J. Korean Math. Soc. **38** (2001), No. 4, pp. 697–709

GENERALIZATIONS OF THE NASH EQUILIBRIUM THEOREM ON GENERALIZED CONVEX SPACES

SEHIE PARK

ABSTRACT. Generalized forms of the von Neumann–Sion type minimax theorem, the Fan–Ma intersection theorem, the Fan–Ma type analytic alternative, and the Nash–Ma equilibrium theorem hold for generalized convex spaces without having any linear structure.

1. Introduction

In 1928, John von Neumann found his celebrated minimax theorem [32] and, in 1937, his intersection lemma [33], which was intended to establish his minimax theorem and his theorem on optimal balanced growth paths. In 1941, Kakutani [9] obtained a fixed point theorem, from which von Neumann's minimax theorem and intersection lemma are easily deduced.

In 1951, John Nash [12] established his celebrated equilibrium theorem. In 1952, Fan [4] and Glicksberg [7] extended Kakutani's theorem to locally convex Hausdorff topological vector spaces, and Fan generalized the von Neumann intersection lemma by applying his own fixed point theorem. In 1964, Fan [5] obtained another intersection theorem for a finite family of sets having convex sections. This was extended, by Ma [11] in 1969, to infinite families by using Fan's generalization of the von Neumann intersection lemma. Ma applied his result to an analytic formulation of Fan type and to Nash's theorem for arbitrary families.

Note that all of the above results are extended in our recent works [16, 17, 20-23, 25, 27, 8] in several directions. In fact, those results are mainly

Received August 31, 2000.
2000 Mathematics Subject Classification: 47H10, 49J53, 91A13.
Key words and phrases: generalized convex (G-convex) space, Φ-map, Γ-convex hull, minimax theorem, intersection theorem, analytic alternative, equilibrium theorem.

This is a refined version of a part of the text of the invited lecture given at International Conference on Mathematics in the New Millennium, Korean Mathematical Society, Oct. 20–22, 2000.

concerned with convex subsets of (Hausdorff) topological vector spaces or convex spaces in the sense of Lassonde. Moreover, the author have developed theory of generalized convex spaces (simply, G-convex spaces) related to the KKM theory and analytical fixed point theory. In the framework of G-convex spaces, we obtained several minimax theorems and the Nash equilibrium theorems in our previous works [20, 21, 25], based on coincidence theorems or intersection theorems for finite families of sets.

Our aim in this paper is to obtain generalized forms of the G-convex space versions of known results due to von Neumann, Sion, Nash, Fan, Ma, and others.

In Section 2, we state basic facts on G-convex spaces in our previous work [18]. Section 3 deals with the Fan–Ma type intersection theorem for G-convex spaces. In Section 4, we deduce a generalized Fan–Ma type analytic alternative and in Section 5, the Nash–Ma type equilibrium theorem and its consequences.

2. Preliminaries

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that, for each $A = \{a_0, a_1, \cdots, a_n\} \in \langle D \rangle$, there exist a subset $\Gamma(A) = \Gamma_A$ of X and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \subset \{0, 1, \cdots, n\}$ implies $\phi_A(\Delta_J) \subset \Gamma(\{a_j : j \in J\})$, where $\langle D \rangle$ denotes the set of all nonempty finite subsets of D, Δ_n an n-simplex with vertices $v_0, v_1, \cdots, v_n$, and $\Delta_J = \mathrm{co}\{v_j : j \in J\}$ the face of Δ_n corresponding to J.

In case to emphasize $X \supset D$, $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$; and if $X = D$, then $(X; \Gamma) := (X, X; \Gamma)$.

There are a large number of examples of G-convex spaces; see [19, 21, 24]. Typical examples are any convex subset of a topological vector space, convex spaces in the sense of Lassonde, C-spaces (or H-spaces) due to Horvath, and many others.

For a topological space X and a G-convex space $(Y, D; \Gamma)$, a multimap $T : X \multimap Y$ is called a *Φ-map* provided that there exists a multimap $S : X \multimap D$ satisfying

(a) for each $x \in X$, $M \in \langle S(x) \rangle$ implies $\Gamma_M \subset T(x)$; and
(b) $X = \bigcup\{\mathrm{Int}\, S^-(y) : y \in D\}$, where $S^-(y) = \{x \in X : y \in S(x)\}$.

