathrm{rad}(M)$ with 0 in the above Theorem 7.7, the proof is completed. □

THEOREM 7.11. *For an openly regular module* $_RM$ *with* $\mathrm{rad}(M) = 0$, *if* $\{I^P | P \in \mathrm{Spec}_I(M)\}$ *is open dense in* $\mathrm{Spec}(S)$, *then the following are equivalent:*

(1) *The endomorphism ring* S *is not prime;*
(2) *The prime spectrum* $\mathrm{Spec}(S)$ *is reducible;*
(3) *The image spectrum* $\mathrm{Spec}_I(M)$ *is reducible;*
(4) 0 *is not meet-prime.*

8. Zariski kernel(null) topologies for closedly regular modules

A module $_RM$ is said to be *closedly regular* if for any submodules $C, D \leq M$, the following properties are satisfied:

(1) $\overline{C} \lneq \overline{D}$ implies that $C \lneq D$,
(2) $\overline{C} = \overline{D}$ implies that $C \leq D$ or $D \lneq C$.

Clearly any self-cogenerated module is closedly regular. There are closedly regular modules which are not self-cogenerated, for example, a closedly regular left $\mathbb{Z}[x]$−module $_{\mathbb{Z}[x]}\mathbb{Z}(p^\infty)[x]$ has non-closed submodules $x^n\mathbb{Z}(p^\infty)[x]$ ($n \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers) including the trivial submodule $\mathbb{Z}(p^\infty)[x] = \overline{x^n\mathbb{Z}(p^\infty)[x]}$. Also $\{x^n\mathbb{Z}(p^\infty)[x] \mid n \in \mathbb{N}\}$ is linearly ordered.

Let $\mathfrak{S}$ be the set of all closed submodules of $_RM$ with a reversing order of set inclusion, then by the Zorn's lemma there are maximal submodules among closed submodules of $_RM$, being closed fully invariant sum-prime submodules of $_RM$. Thus it follows that

$$\mathfrak{S} = \{Q \leq M \mid Q \text{ is a sum-prime submodule of } {}_RM\} \neq \emptyset$$

but

$$\{Q \leq M \mid 0 \neq Q \text{ is a nonzero sum-prime submodule of } {}_RM\} \neq \emptyset$$

is not held, in general. With a risk of being empty set, we will introduce a topological space on the set of all nonzero fully invariant sum-prime submodules of any closedly regular module over any ring as follows.

Let $\Xi = \{P_\alpha \neq 0 | P_\alpha$ is a nonzero fully invariant sum-prime submodule of $_RM\}$ be the set of all non-zero fully invariant sum-prime submodules of $_RM$. Then we have the following propostition.

PROPOSITION 8.1. *For a closedly regular left $R-$module ${}_RM$, Ξ is a topological space, if as closed sets we take all sets of form*

$$w(E) = \{P \in \Xi \mid P \subseteq E\},$$

where $E \subseteq M$ is any subset of ${}_RM$. Precisely, the sets $w(E)$ satisfy the axioms for closed sets in a topological space:

(1) *For any subset $E \subseteq M$, if $\langle E \rangle$ is the submodule of M generated by E, then $w(E) = w(\langle E \rangle) = w(\mathrm{soc}(E))$, where $\mathrm{soc}(E) = \sum_{E \supseteq P_\alpha \in \Xi} P_\alpha$ is the prime socle of E.*
(2) $w(M) = w(\mathrm{soc}(M)) = \Xi$, $w(0) = \emptyset$.
(3) $w(\cap_{i \in I} E_i) = \cap_{i \in I} w(E_i)$ *for* $E_i \subseteq M (i \in I)$.
(4) $w(A \cup B) = w(\langle A \rangle + \langle B \rangle) = w(A) \cup w(B)$ *for* $A, B \subseteq M$.

