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A NOTE ON UNIQUENESS AND STABILITY FOR THE INVERSE CONDUCTIVITY PROBLEM WITH ONE MEASUREMENT

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ABSTRACT. We consider the inverse conductivity problem to identify the unknown conductivity k as well as the domain D. We show that, unlike the case when k is known, even a two or three dimensional ball may not be identified uniquely if the conductivity constant k is not known. We find a necessary and sufficient condition on the Cauchy data $(u|_{\partial\Omega,g})$ for the uniqueness in identification of k and D. We also discuss on failure of stability.

1. Introduction

Let Ω be a simply connected domain with Lipschitz boundary $\partial\Omega$ in $\mathbb{R}^n$ $(n = 2, 3)$. Let D be a subdomain compactly contained Ω. Let $k > 0$ $(k \neq 1)$ be a constant. We consider the inverse problem of identifying the unknown conductivity constant k as well as the unknown domain D from the relation between a current density g (Neumann data) applied to the boundary $\partial\Omega$ and the resulting voltage potential u (Dirichlet data) measured on $\partial\Omega$. For a given current density $g \in L^2(\partial\Omega)$ with $\int_{\partial\Omega} g = 0$, the voltage potential u in Ω satisfies the following Neumann problem

$$(1.1)\quad P[k, D, g] \begin{cases} \nabla \cdot ((1 + (k-1)\chi(D))\nabla u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g \in L^2_0(\partial\Omega) \quad \text{on } \partial\Omega, & \displaystyle\int_{\partial\Omega} u d\sigma = 0, \end{cases}$$

where $\chi(D)$ is the characteristic function of D and ν is the outward unit normal vector to $\partial\Omega$. Define

$$(1.2)\qquad \Lambda_{k,D}(g) = u|_{\partial\Omega} \qquad \text{on } \partial\Omega$$

where u is the solution to $P[k, D, g]$.

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When k is known, several classes of domains D within which the global uniqueness with single measurement holds have been found. Among them are classes of ploygons, polyhedra, discs, and balls [7, 4, 10, 11, 12, 14]. Note that all the domains D belonging to the above mentioned classes are simply connected and hence $\Omega \setminus \overline{D}$ are connected. In one dimension where the uniqueness fails completely [8], $\Omega \setminus \overline{D}$ is not connected. Even for $n \geq 2$, there are examples of two different domains D_1 and D_2 such that $\Lambda_{D_1}(g) = \Lambda_{D_2}(g)$, i.e., the uniqueness fails. One of them is simply connected, but the other is not. See [1]. By perturbing one of domains, it is not hard to prove that the stability does not hold even within the class of simply connected domains. These examples are given in Section 4 at the end of this paper.

The main interest of this paper lies in the uniqueness question when k is also unknown: whether $\Lambda_{k_1,D_1}(g) = \Lambda_{k_2,D_2}(g)$ implies $D_1 = D_2$ and $k_1 = k_2$. We show that, unlike the case when k is known, even a two or three dimensional ball may not be identified uniquely if the conductivity constant k is not known. In fact, we find a necessary and sufficient condition on the Cauchy data $(u|_{\partial\Omega}, g)$ for the uniqueness in identification of k and D. In particular, we show that there are infinitely many pairs (k, D) which produce the same Cauchy data on $\partial\Omega$. This result forms a sharp contrast to the previous results of uniqueness of balls when k is known [11, 12]. There it is proved that a single measurement corresponding to any nonzero Neumann data is enough for the unique identification. As a consequence of the result, we will give a sufficient condition on the Neumann data g for the unique identification of k and D. These results are given in Section 3.

This paper is organized as follows: In Section 2, we review the representation formula for the solution of the problem $P[k, D, g]$. In Section 3, we show the failure of the uniqueness in identifying k and D. In Section 4, we remark on the failure of stability.

2. Layer potential approach

Since the arguments of this paper rely on the representation formula of the solution to $P[k, D, g]$ obtained in [11], we first recall it and derive some interesting consequences of it.