We need the following selection theorem:

THEOREM 1 [18]. *Let X be a Hausdorff space, $(Y, D; \Gamma)$ a G-convex space, and $T : X \multimap Y$ a Φ-map.*

Then for any nonempty compact subset K of X, we have the following:

(i) *$T|_K$ has a continuous selection $f : K \to Y$ such that $f(K) \subset \Gamma_A$ for some $A \in \langle D \rangle$. More precisely, there exist two continuous functions $p : K \to \Delta_n$ and $\phi_A : \Delta_n \to \Gamma_A$ such that $f = \phi_A \circ p$ for some $A \in \langle D \rangle$ with $|A| = n + 1$.*
(ii) *If $g : Y \to K$ is a continuous map, then there exists a $y_0 \in Y$ such that $y_0 \in T(g(y_0))$.*
(iii) *If $R : K \multimap Y$ is a multimap such that $R^- : Y \multimap K$ has a continuous selection, then R and $T|_K$ have a coincidence point $x_0 \in K$; that is, $R(x_0) \cap T(x_0) \neq \emptyset$.*
(iv) *For any compact subset L of X containing K, there exists a continuous extension $\tilde{f} : L \to Y$ of the map f in* (i) *such that $\tilde{f}(x) \in T(x)$ for each $x \in L$ and $\tilde{f}(L) \subset \Gamma_B$ for some $B \in \langle D \rangle$.*

The following is known:

LEMMA. *Let $\{(X_i, D_i; \Gamma_i)\}_{i \in I}$ be a family of G-convex spaces, $X = \Pi_{i \in I} X_i$, $D = \Pi_{i \in I} D_i$, and $\pi_i : D \to D_i$ the projection for each $i \in I$. Define*

$$\Gamma(A) := \prod_{i \in I} \Gamma_i(\pi_i(A)) \quad \text{for each} \quad A \in \langle D \rangle.$$

Then $(X, D; \Gamma)$ is a G-convex space.

From Theorem 1 and Lemma, we deduced the following collectively fixed point theorem:

THEOREM 2 [18]. *Let $\{(X_i; \Gamma_i)\}_{i \in I}$ be a family of Hausdorff compact G-convex spaces, $X = \Pi_{i \in I} X_i$, and for each $i \in I$, $T_i : X \multimap X_i$ a Φ-map. Then there exists a point $x \in X$ such that $x \in T(x) := \Pi_{i \in I} T_i(x)$; that is, $x_i = \pi_i(x) \in T_i(x)$ for each $i \in I$.*

REMARKS 1. If I is a singleton, X is a convex space, and $S_i = T_i$, then Theorem 3 reduces to the well-known Fan–Browder fixed point theorem; see Park [15].

2. For the case I is a singleton, Theorem 3 for a convex space X was obtained by Ben-El-Mechaiekh *et al.* [1, Theorem 1] and Simons [28, Theorem 4.3]. This was extended by several authors; see Park [15].

3. In case when $(X_i;\Gamma_i)$ are all C-spaces, Theorem 2 reduces to Tarafdar [31, Theorem 2.3]. This is applied to sets with C-convex sections [31, Theorem 3.1] and to existence of an equilibrium point of an abstract economy [31, Theorem 4.1 and Corollary 4.1]. These results can also be extended to G-convex spaces and we will not repeat here.

For a G-convex space $(X \supset D;\Gamma)$, a subset $Y \subset X$ is said to be Γ-*convex* if for each $N \in \langle D\rangle$, $N \subset Y$ implies $\Gamma_N \subset Y$; and for any subset $Y \subset X$, the Γ-*convex hull* of Y is defined as follows:

$$\Gamma\text{-co}\,Y := \bigcap\{Z : Z \text{ is a } \Gamma\text{-convex subset of } X \text{ containing } Y\}.$$

It is easily seen that $\Gamma\text{-co}\,Y = \bigcup\{\Gamma\text{-co}\,N : N \in \langle Y\rangle\}$.

For a G-convex space $(X \supset D;\Gamma)$, a real function $f : X \to \mathbb{R}$ is said to be *quasiconcave* [resp. *quasiconvex*] if $\{x \in X : f(x) > r\}$ [resp. $\{x \in X : f(x) < r\}$] is Γ-convex for each $r \in \mathbb{R}$.