Proof. (4): Trivially it is true that $w(\langle A \rangle + \langle B \rangle) = w(A) \cup w(B) \subseteq w(A \cup B)$. It remains to show that $w(A \cup B) \subseteq w(\langle A \rangle + \langle B \rangle) = w(A) \cup w(B)$. Let P be any sum-prime submodule of ${}_RM$ such that $P \leq A \cup B$, then $P \leq \langle A \rangle + \langle B \rangle \leq \overline{\langle A \rangle} + \overline{\langle B \rangle}$ and then $P \leq \overline{\langle A \rangle}$ or $P \leq \overline{\langle B \rangle}$ by (2) of the Lemma 5.2. Since ${}_RM$ is closedly regular and since $\overline{P} \leq \overline{\langle A \rangle} + \overline{\langle B \rangle} \Longleftrightarrow P \leq \langle A \rangle + \langle B \rangle$ we have that $P \leq A$ or $P \leq B$(otherwise if $P \gneq \langle A \rangle$ and if $P \gneq \langle B \rangle$, then $P \gneq \langle A \rangle + \langle B \rangle = \langle A \cup B \rangle$ and it is contradicted to $P \leq A \cup B$.) Thus we have $w(A \cup B) \subseteq w(\langle A \rangle + \langle B \rangle) = w(A) \cup w(B)$. □

PROPOSITION 8.2. *Ξ is a topological space, if as open sets we take all sets of form $\tau A = \{\, P \in \Xi \mid P \nsubseteq A \,\}$, where $A \subseteq M$ is any subset of ${}_RM$.*

Before a proof, it is convenient to note that

$$\tau A = \{\, P \in \Xi \mid P \nsubseteq A \,\} = \{\, P \in \Xi \mid P \nleq \langle A \rangle \,\},$$

where A is any subset of M and $\langle A \rangle$ is the submodule of ${}_RM$ generated by the set A. Additionally notice that for any subset A of ${}_RM$

$$\begin{aligned} \tau A &= \cup_{a \in A} \tau a \\ &= \cup_{a \in A} \tau \langle a \rangle \\ &= \tau(\sum_{a \in A} \langle a \rangle) \\ &= \{P \in \Xi \mid P \nsubseteq A\} \\ &= \{P \in \Xi \mid P \nleq \langle A \rangle \,\} \\ &= \tau(\cap_{P_\beta \nsubseteq A} P_\beta), \end{aligned}$$

for which P_β is a non-zero closed fully invariant sum-prime submodule of ${}_RM$.

Proof. The similar proof of the proposition 6.2 completes the proof.□

The resulting topology is called the *Zariski kernel(or null) topology* for ${}_RM$ named after the Zariski topology on the prime spectrum of a commutative ring. The topological space Ξ is called the *kernel(or null) spectrum* of M, denoted by $\text{Spec}_N(M)$. Also we define the *prime socle* $\text{soc}(M)$ by the sum of all *sum-prime* submodules of ${}_RM$, in other words, $\text{soc}(M) = \sum_{P_\alpha \in \Xi} P_\alpha$ (cf. the Socle $Soc(M)$ the sum of all minimal submodules of ${}_RM$). Clearly in any closedly regular module it follows easily that $\text{soc}(M) \leq Soc(M)$.

If the prime socle $\text{soc}(M)$ is a sum-prime submodule of ${}_RM$, then $\text{Spec}_N(M) = \{ L \neq 0 \mid L \leq \text{soc}(M) \}$ contains $\text{soc}(M)$ since the prime radical $\text{soc}(M)$ is closed and fully invariant in ${}_RM$.

THEOREM 8.3. *For any closedly regular module ${}_RM$, if a submodule $K \geq \text{soc}(M)$ of ${}_RM$ is in $\text{Spec}_N(M)$, then we have that $K = \text{soc}(M)$ and $\text{Spec}_N(M)$ is irreducible.*

Proof. If $K \in \text{Spec}_N(M)$, then K is fully invariant sum-prime, then the closed submodule $\overline{K}$ is also fully invariant sum-prime in ${}_RM$. Thus $\text{soc}(M) \leq K \leq \overline{K} \in \text{Spec}_N(M)$ implies that $\text{soc}(M) = \overline{K} = K \in \text{Spec}_N(M)$. And every basic open set in the kernel(null) spectrum $\text{Spec}_N(M)$ contains $\text{soc}(M)$, in other words, $\text{Spec}_N(M)$ is irreducible. And by the hypothesis of $K \geq \text{soc}(M)$, we have a closed submodule $\text{soc}(M) = K$ which is in $\text{Spec}(M)$. □

COROLLARY 8.4. *For any closedly regular module ${}_RM$, the following are equivalent:*

(1) $\text{Spec}_N(M)$ *is irreducible;*
(2) $\text{soc}(M) \in \text{Spec}_N(M)$.

For any module ${}_RM$, we have a surjective mapping from the kernel(null) spectrum $\text{Spec}_N(M)$ onto a subset

$$\{I_P \mid P \in \text{Spec}_N(M)\} \subseteq \text{Spec}(S)$$

of the prime spectrum Spec(S) of the endomorphism ring S of ${}_RM$. Let this subspace $\{I_P | P \in \mathrm{Spec}_N(M)\}$ be a topological subspace of the Zariski topology of the spectrum Spec(S) of the endomorphism ring. Then we have the next theorem.