Let D, D_1, and D_2 be Lipschitz domains compactly contained in Ω and u be the weak solution to the problem $P[k, D, g]$. Let $\mathcal{S}_\Omega$ and $\mathcal{D}_\Omega$ be the single and double layer integral operators on $\partial\Omega$ for the Laplacian, respectively. $\mathcal{S}_D$ also denotes the single layer operator on ∂D. Then the solution to the problem $P[k, D, g]$ can be uniquely represented as

$$u = H + \mathcal{S}_D(\varphi) \qquad \text{in } \Omega \tag{2.1}$$

where H is a harmonic function in Ω defined by

$$H(x) = -\mathcal{S}_\Omega(g)(x) + \mathcal{D}_\Omega(\Lambda_{k,D}(g))(x) \qquad x \in \Omega, \tag{2.2}$$

and the density function φ is determined by H and D:

$$\varphi = \left(\frac{k+1}{2(k-1)} I - \mathcal{K}_D^*\right)^{-1} \left(\frac{\partial H}{\partial \nu}|_{\partial D}\right) \quad \text{on } \partial D. \tag{2.3}$$

If we put $u^e = u|_{\Omega \setminus \overline{D}}$ and $u^i = u|_D$, then

$$\varphi = \frac{k-1}{k} \frac{\partial u^e}{\partial \nu} = (k-1) \frac{\partial u^i}{\partial \nu} \quad \text{in } L^2(\partial D)\text{-sense.}$$

For detailed proofs of these facts and definitions of operators, we refer to [11] and [13].

Suppose now that $k_1 = k_2 = k$ and that $\Lambda_{D_1}(g) = \Lambda_{D_2}(g)$. Let $u_j (j = 1, 2)$ be the solution of $P[k, D_j, g]$. Then it follows from the unique continuation that $\mathcal{S}_{D_1}(\frac{\partial u_1^e}{\partial \nu}) = \mathcal{S}_{D_2}(\frac{\partial u_2^e}{\partial \nu})$ in the connected component of $\mathbb{R}^n \setminus \overline{D_1 \cup D_2}$ containing $\partial\Omega$. Hence, for any simply connected Lipschitz domain Ω_0 containing $\overline{D_1 \cup D_2}$, we obtain

$$\int_{\partial\Omega_0} \phi(x) \mathcal{S}_{D_1}(\frac{\partial u_1^e}{\partial \nu})(x) d\sigma_x = \int_{\partial\Omega_0} \phi(x) \mathcal{S}_{D_2}(\frac{\partial u_2^e}{\partial \nu})(x) d\sigma_x$$

for all $\phi \in L^2(\partial\Omega_0)$ and therefore

$$\int_{\partial D_1} \frac{\partial u_1^e}{\partial \nu}(y) \mathcal{S}_{\Omega_0}\phi(y) d\sigma_y = \int_{\partial D_2} \frac{\partial u_2^e}{\partial \nu}(y) \mathcal{S}_{\Omega_0}\phi(y) d\sigma_y$$

for all $\phi \in L^2(\partial\Omega_0)$. Hence for all harmonic function h in Ω_0

$$\int_{\partial D_1} \frac{\partial u_1^e}{\partial \nu}(y) h(y) d\sigma_y = \int_{\partial D_2} \frac{\partial u_2^e}{\partial \nu}(y) h(y) d\sigma_y. \tag{2.4}$$

Using the above identity and the Runge approximation, we may obtain the uniqueness result from full measurements: $\Lambda_{D_1}(g) = \Lambda_{D_1}(g)$ for all $g \in L_0^2(\partial\Omega)$ implies $D_1 = D_2$. We will not give the detail of the proof because the uniqueness with full measurements has been proved by Isakov in the paper [8].

Now let us suppose that k is close to 1. Observe that

$$\frac{\partial u_j^e}{\partial \nu} = \frac{k}{k-1} (\lambda I - \mathcal{K}_{D_j}^*)^{-1} (\frac{\partial H}{\partial \nu}|_{\partial D_j}) \quad \text{on } \partial D_j$$

where $\lambda = \frac{k+1}{2(k-1)}$. By (2.4),

$$\int_{\partial D_1} (\lambda I - \mathcal{K}_{D_1}^*)^{-1}(\frac{\partial H}{\partial \nu}|_{D_1})(y)h(y)d\sigma_y = \int_{\partial D_2} (\lambda I - \mathcal{K}_{D_2}^*)^{-1}(\frac{\partial H}{\partial \nu}|_{D_2})(y)h(y)d\sigma_y.$$