Recall that a real function $f : X \to \mathbb{R}$, where X is a topological space, is *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] if $\{x \in X : f(x) > r\}$ [resp. $\{x \in X : f(x) < r\}$] is open for each $r \in \mathbb{R}$.

Let $\{X_i\}_{i\in I}$ be a family of sets, and let $i \in I$ be fixed. Let

$$X := \prod_{j\in I} X_j, \quad X^i := \prod_{j\in I\setminus\{i\}} X_j.$$

If $x^i \in X^i$ and $j \in I\setminus\{i\}$, let x^i_j denote the jth coordinate of x^i. If $x^i \in X^i$ and $x_i \in X_i$, let $[x^i, x_i] \in X$ be defined as follows: its ith coordinate is x_i and, for $j \neq i$ the jth coordinate is x^i_j. Therefore, any $x \in X$ can be expressed as $x = [x^i, x_i]$ for any $i \in I$, where x^i denotes the projection of x in X^i.

For $A \subset X$, $x^i \in X^i$, and $x_i \in X_i$, let

$$A(x^i) := \{y_i \in X_i : [x^i, y_i] \in A\}, \quad A(x_i) := \{y^i \in X^i : [y^i, x_i] \in A\}.$$

3. Intersection theorems for sets with convex sections

In our previous work [20], from a G-convex space version of the Fan–Browder fixed point theorem, we deduced a Ky Fan type intersection theorem for n subsets of a cartesian product of n compact G-convex spaces which are not necessarily Hausdorff. This was applied to obtain a von Neumann–Sion type minimax theorem and a Nash type equilibrium theorem for G-convex spaces.

In the present section, we generalize the above-mentioned intersection theorem to arbitrary number of subsets. From now on, we assume that all topological spaces are Hausdorff. This is mainly because of that we can not get rid of the Hausdorffness in Theorem 2.

The collectively fixed point theorem in Section 2 can be reformulated to a generalization of various Ky Fan type intersection theorems for sets with convex sections as follows:

THEOREM 3. *Let $\{(X_i;\Gamma_i)\}_{i\in I}$ be a family of compact G-convex spaces and, for each $i \in I$, let A_i and B_i are subsets of $X = \prod_{i\in I} X_i$ satisfying the following:*

(1) *for each $x^i \in X^i$, $\emptyset \neq \Gamma_i\text{-co}\, B_i(x^i) \subset A_i(x^i)$; and*

(2) *for each $y_i \in X_i$, $B_i(y_i)$ is open in X^i.*

Then we have $\bigcap_{i\in I} A_i \neq \emptyset$.

Proof. We apply Theorem 2 with multimaps $S_i, T_i : X \multimap X_i$ given by $S_i(x) := B_i(x^i)$ and $T_i(x) := A_i(x^i)$ for each $x \in X$. Then for each $i \in I$ we have the following:

(a) For each $x \in X$, we have $\emptyset \neq \Gamma_i\text{-co}\, S_i(x) \subset T_i(x)$.

(b) For each $y_i \in X_i$, we have

$$\begin{aligned} x \in S_i^-(y_i) &\iff y_i \in S_i(x) = B_i(x^i) \\ &\iff [x^i, y_i] \in B_i \subset X^i \times X_i = X. \end{aligned}$$

Hence,

$$\begin{aligned} S_i^-(y_i) &= \{x = [x^i, x_i] \in X : x^i \in B_i(y_i),\ x_i \in X_i\} \\ &= B_i(y_i) \times X_i. \end{aligned}$$

Note that $S_i^-(y_i)$ is open in $X = X^i \times X_i$ and that T_i is a Φ-map. Therefore, by Theorem 2, there exists an $\widehat{x} \in X$ such that $\widehat{x}_i \in T_i(\widehat{x}) = A_i(\widehat{x}^i)$ for all $i \in I$. Hence $\widehat{x} = [\widehat{x}^i, \widehat{x}_i] \in \bigcap_{i\in I} A_i \neq \emptyset$. This completes our proof. □

EXAMPLES. For convex spaces X_i, particular forms of Theorem 3 have appeared as follows:

1. Ky Fan [5, Théorème 1]: I is finite and $A_i = B_i$ for all $i \in I$.
2. Ky Fan [6, Theorem 1′]: $I = \{1, 2\}$ and $A_i = B_i$ for all $i \in I$.