LEMMA 8.5. *For any closedly regular module ${}_RM$ let*

$$Y = \{I_P | P \in \mathrm{Spec}_N(M)\} \subseteq \mathrm{Spec}(S),$$

then we have the following.

(1) *If Y is open in* Spec(S) *and if the prime spectrum* Spec(S) *is irreducible. then the kernel(null) spectrum* Spec$_N(M)$ *is irreducible.*
(2) *If Y is dense in* Spec(S) *and if the kernel(null) spectrum* Spec$_N(M)$ *is irreducible, then the prime spectrum* Spec(S) *is irreducible.*
(3) *If Y is open dense in* Spec(S). *Then the prime spectrum* Spec(S) *is irreducible if and only if the kernel(null) spectrum* Spec$_N(M)$ *is irreducible.*

Proof. (1): By the hypothesis of irreducibility of Spec(S), it follows that its subspace is irreducible since the closure of an open set in the subspace $\{I_P | P \in \mathrm{Spec}_N(M)\}$ is the intersection of the closure of the open set in Spec(S) and the subspace $\{I_P | P \in \mathrm{Spec}_N(M)\}$ is inherited from the Zariski topology. The Zariski kernel topology Spec$_N(M)$ is the same that the topology with an onto mapping $P \mapsto I_P : \mathrm{Spec}_N(M) \to Y$ satisfies that each basic open set contains preimage of a basic open set in $Y = \{I_P | P \in \mathrm{Spec}_N(M)\}$. Therefore Spec$_N(M)$ is also irreducible.

(2): Assume that the prime spectrum Spec(S) is reducible. Then there are two nonempty disjoint open subsets in Spec(S) inducing two disjoint nonempty open subsets in Y since Y is dense in Spec(S). Therefore it follows easily that Spec$_N(M)$ is reducible.

(3): From (1) and (2) it follows immediately. □

COROLLARY 8.6. *For any openly regular module ${}_RM$, if*

$$\{I_P | P \in \mathrm{Spec}_N(M)\}$$

is open dense in Spec(S), *then the following are equivalent:*

(1) *The prime spectrum* Spec(S) *is reducible;*
(2) *The kernel(null) spectrum* Spec$_N(M)$ *is reducible.*

REMARK 8.7. The openness and density of $\{I_P | P \in \mathrm{Spec}_N(M)\}$ in the hypotheses of the Proposition 8.5 and Corollary 8.6 is essential. For example, a $\mathbb{Z}-$module ${}_{\mathbb{Z}}\mathbb{Z}(p^\infty)$ for a prime number p) has a non-sum-prime submodule $\mathrm{soc}(\mathbb{Z}(p^\infty)) = \mathbb{Z}(p^\infty) \notin \mathrm{Spec}_N(\mathbb{Z}(p^\infty))$, in other words, $\{I_K | K$ is a nonzero fully invariant sum-prime submodule of ${}_{\mathbb{Z}}\mathbb{Z}(p^\infty)\}$ is not an open set in the prime spectrum $\mathrm{Spec}(S) \ni 0 = I_{\mathrm{soc}(\mathbb{Z}(p^\infty))=\mathbb{Z}(p^\infty)}$. Considering a module ${}_{\mathbb{Z}}\mathbb{Z}$ being a closely simple module, then we have an empty set

$$Y = \{I_P | P \text{ is a sum-prime submodule of } \mathbb{Z}\} = \emptyset \subseteq \mathrm{Spec}(\mathrm{End}_{\mathbb{Z}}(\mathbb{Z})).$$

And Y is reducible and $\mathrm{Spec}(\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}))$ is irreducible. Therefore without the density of Y the reducibility of Y does not imply that of $\mathrm{Spec}(\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}))$.

Considering the socle $\mathrm{soc}(M) \leq M$ as an $R-$submodule of any module ${}_RM$, let T denote the endomorphism ring $\mathrm{End}_R(\mathrm{soc}(M))$ of ${}_R\mathrm{soc}(M)$.