Since

$$(\lambda I - \mathcal{K}_{D_j}^*)^{-1}(\frac{\partial H}{\partial \nu}|_{D_j}) = \frac{1}{\lambda}\frac{\partial H}{\partial \nu} + \frac{1}{\lambda^2}\mathcal{K}_{D_j}^*(I - \frac{1}{\lambda}\mathcal{K}_{D_j}^*)^{-1}(\frac{\partial H}{\partial \nu}|_{D_j}),$$

we obtain

$$\int_{D_1} \nabla H \cdot \nabla h - \int_{D_2} \nabla H \cdot \nabla h = \frac{1}{\lambda}E \tag{2.5}$$

where

$$E = \int_{\partial D_1} \mathcal{K}_{D_1}^*(I - \frac{1}{\lambda}\mathcal{K}_{D_1}^*)^{-1}(\frac{\partial H}{\partial \nu}|_{D_1}) - \int_{\partial D_2} \mathcal{K}_{D_2}^*(I - \frac{1}{\lambda}\mathcal{K}_{D_2}^*)^{-1}(\frac{\partial H}{\partial \nu}|_{D_2}).$$

It follows from the L^2-boundedness of $\mathcal{K}_{D_j}$ [6, 15] that there is a positive constant C depending only on the Lipschitz character of D_j so that

$$\int_{\partial D_j} \left|\mathcal{K}_{D_j}^*(I - \frac{1}{\lambda}\mathcal{K}_{D_j}^*)^{-1}(\frac{\partial H}{\partial \nu}|_{D_j})\right|^2 \le C \int_{\partial D_j} \left|\frac{\partial H}{\partial \nu}\right|^2.$$

Hence

$$E \le C\|\nabla H\|_{L^2(\partial D_1 \cup \partial D_2)}\|h\|_{L^2(\partial D_1 \cup \partial D_2)}.$$

Using an idea of P. Novikov as appeared in [9, Theorem 2.2.1], we have the following Lemma for a special H.

LEMMA 2.1. *Suppose that D_j is a star-shaped region with respect to the origin. If H is a non-constant linear function, then*

$$|D_1 \setminus D_2| + |D_2 \setminus D_1| \le \frac{C}{|\lambda|}. \tag{2.6}$$

In particular, if $k \to 1$, then the measure of the symmetric difference of D_1 and D_2 converges to zero.

Proof. We may assume that $H = x_j$. Then the equation (2.5) becomes

$$\int_{D_1 \setminus \overline{D_2}} D_j h - \int_{D_2 \setminus \overline{D_1}} D_j h = \frac{1}{|\lambda|}E$$

for all harmonic function in a neighborhood of $D_1 \cup D_2$. Since $x \cdot \nabla(D_j h)$ is also a harmonic function,

$$\int_{\partial(D_1 \setminus \overline{D_2})} x \cdot \nu D_j h - \int_{\partial(D_2 \setminus \overline{D_1})} x \cdot \nu D_j h = \frac{1}{|\lambda|} E. \tag{2.7}$$

Let $\Sigma_1 = \partial D_1 \setminus D_2$ and $\Sigma_2 = \partial D_2 \setminus D_1$. As in [9, Lemma 1.7.4], we can choose a sequence of harmonic function $\{h_m\}$ so that

$$\begin{aligned} &\lim_{m \to \infty} D_j h_m = 1 \qquad \text{in } L^1(\Sigma_1), \\ &\lim_{m \to \infty} D_j h_m = 0 \qquad \text{in } L^1(\Sigma_2). \end{aligned}$$

Since

$$\limsup_{m \to \infty} \int_{\partial(D_2 \setminus \overline{D_1})} x \cdot \nu D_j h_m \le 0,$$

by passing to the limit in (2.7), we have

$$\int_{\partial(D_1 \setminus \overline{D_2})} x \cdot \nu \le \frac{C}{|\lambda|},$$

or

$$n |D_1 \setminus \overline{D_2}| \le \frac{C}{|\lambda|} E.$$

In the same way, we can prove that

$$n |D_2 \setminus \overline{D_1}| \le \frac{C}{|\lambda|} E.$$

This completes the proof. □

3. Identification of k and D

For this section the conductivity constant k is also unknown to be identified and we turn to the question of uniqueness in identification of k and D.