From these results, Ky Fan [6] deduced an analytic formulation, fixed point theorems, extension theorems of monotone sets, and extension theorems for invariant vector subspaces.

3. Ma [11, Theorem 2]: The case $A_i = B_i$ for all $i \in I$ with a different proof.

4. Chang [3, Theorem 4.2] first obtained Theorem 3 with a different proof. She also obtained a noncompact version of Theorem 3 as [3, Theorem 4.3].

5. Park [22, Theorem 4.2]: X_i are convex spaces.

For particular types of G-convex spaces and a finite set I, Theorem 3 was known as follows:

6. Bielawski [2, Proposition (4.12) and Theorem (4.15)]: X_i has the finitely local convexity.

7. Kirk, Sims, and Yuan [10, Theorem 5.2]: X_i are hyperconvex metric spaces.

8. Park [20, Theorem 4], [21, Theorem 19]: I is finite.

4. The Fan type analytic alternative

From the intersection theorem 3, we can deduce the following equivalent form of a generalized Fan type minimax theorem or an analytic alternative. Our method is based on that of Fan [5, 6] and Ma [11].

THEOREM 4. *Let $\{(X_i;\Gamma_i)\}_{i\in I}$ be a family of compact G-convex spaces and, for each $i \in I$, let $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ be real functions satisfying*

(1) *$g_i(x) \le f_i(x)$ for each $x \in X$;*

(2) *for each $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ is quasiconcave on X_i; and*

(3) *for each $x_i \in X_i$, $x^i \mapsto g_i[x^i, x_i]$ is l.s.c. on X^i.*

Let $\{t_i\}_{i\in I}$ be a family of real numbers. Then either

(a) *there exist an $i \in I$ and an $x^i \in X^i$ such that*

$$g_i[x^i, y_i] \le t_i \quad \text{for all } y_i \in X_i; \text{ or}$$

(b) *there exists an $x \in X$ such that*

$$f_i(x) > t_i \quad \text{for all } i \in I.$$

Proof. Suppose that (a) does not hold; that is, for any $i \in I$ and any $x^i \in X^i$, there exists an $x_i \in X_i$ such that $g_i[x^i, x_i] > t_i$. Let

$$A_i := \{x \in X : f_i(x) > t_i\} \quad \text{and} \quad B_i := \{x \in X : g_i(x) > t_i\}$$

for each $i \in I$. Then

(4) for each $x^i \in X^i$, $\emptyset \neq B_i(x^i) \subset A_i(x^i)$;

(5) for each $x^i \in X^i$, $A_i(x^i)$ is Γ_i-convex; and

(6) for each $y_i \in X_i$, $B_i(y_i)$ is open in X^i.

Therefore, by Theorem 3, there exists an $x \in \bigcap_{i \in I} A_i$. This is equivalent to (b). □

EXAMPLES. 1. Ky Fan [5, Théorème 2; 6, Theorem 3]: X_i are convex spaces, I is finite, and $f_i = g_i$ for all $i \in I$. From this, Ky Fan [5, 6] deduced Sion's minimax theorem [29], the Tychonoff fixed point theorem, solutions to systems of convex inequalities, extremum problems for matrices, and a theorem of Hardy–Littlewood–Pólya.

2. Ma [11, Theorem 3]: X_i are convex spaces and $f_i = g_i$ for all $i \in I$.

3. Park [22, Theorem 8.1]: X_i are convex spaces.

REMARKS. 1. We obtained Theorem 4 from Theorem 3. As was pointed out by Ky Fan [5] for his case, we can deduce Theorem 3 from Theorem 4 by considering the characteristic functions of the sets A_i and B_i.

2. The conclusion of Theorem 4 can be stated as follows: If

$$\min_{x^i \in X^i} \sup_{x_i \in X_i} g_i[x^i, x_i] > t_i \quad \text{for all} \quad i \in I,$$

then (b) holds; see Fan [5, 6].