THEOREM 8.8. *For a closedly regular module ${}_RM$ with the prime socle $\mathrm{soc}(M)$, if $\{I_L | L \in \mathrm{Spec}_N(\mathrm{soc}(M))\}$ is open dense in $\mathrm{Spec}(T)$, where $T = \mathrm{End}_R(\mathrm{soc}(M))$ is the endomorphism ring of the submodule $\mathrm{soc}(M)$, the following are equivalent:*

(1) *The endomorphism ring $\mathrm{End}_R(\mathrm{soc}(M))$ is prime;*
(2) *The prime spectrum $\mathrm{Spec}(T)$ is irreducible;*
(3) *The kernel(null) spectrum $\mathrm{Spec}_N(\mathrm{soc}(M))$ is irreducible;*
(4) *The prime socle $\mathrm{soc}(M)$ of ${}_RM$ is sum-prime.*

Proof. (1) $\Longrightarrow$ (2): It is trivial.

(2) $\Longrightarrow$ (3): Assume (1), then the prime spectrum $\mathrm{Spec}(T)$ is irreducible. Thus by the above Lemma 8.4 we have the irreducible kernel(null) spectrum $\mathrm{Spec}_N(\mathrm{soc}(M))$.

(3) $\Longrightarrow$ (4): From Corollary 8.4 it follows immediately.

(4) $\Longrightarrow$ (1): Assume that the kernel(null) spectrum $\mathrm{Spec}_N(\mathrm{soc}(M))$ is irreducible, then the prime socle $\mathrm{soc}(\mathrm{soc}(M)) = \mathrm{soc}(M) \in \mathrm{Spec}_N(\mathrm{soc}(M))$ is a fully invariant sum-prime submodule of a closedly regular module $\mathrm{soc}(M)$. Therefore we obtain a prime ideal $I_{\mathrm{soc}(M)} = I_M = 0 \trianglelefteq T$ of the endomorphism ring of $\mathrm{soc}(M)$. Therefore the endomorphism ring T is a prime ring. □

Theorem 8.9. *For a closedly regular module ${}_RM$ with the prime socle* $\mathrm{soc}(M)$, *if* $\{I_L | L \in \mathrm{Spec}_N(\mathrm{soc}(M))\}$ *is open dense in* $\mathrm{Spec}(T)$, *where* $T = \mathrm{End}_R(\mathrm{soc}(M))$ *is the endomorphism ring of the submodule* $\mathrm{soc}(M)$, *the following are equivalent:*

(1) *The endomorphism ring* $\mathrm{End}_R(\mathrm{soc}(M))$ *is not prime;*
(2) *The prime spectrum* $\mathrm{Spec}(T)$ *is reducible;*
(3) *The kernel(null) spectrum* $\mathrm{Spec}_N(\mathrm{soc}(M))$ *is reducible;*
(4) *The prime socle* $\mathrm{soc}(M)$ *of* ${}_RM$ *is not sum-prime.*

Theorem 8.10. *For a closedly regular module ${}_RM$ with* $\mathrm{soc}(M) = M$, *if* $\{I_P | P \in \mathrm{Spec}_N(M)\}$ *is open dense in* $\mathrm{Spec}(S)$, *then the following are equivalent:*

(1) *The endomorphism ring S is prime;*
(2) *The prime spectrum* $\mathrm{Spec}(S)$ *is irreducible;*
(3) *The kernel(null) spectrum* $\mathrm{Spec}_N(M)$ *is irreducible;*
(4) ${}_RM$ *is sum-prime.*

Proof. Replacing $\mathrm{soc}(M)$ with M in the above Theorem 8.7, the proof is completed. □

Theorem 8.11. *For a closedly regular module ${}_RM$ with* $\mathrm{soc}(M) = M$, *if* $\{I_P | P \in \mathrm{Spec}_N(M)\}$ *is open dense in* $\mathrm{Spec}(S)$, *then the following are equivalent:*

(1) *The endomorphism ring S is not prime;*
(2) *The prime spectrum* $\mathrm{Spec}(S)$ *is reducible;*
(3) *The kernel(null) spectrum* $\mathrm{Spec}_N(M)$ *is reducible;*
(4) ${}_RM$ *is not sum-prime.*