Let D_0 be a ball in $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$. Let u_0 be the solution to the Neumann problem $P[k, D_0, g]$ with a given nonzero Neumann data g. As in the equation (2.2), let

$$H_0 = -\mathcal{S}_\Omega(g) + \mathcal{D}_\Omega(u_0|_{\partial\Omega}) \qquad \text{in } \Omega. \tag{3.1}$$

THEOREM 3.1. (i) *H_0 is homogeneous with respect to the center x_0 of D_0, i.e., $H_0(r(x - x_0)) = r^n H_0(x - x_0)$ for all $r > 0$ and for some integer $n > 0$, if and only if there are infinitely many pairs of (k, D) where D is ball contained in Ω such that $\Lambda_{k,D}(g) = \Lambda_{k_0,D_0}(g)$ on $\partial\Omega$.*

(ii) *If H_0 is not homogeneous with respect to the center x_0 of D_0 if and only if $\Lambda_{k,D}(g) = \Lambda_{k_0,D_0}(g)$ on $\partial\Omega$ (D is a ball) implies $k = k_0$ and $D = D_0$.*

REMARK. A part of what Theorem 3.1 claims is that there are Neumann data g such that corresponding harmonic functions H are homogeneous. As mentioned in Introduction, it is interesting to compare Theorem 3.1 with the uniqueness of the balls obtained in [11] and [12]. It is proved that if k is known, then a ball can be uniquely determined by a single measurement corresponding to any nonzero Neumann data g.

According to Theorem 3.1, in order to identify a ball D and k, we need to choose the Neumann data g so that the corresponding harmonic function H is not homogeneous with respect to any point $x \in \Omega$. For example we have the following corollary.

COROLLARY 3.2. *If $g \in L^2_0(\partial\Omega)$ is not continuous at a point $p \in \partial\Omega$ where $\partial\Omega$ is continuously differentiable, then $\Lambda_{k_1,D_1}(g) = \Lambda_{k_2,D_2}(g)$ implies $k_1 = k_2$ and $D_1 = D_2$.*

We prove Theorem 3.1 and Corollary 3.2 in the following sequence of lemmas.

Put $u^e = u|_{\Omega\setminus\overline{D}}$ and $u^i = u|_D$. Then the transmission conditions $\frac{\partial u^e}{\partial\nu} = k\frac{\partial u^i}{\partial\nu}$ and $u^e = u^i$ hold on $\partial\Omega$ in the L^2 sense. The following lemma gives a general solution to the equation $\nabla\cdot((1+(k-1)\chi(D))\nabla u) = 0$ in Ω when D is a 2 or 3 dimensional ball. This result is obtained in [12].

LEMMA 3.3. *Let $D = B_d(a)$ be a ball in $\Omega \subset \mathbb{R}^n$ ($n = 2,3$). Then the solution u to $P[k,D,g]$ is of the following form.*

If $n = 2$, then

$$
(3.2)\qquad \begin{cases} u^i(x) = H(x) - \lambda(H(x) - H(a)) & x \in D,\\ u^e(x) = H(x) - \lambda\displaystyle\sum_{n=1}^{\infty}\frac{d^{2n}}{|x-a|^{2n}}H^{(n)}(x,a) & x \in \Omega\setminus D, \end{cases}
$$

where H is a harmonic function in Ω and

$$
H^{(n)}(x,a) = \sum_{|\alpha|=n}\frac{D^\alpha H(a)}{\alpha!}(x-a)^\alpha, \qquad \lambda = \frac{k-1}{k+1}.
$$

If $n = 3$, then

(3.3)
$$\begin{cases} u^i(x) = H(x) - \lambda \sum_{n=1}^{\infty} \frac{n}{3n+1} H^{(n)}(x,a) & x \in D, \\ u^e(x) = H(x) - \lambda \sum_{n=1}^{\infty} \frac{n}{3n+1} \frac{d^{2n+1}}{|x-a|^{2n+1}} H^{(n)}(x,a) & x \in \Omega \setminus D, \end{cases}$$

where H, $H^{(n)}(x,a)$, and λ are as above.