5. The Nash type equilibrium theorem

From Theorem 3, we obtain the following generalization of the Nash-Ma type equilibrium theorems:

THEOREM 5. *Let $\{(X_i; \Gamma_i)\}_{i \in I}$ be a family of compact G-convex spaces and, for each $i \in I$, let $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ be real functions such that*

(0) *$g_i(x) \leq f_i(x)$ for each $x \in X$;*

(1) *for each $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ is quasiconcave on X_i;*

(2) *for each $x^i \in X^i$, $x_i \mapsto g_i[x^i, x_i]$ is u.s.c. on X_i; and*

(3) *for each $x_i \in X_i$, $x^i \mapsto g_i[x^i, x_i]$ is l.s.c. on X^i.*

Then there exists a point $\hat{x} \in X$ such that

$$f_i(\hat{x}) \geq \max_{y_i \in X_i} g_i[\hat{x}^i, y_i] \quad \text{for all } i \in I.$$

Proof. For any $\varepsilon > 0$, we define

$$A_{\varepsilon,i} := \{x \in X : f_i(x) > \max_{y_i \in X_i} g_i[x^i, y_i] - \varepsilon\},$$
$$B_{\varepsilon,i} := \{x \in X : g_i(x) > \max_{y_i \in X_i} g_i[x^i, y_i] - \varepsilon\}$$

for each i. Then

(1) for each $x^i \in X^i$, $B_{\varepsilon,i}(x^i) \subset A_{\varepsilon,i}(x^i)$;

(2) for each $x^i \in X^i$, $A_{\varepsilon,i}(x^i)$ is Γ_i-convex;

(3) for each $x^i \in X^i$, $B_{\varepsilon,i}(x_i) \neq \emptyset$ since $x_i \mapsto g_i[x^i, x_i]$ is u.s.c. on the compact space X_i; and

(4) for each $x_i \in X_i$, $B_{\varepsilon,i}(x_i)$ is open since $x^i \mapsto g_i[x^i, x_i]$ is l.s.c. on X^i.

Therefore, by applying Theorem 3, we have

$$\bigcap_{i \in I} A_{\varepsilon,i} \neq \emptyset \quad \text{for every} \quad \varepsilon > 0.$$

Since X is compact, there exists an $\hat{x} \in X$ such that

$$f_i(\hat{x}) \geq \max_{y_i \in X_i} g_i[\hat{x}^i, y_i] \quad \text{for all} \quad i \in I. \qquad \square$$

EXAMPLES. 1. In case when X_i are convex spaces, $f_i = g_i$, and I is finite, Theorem 5 reduces to Tan *et al.* [30, Theorem 2.1].

2. Park [22, Theorem 8.2]: X_i are convex spaces.

From Theorem 5, we obtain the following generalization of the Nash equilibrium theorem for G-convex spaces:

THEOREM 6. *Let $\{(X_i; \Gamma_i)\}_{i \in I}$ be a family of compact G-convex spaces and, for each $i \in I$, let $f_i : X \to \mathbb{R}$ be a function such that*

(1) *for each $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ is quasiconcave on X_i;*

(2) *for each $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ is u.s.c. on X_i; and*

(3) *for each $x_i \in X_i$, $x^i \mapsto f_i[x^i, x_i]$ is l.s.c. on X^i.*

Then there exists a point $\hat{x} \in X$ such that

$$f_i(\hat{x}) = \max_{y_i \in X_i} f_i[\hat{x}^i, y_i] \quad \text{for all } i \in I.$$

EXAMPLES. For continuous functions f_i, a number of particular forms of Theorem 6 have appeared for convex subsets X_i of topological vector spaces as follows:

1. Nash [12, Theorem 1]: I is finite and X_i are subsets of Euclidean spaces.
2. Nikaido and Isoda [13, Theorem 3.2]: I is finite.
3. Ky Fan [6, Theorem 4]: I is finite.
4. Ma [11, Theorem 4]: I is arbitrary.

For particular types of G-convex spaces X_i, continuous functions f_i, and a finite index set I, particular forms of Theorem 6 have appeared as follows:

5. Bielawski [2, Theorem (4.16)]: X_i have the finitely local convexity.
6. Kirk, Sims, and Yuan [10, Theorem 5.3]: X_i are hyperconvex metric spaces.
7. Park [20, Theorem 6], [21, Theorem 20]: I is finite and f_i are continuous.

The point $\widehat{x}$ in the conclusion of Theorem 6 is called a *Nash equilibrium*. This concept is a natural extension of the local maxima and the saddle point as follows.