9. Zariski topologies for commutators of rings

For a left $R-$module ${}_RM$ over a ring R, let Z denote the commutator of the ground ring R over which ${}_RM$ is a left $R-$module,

$$\text{that is, } Z = \{a \in R \mid ar = ra, \text{ for each } r \in R\}.$$

We are regarding any left multiplication by $a \in Z$, denoted by $\rho(a)$: ${}_RM \to {}_RM$ defined by $m\rho(a) = am$ for every element $m \in M$ as an endomorphism, in other words, $\rho(Z) = \{\rho(a) \mid a \in Z\} \leq \mathrm{End}_R(M)$ is a subring with identity of the endomorphism $\mathrm{End}_R(M)$. Moreover for any left $R-$module $_RM$ over a commutative ring R with identity, clearly it follows that $Z = R$ and $\rho(R) = \{\rho(r) \mid r \in R\} \leq \mathrm{End}_R(M)$ is a subring of the endomorphism $\mathrm{End}_R(M)$. Thus if $P \leq {}_RM$ is a meet-[resp. sum-]prime submodule of $_RM$, we have a prime ideal $I^P \cap \rho(Z)$ [resp. $I_P \cap \rho(Z)$] $\trianglelefteq \rho(Z)$ of the subring $\rho(Z)$ of the endomorphism $\mathrm{End}_R(M)$, for all modules over any ring R with identity.

It is well-known that any commutative ring R can construct the Zariski topology of the prime spectrum $\mathrm{Spec}(R) = \{J \trianglelefteq R \mid J$ is a prime ideal of $R\}$, by the same method we can construct the Zariski topology of the prime spectrum $\mathrm{Spec}(\rho(Z))$, if as closed sets we take all sets of form $v(E) = \{\ I \in \mathrm{Spec}(\rho(Z)) \mid E \subseteq I\ \}$, where E is any subset of $\rho(Z)$. Precisely, the sets $v(E)$ satisfy the axioms for closed sets in a topological space:

(1) For any subset $E \subsetneq \rho(Z)$, if $\langle E\rangle$ is the ideal of $\rho(Z)$ generated by E, then $v(E) = v(\langle E\rangle) = v(r(E))$, where $r(E) = \cap_{E\subseteq J_\alpha \in \mathrm{Spec}(\rho(Z))} J_\alpha$ is the prime radical of E.
(2) $v(0) = \mathrm{Spec}(\rho(Z))$, $v(\rho(Z)) = \emptyset$.
(3) $v(\cup_{i\in I} E_i) = \cap_{i\in I} v(E_i)$, for each $E_i \subseteq \rho(Z)$.
(4) $v(AB) = v(A) \cup v(B)$ for $A, B \subseteq \rho(Z)$.

THEOREM 9.1. *For any module $_RM$ over a ring R with identity, the following are equivalent:*

(1) $\mathrm{Spec}(\rho(Z))$ *is irreducible;*
(2) *The prime radical* $\mathrm{rad}(\rho(Z)) = \cap_{J\in \mathrm{Spec}(\rho(Z))} J$ *is in* $\mathrm{Spec}(\rho(Z))$, *that is,* $\mathrm{rad}(\rho(Z))$ *is a prime ideal of* $\rho(Z)$.

In fact, it is true that the prime radical

$$\mathrm{rad}(\rho(Z)) = \cap_{J\in \mathrm{Spec}(\rho(Z))} J = \mathrm{rad}(S) \cap \rho(Z),$$

where $\mathrm{rad}(\rho(Z))$ is the prime radical of $\rho(Z)$ and $\mathrm{rad}(S) = \cap_{J\in \mathrm{Spec}(S)} J$ is the prime radical of the endomorphism ring S of $_RM$. The following note is rewritten for a faithful module $_RM$ over a commutative ring R in terms of $\rho(Z) \equiv R$.

NOTE 9.2. For (any faithful module ${}_RM$ over) a commutative ring R with identity, the following are equivalent:

(1) $\mathrm{Spec}(R)$ is irreducible;
(2) The prime radical $\mathrm{rad}(R) = \cap_{J \in \mathrm{Spec}(R)} J$ is in $\mathrm{Spec}(R)$, i.e., $\mathrm{rad}(R)$ is a prime ideal of R.

Since ${}_RM$ is faithful we can identify the subring $\rho(Z)$ of S with the ground ring R. Replace $\rho(Z)$ by R.

10. On openly regular modules

For any fully invariant meet-prime submodule $P \leq M$ of a module ${}_RM$, we have prime ideals $I^P \trianglelefteq S$ and $I^P \cap \rho(Z) \trianglelefteq \rho(Z)$.

For any module ${}_RM$, we have a surjective mapping from the image spectrum $\mathrm{Spec}_I(M)$ onto a subset $\{ I^P \mid P \in \mathrm{Spec}_I(M) \} \subseteq \mathrm{Spec}(S)$ of the prime spectrum $\mathrm{Spec}(S)$ of the endomorphism ring S of ${}_RM$. Also we have a surjective mapping from the image spectrum $\mathrm{Spec}_I(M)$ onto $\{ I^P \cap \rho(Z) \mid P \in \mathrm{Spec}_I(M) \} \subseteq \mathrm{Spec}(\rho(Z))$ of the prime spectrum $\mathrm{Spec}(\rho(Z))$ of the commutator ring $\rho(Z)$ of a ring R with identity.

Let this subspace $\{ I^P \mid P \in \mathrm{Spec}_I(M) \}$ be inherited from the Zariski topology of the spectrum $\mathrm{Spec}(S)$ of the endomorphism ring. Then we have the next results. No proof will be given.

THEOREM 10.1. *For any openly regular module ${}_RM$ if*

$$\{ I^P \mid P \in \mathrm{Spec}_I(M) \} \ \textit{and} \ \{ I^P \cap \rho(Z) \mid P \in \mathrm{Spec}_I(M) \}$$

are open dense sets in the prime spectra $\mathrm{Spec}(S)$ and $\mathrm{Spec}(\rho(Z))$, respectively, then the following are equivalent:

(1) *The prime spectrum $\mathrm{Spec}(S)$ is irreducible;*
(2) *The image spectrum $\mathrm{Spec}_I(M)$ is irreducible;*
(3) *The prime spectrum $\mathrm{Spec}(\rho(Z))$ is irreducible.*

Note here if the commutator $\rho(Z)$ is not a prime ring, then immediately follows that neither S nor R is a prime ring. Thus we have the following corollary of the contraposition of Theorem 10.1 as follows:

COROLLARY 10.2. *For any openly regular module* $_RM$*, if*

$$\{\ I^P \mid P \in \mathrm{Spec}_I(M)\ \} \ \text{ and } \ \{\ I^P \cap \rho(Z) \mid P \in \mathrm{Spec}_I(M)\ \}$$

are open dense sets in the prime spectra Spec(S) *and* Spec($\rho(Z)$), *respectively, then the following are equivalent:*

(1) *The prime spectrum* Spec(S) *is reducible;*
(2) *The image spectrum* $\mathrm{Spec}_I(M)$ *is reducible;*
(3) *The prime spectrum* Spec($\rho(Z)$) *is reducible.*

REMARK 10.3. The opennesses and density of $\{\ I^P \mid P \in \mathrm{Spec}_I(M)\ \}$ and $\{\ I^P \cap \rho(Z) \mid P \in \mathrm{Spec}_I(M)\ \}$ in the hypotheses of the Theorem 10.1 and Corollary 10.2 is essential.

THEOREM 10.4. *For any openly regular module* $_RM$ *with* $\mathrm{rad}(M) = 0$, *if* $\{\ I^P \mid P \in \mathrm{Spec}_I(M)\ \}$ *and* $\{\ I^P \cap \rho(Z) \mid P \in \mathrm{Spec}_I(M)\ \}$ *are open dense sets in the prime spectra* Spec(S) *and* Spec($\rho(Z)$), *respectively, then the following are equivalent:*

(1) *The commutator* $\rho(Z)$ *has a prime annihilator ideal* $\mathrm{Ann}_R(M) \cap \rho(Z)$;
(2) *The endomorphism ring* S *is prime;*
(3) *The prime spectrum* Spec(S) *is irreducible;*
(4) *The prime spectrum* Spec($\rho(Z)$) *is irreducible;*
(5) *The image spectrum* $\mathrm{Spec}_I(M)$ *is irreducible;*
(6) $0 \leq M$ *is meet-prime.*

THEOREM 10.5. *For any openly regular module* $_RM$ *with* $\mathrm{rad}(M) = 0$, *if* $\{\ I^P \mid P \in \mathrm{Spec}_I(M)\ \}$ *and* $\{\ I^P \cap \rho(Z) \mid P \in \mathrm{Spec}_I(M)\ \}$ *are open dense sets in the prime spectra* Spec(S) *and* Spec($\rho(Z)$), *respectively, then the following are equivalent:*

(1) *The commutator* $\rho(Z)$ *has a nonprime ideal* $\mathrm{Ann}_R(M) \cap \rho(Z)$;
(2) *The endomorphism ring* S *is not prime;*
(3) 0 *is not meet-prime;*
(4) *The prime spectrum* Spec(S) *is reducible;*
(5) *The prime spectrum* Spec($\rho(Z)$) *is reducible;*
(6) *The image spectrum* $\mathrm{Spec}_I(M)$ *is reducible.*

For any faithful module ${}_RM$, the annihilator $\mathrm{Ann}_R(M) = 0$ is trivial. Thus we have immediate consequences of Theorem 10.4 and Corollary 10.5 as follows.