In case I is a singleton, we obtain the following:

COROLLARY 1. *Let X be a closed bounded convex subset of a reflexive Banach space E and $f : X \to \mathbb{R}$ a quasiconcave u.s.c. function. Then f attains its maximum on X; that is, there exists an $\widehat{x} \in X$ such that $f(\widehat{x}) \geq f(x)$ for all $x \in X$.*

Proof. Let E be equipped with the weak topology. Then, by the Hahn-Banach theorem, f is still u.s.c. because f is quasiconcave, and X is still closed. Being bounded, X is contained in some closed ball which is weakly compact. Since any closed subset of a compact set is compact, so X is (weakly) compact. Now, by Theorem 6 for a singleton I, we have the conclusion. □

Corollary 1 is due to Mazur and Schauder in 1936. Several generalized forms of Corollary 1 were known by Park *et al.* [26, 14].

For $I = \{1, 2\}$, Theorem 6 reduces to the following:

COROLLARY 2. *Let $(X;\Gamma)$ and $(Y;\Gamma')$ be compact G-convex spaces and $f : X \times Y \to \mathbb{R} \cup \{+\infty\}$ a function such that*

(1) *for each $x \in X$, $f(x, \cdot)$ is l.s.c. and quasiconvex on Y; and*
(2) *for each $y \in Y$, $f(\cdot, y)$ is u.s.c. and quasiconcave on X.*

Then

(i) *f has a saddle point $(x_0, y_0) \in X \times Y$; and*
(ii) *we have*

$$\max_{x\in X}\min_{y\in Y} f(x,y) = \min_{y\in Y}\max_{x\in X} f(x,y).$$

Proof. Let $f_1(x, y) := -f(x, y)$ and $f_2(x, y) := f(x, y)$. Then all of the requirements of Theorem 6 are satisfied. Therefore, by Theorem 6, there exists a point $(x_0, y_0) \in X \times Y$ such that

$$f_1(x_0, y_0) = \max_{y\in Y} f_1(x_0, y) \quad \text{and} \quad f_2(x_0, y_0) = \max_{x\in X} f_2(x, y_0).$$

Therefore, we have

$$\begin{aligned} -f(x_0, y_0) &= f_1(x_0, y_0) \\ &\geq f_1(x_0, y) \\ &= -f(x_0, y) \quad \text{for all } y \in Y, \end{aligned}$$

and

$$\begin{aligned} f(x_0, y_0) &= f_2(x_0, y_0) \\ &\geq f_2(x, y_0) \\ &= f(x, y_0) \quad \text{for all } x \in X. \end{aligned}$$

Hence

$$\begin{aligned} f(x, y_0) &\leq f(x_0, y_0) \\ &\leq f(x_0, y) \quad \text{for all } (x, y) \in X \times Y. \end{aligned}$$

Therefore

$$\max_{x\in X} f(x, y_0) \leq f(x_0, y_0) \leq \min_{y\in X} f(x_0, y).$$

This implies

$$\min_{y\in X}\max_{x\in X} f(x,y) \leq f(x_0,y_0) \leq \max_{x\in X}\min_{y\in X} f(x,y).$$

On the other hand, we have trivially

$$\min_{y\in X} f(x,y) \leq \max_{x\in X} f(x,y)$$

and hence

$$\max_{x\in X}\min_{y\in X} f(x,y) \leq \min_{y\in X}\max_{x\in X} f(x,y)$$

Therefore, we have the conclusion. □

REMARK. A little better results than Corollary 2 were already obtained by the author [20, Theorems 2, 3, and 5] with different proofs.

EXAMPLES 1. von Neumann [32]: X and Y are subsets of Euclidean spaces and f is continuous in Corollary 2.

2. Sion [29]: X and Y are compact convex subsets in topological vector spaces (not necessarily Hausdorff) in Corollary 2.

References

[1] H. Ben-El-Mechaiekh, P. Deguire, et A. Granas, *Une alternative non linéaire en analyse convexe et applications*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), 257–259.

[2] R. Bielawski, *Simplicial convexity and its applications*, J. Math. Anal. Appl. **127** (1987), 155–171.

[3] S. Y. Chang, *A generalization of KKM principle and its applications*, Soochow J. Math. **15** (1989), 7–17.

[4] K. Fan, *Fixed point and minimax theorems in locally convex linear spaces*, Proc. Natl. Acad. Sci. USA **38** (1952), 121–126.

[5] ———, *Sur un théorème minimax*, C. R. Acad. Sci. Paris Sér. I Math. **259** (1964), 3925–3928.