COROLLARY 10.6. *For any openly regular faithful module* ${}_RM$ *with* $\mathrm{rad}(M) = 0$, *if* $\{ I^P \mid P \in \mathrm{Spec}_I(M) \}$ *and* $\{ I^P \cap \rho(Z) \mid P \in \mathrm{Spec}_I(M) \}$ *are open dense sets in the prime spectra* $\mathrm{Spec}(S)$ *and* $\mathrm{Spec}(\rho(Z))$, *respectively, then the following are equivalent:*

(1) *The commutator* $\rho(Z)$ *is prime;*
(2) *The endomorphism ring* S *is prime;*
(3) *The prime spectrum* $\mathrm{Spec}(S)$ *is irreducible;*
(4) *The prime spectrum* $\mathrm{Spec}(\rho(Z))$ *is irreducible;*
(5) *The image spectrum* $\mathrm{Spec}_I(M)$ *is irreducible;*
(6) $0 \leq M$ *is meet-prime.*

COROLLARY 10.7. *For any openly regular faithful module* ${}_RM$ *with* $\mathrm{rad}(M) = 0$, *if* $\{ I^P \mid P \in \mathrm{Spec}_I(M) \}$ *and* $\{ I^P \cap \rho(Z) \mid P \in \mathrm{Spec}_I(M) \}$ *are open dense sets in the prime spectra* $\mathrm{Spec}(S)$ *and* $\mathrm{Spec}(\rho(Z))$, *respectively, then the following are equivalent:*

(1) *The commutator* $\rho(Z)$ *is not prime;*
(2) *The endomorphism ring* S *is not prime;*
(3) $0 \leq M$ *is not meet-prime;*
(4) *The prime spectrum* $\mathrm{Spec}(S)$ *is reducible;*
(5) *The prime spectrum* $\mathrm{Spec}(\rho(Z))$ *is reducible;*
(6) *The image spectrum* $\mathrm{Spec}_I(M)$ *is reducible.*

11. On closedly regular modules

For any module ${}_RM$, we have a surjective mapping from the kernel(null) spectrum $\mathrm{Spec}_N(M)$ onto a subset $\{ I_P \mid P \in \mathrm{Spec}_N(M) \} \subseteq \mathrm{Spec}(S)$ of the prime spectrum $\mathrm{Spec}(S)$ of the endomorphism ring S of M.

Also we have a surjective mapping from the kernel(null) spectrum $\mathrm{Spec}_N(M)$ onto a subset $\{ I_P \cap \rho(Z) \mid P \in \mathrm{Spec}_N(M) \} \subseteq \mathrm{Spec}(\rho(Z))$ of the prime spectrum $\mathrm{Spec}(\rho(Z))$ of the commutator of ring R.

Let this subspace $\{ I_P \mid P \in \mathrm{Spec}_N(M) \}$ be inherited from the Zariski topology of the spectrum $\mathrm{Spec}(S)$ of the endomorphism ring S. Then we have the next theorem.

LEMMA 11.1. *For any closedly regular module* ${}_RM$, *if* $\{ I_P \mid P \in \mathrm{Spec}_N(M) \}$ *and* $\{ I_P \cap \rho(Z) \mid P \in \mathrm{Spec}_N(M) \}$ *are open dense sets in the prime spectra* $\mathrm{Spec}(S)$ *and* $\mathrm{Spec}(\rho(Z)$, *respectively, then the following are equivalent:*

(1) *The prime spectrum* $\mathrm{Spec}(\rho(Z))$ *is irreducible;*
(2) *The prime spectrum* $\mathrm{Spec}(S)$ *is irreducible;*
(3) *The kernel(null) spectrum* $\mathrm{Spec}_N(M)$ *is irreducible.*

COROLLARY 11.2. *For any openly regular module* ${}_RM$, *if* $\{ I_P \mid P \in \mathrm{Spec}_N(M) \}$ *and* $\{ I_P \cap \rho(Z) \mid P \in \mathrm{Spec}_N(M) \}$ *are open dense sets in the prime spectra* $\mathrm{Spec}(S)$ *and* $\mathrm{Spec}(\rho(Z))$, *respectively, then the following are equivalent:*

(1) *The prime spectrum* $\mathrm{Spec}(\rho(Z))$ *is reducible;*
(2) *The prime spectrum* $\mathrm{Spec}(S)$ *is reducible;*
(3) *The kernel(null) spectrum* $\mathrm{Spec}_N(M)$ *is reducible.*

REMARK 11.3. The opennesses and density of $\{I_P \mid P \in \mathrm{Spec}_N(M)\}$ and $\{ I_P \cap \rho(Z) \mid P \in \mathrm{Spec}_N(M) \}$ in the hypotheses of the Theorem 11.1 and Corollary 11.2 is essential.