[6] ———, *Applications of a theorem concerning sets with convex sections*, Math. Ann. **163** (1966), 189–203.

[7] I. L. Glicksberg, *A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points*, Proc. Amer. Math. Soc. **3** (1952), 170–174.

[8] A. Idzik and S. Park, *Leray–Schauder type theorems and equilibrium existence theorems*, Differential Inclusions and Optimal Control, Lect. Notes in Nonlinear Anal. **2** (1998), 191–197.

[9] S. Kakutani, *A generalization of Brouwer's fixed-point theorem*, Duke Math. J. **8** (1941), 457–459.

[10] W. A. Kirk, B. Sims, and G. X.-Z. Yuan, *The Knaster-Kuratowski and Mazurkiewicz theory in hyperconvex metric spaces and some of its applications*, Nonlinear Anal. **39** (2000), 611–627.

[11] T.-W. Ma, *On sets with convex sections*, J. Math. Anal. Appl. **27** (1969), 413–416.

[12] J. Nash, *Non-cooperative games*, Ann. of Math. **54** (1951), 286–293.

[13] H. Nikaido and K. Isoda, *Note on non-cooperative games*, Pacific J. Math. **5** (1955), 807–815.

[14] Sehie Park, *Variational inequalities and extremal principles*, J. Korean Math. Soc. **28** (1991), 45–56.

[15] S. Park, *Foundations of the KKM theory via coincidences of composites of upper semicontinuous maps*, J. Korean Math. Soc. **31** (1994), 493–519.

[16] ———, *Applications of the Idzik fixed point theorem*, Nonlinear Funct. Anal. Appl. **1** (1996), 21–56.

[17] ———, *Remarks on a social equilibrium existence theorem of G. Debreu*, Appl. Math. Lett. **11** (1998), no. 5, 51–54.

[18] ———, *Continuous selection theorems in generalized convex spaces*, Numer. Funct. Anal. Optim. **20** (1999), 567–583.

[19] ———, *Ninety years of the Brouwer fixed point theorem*, Vietnam J. Math. **27** (1999), 187–222.

[20] ———, *Minimax theorems and the Nash equilibria on generalized convex spaces*, Josai Math. Monogr. **1** (1999), 33–46.

[21] ———, *Elements of the KKM theory for generalized convex spaces*, Korean J. Comp. Appl. Math. **7** (2000), 1–28.

[22] ———, *Fixed points, intersection theorems, variational inequalities, and equilibrium theorems*, Int. J. Math. Math. Sci. **24** (2000), 73–93.

[23] ———, *Acyclic versions of the von Neumann and Nash equilibrium theorems*, J. Comput. Appl. Math. **113** (2000), 83–91.

[24] ———, *Fixed points of better admissible maps on generalized convex spaces*, J. Korean Math. Soc. **37** (2000), 885–899.

[25] ———, *New topological versions of the Fan–Browder fixed point theorem*, Nonlinear Anal., TMA (to appear).

[26] S. Park and S. K. Kim, *On generalized extremal principles*, Bull. Korean Math. Soc. **27** (1990), 49–52.

[27] S. Park and J. A. Park, *The Idzik type quasivariational inequalities and noncompact optimization problems*, Colloq. Math. **71** (1996), 287–295.

[28] S. Simons, *Two-function minimax theorems and variational inequalities for functions on compact and noncompact sets, with some comments on fixed-point theorems*, Proc. Sympos. Pure Math., Amer. Math. Soc. **45** (Part 2) (1986), 377–392.

[29] M. Sion, *On general minimax theorems*, Pacific J. Math. **8** (1958), 171–176.

[30] K.-K. Tan, J. Yu, and X.-Z. Yuan, *Existence theorems of Nash equilibria for non-cooperative N-person games*, Internat. J. Game Theory **24** (1995), 217–222.

[31] E. Tarafdar, *Fixed point theorems in H-spaces and equilibrium points of abstract economies*, J. Austral. Math. Soc. (Ser. A) **53** (1992), 252–260.

[32] J. von Neumann, *Zur Theorie der Gesellschaftsspiele*, Math. Ann. **100** (1928), 295–320.

[33] ———, *Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes*, Ergeb. Math. Kolloq. **8** (1937), 73–83.

Department of Mathematics
Seoul National University
Seoul 151–742, Korea