THEOREM 11.4. *For any closedly regular module* ${}_RM$ *with* $\mathrm{soc}(M) = M$, *if* $\{ I_P \mid P \in \mathrm{Spec}_N(M) \}$ *and* $\{ I_P \cap \rho(Z) \mid P \in \mathrm{Spec}_N(M) \}$ *are open dense sets in the prime spectra* $\mathrm{Spec}(S)$ *and* $\mathrm{Spec}(\rho(Z))$, *respectively, then the following are equivalent:*

(1) *The commutator* $\rho(Z)$ *has a prime ideal* $\mathrm{Ann}_R(M) \cap \rho(Z)$;
(2) *The endomorphism ring* S *is prime;*
(3) *The prime spectrum* $\mathrm{Spec}(S)$ *is irreducible;*
(4) *The prime spectrum* $\mathrm{Spec}(\rho(Z))$ *is irreducible;*
(5) *The kernel(null) spectrum* $\mathrm{Spec}_N(M)$ *is irreducible;*
(6) $M \leq M$ *is sum-prime.*

THEOREM 11.5. *For any closedly regular module* ${}_RM$ *with* $\mathrm{soc}(M) = M$, *if* $\{ I_P \mid P \in \mathrm{Spec}_N(M) \}$ *and* $\{ I_P \cap \rho(Z) \mid P \in \mathrm{Spec}_N(M) \}$ *are open dense sets in the prime spectra* $\mathrm{Spec}(S)$ *and* $\mathrm{Spec}(\rho(Z))$, *respectively, then the following are equivalent:*

(1) *The commutator* $\rho(Z)$ *has a nonprime ideal* $\mathrm{Ann}_R(M) \cap \rho(Z)$;

(2) *The endomorphism ring S is not prime;*
(3) *The prime spectrum* Spec(S) *is reducible;*
(4) *The prime spectrum* Spec($\rho(Z)$) *is reducible;*
(5) *The kernel(null) spectrum* $\mathrm{Spec}_N(M)$ *is reducible;*
(6) $M \leq M$ *is not sum-prime.*

THEOREM 11.6. *For any closedly regular faithful module $_RM$ with* $\mathrm{soc}(M) = M$, *if* $\{ I_P \mid P \in \mathrm{Spec}_N(M) \}$ *and* $\{ I_P \cap \rho(Z) \mid P \in \mathrm{Spec}_N(M) \}$ *are open dense sets in the prime spectra* Spec(S) *and* Spec($\rho(Z)$), *respectively, then the following are equivalent:*

(1) *The commutator $\rho(Z)$ is prime;*
(2) *The endomorphism ring S is prime;*
(3) *The prime spectrum* Spec(S) *is irreducible;*
(4) *The prime spectrum* Spec($\rho(Z)$) *is irreducible;*
(5) *The kernel(null) spectrum* $\mathrm{Spec}_N(M)$ *is irreducible;*
(6) $M \leq M$ *is sum-prime.*

THEOREM 11.7. *For any closedly regular faithful module $_RM$ with* $\mathrm{soc}(M) = M$, *if* $\{ I_P \mid P \in \mathrm{Spec}_N(M) \}$ *and* $\{ I_P \cap \rho(Z) \mid P \in \mathrm{Spec}_N(M) \}$ *are open dense sets in the prime spectra* Spec(S) *and* Spec($\rho(Z)$), *respectively, then the following are equivalent:*

(1) *The commutator $\rho(Z)$ is not prime;*
(2) *The endomorphism ring S is not prime;*
(3) *The prime spectrum* Spec(S) *is reducible;*
(4) *The kernel(null) spectrum* $\mathrm{Spec}_N(M)$ *is reducible;*
(5) *The prime spectrum* Spec($\rho(Z)$) *is reducible;*
(6) $M \leq M$ *is not sum-prime.*

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