Proof. It is a straightforward computation to check that u^i and u^e in (3.2) and (3.3) satisfy the transmission condition. Conversely, if u is the solution to $P[k, D, g]$, then by the representation formula (2.1) and (2.2),

$$u = H + \mathcal{S}_D(\varphi), \qquad H = -\mathcal{S}_\Omega(g) + \mathcal{D}_\Omega(\Lambda_{k,D}(g)).$$

By the uniqueness of this representation, u must be of the form (3.2) if $n = 2$ or (3.3) if $n = 3$. This completes the proof. □

THEOREM 3.4. *Let D_1 and D_2 be two balls in Ω. Let $g \in L_0^2(\partial\Omega)$ be any nonzero Neumann data. If $\Lambda_{k_1,D_1}(g) = \Lambda_{k_2,D_2}(g)$ on $\partial\Omega$, then D_1 and D_2 are concentric.*

Proof. This is proved in [12] when $k_1 = k_2$. However, the argument in [12] does not rely on the conductivity constants k_j. □

Proof of Theorem 1.1. We will only prove the 3 dimensional case. 2 dimensional case is even simpler and can be proved in the same way.

Let D be a ball in $\Omega \subset \mathbb{R}^3$ with conductivity k. Let u be the solution to $P[k, D, g]$.

Suppose now that $\Lambda_{k_0,D_0}(g) = \Lambda_{k,D}(g)$. By Lemma 3.4, D_0 and D are concentric. Assume without loss of generality that $D_0 = B_{d_0}(0)$ and $D = B_d(0)$.

Let H be the harmonic function given in (2.2), namely,

$$H = -\mathcal{S}_\Omega(g) + \mathcal{D}_\Omega(u|_{\partial\Omega}).$$

Then $H = H_0$ in Ω. Let $\{Y_n^m : m = 0, \cdots, 2n,\ n = 1, 2, \cdots\}$ be the spherical harmonics in S^2 (see [5] for spherical harmonics). If

$$H(x) = H(0) + \sum_{n=1}^{\infty} r^n \sum_{m=0}^{2n} \alpha_n^m = Y_n^m(\hat{x}), \qquad r = |x|,\ \hat{x} = \frac{x}{|x|},$$

then by Lemma 3.3

$$u^e(x) = H(0) + \sum_{n=1}^{\infty} \left[r^n - \lambda \frac{n}{3n+1} \frac{d^{2n+1}}{r^{n+1}} \right] \sum_{m=0}^{2n} \alpha_n^m Y_n^m(\hat{x}) \quad |x| > d$$

where $\lambda = \frac{k-1}{k+1}$. Therefore $u^e = u_0^e$ in $|x| > \max\{d, d_0\}$ if and only if

$$\lambda d^{2n+1} \alpha_n^m = \lambda_0 d_0^{2n+1} \alpha_n^m, \quad \text{for every } n, m \tag{3.4}$$

where $\lambda_0 = \frac{k_0 - 1}{k_0 + 1}$.

If H_0 is homogeneous with respect to 0, then there is only one n such that $\alpha_n^m \neq 0$. This means that there are infinitely many pair (k, D) which satisfies (3.4). On the other hand, if H_0 is not homogeneous with respect to 0, then there are at least two different n's, say n_1 and n_2, such that $\alpha_{n_1}^{m_1}$ and $\alpha_{n_2}^{m_2}$ are not zero for some m_1 and m_2. Again by (3.4), one can easily see that

$$\left(\frac{d_0}{d}\right)^{2n_1+1} = \frac{\lambda}{\lambda_0} = \left(\frac{d_0}{d}\right)^{2n_2+1}.$$

It is possible only when $d = d_0$ and $k = k_0$. This completes the proof. □

REMARK. Let $\Omega = B(0,1) \subset \mathbb{R}^2$. According to Lemma 3.3,

$$\begin{cases} u^i(r,\theta) = \left(1 - \dfrac{k-1}{k+1}\right) r^n \cos n\theta & r \leq d, \\ u^e(r,\theta) = r^n \cos n\theta - \dfrac{k-1}{k+1} \dfrac{d^{2n}}{r^n} \cos n\theta & d \leq r < 1 \end{cases}$$

satisfies $\nabla \cdot ((1 + (k-1)\chi(D))\nabla u) = 0$ in Ω where $D = B_d(0)$. Note that

$$\frac{\partial u^e}{\partial \nu} = \left(n + \frac{k-1}{k+1} d^{2n}\right) \cos n\theta \qquad \text{on } \partial\Omega.$$

Note also that g has the index 0 if $n = 1$, i.e., the set $\{g \geq 0\}$ is connected. So, even if the Neumann data g has index zero, $\Lambda_{k_1,D_1}(g) = \Lambda_{k_2,D_2}(g)$ as long as $\lambda_1 d_1^2 = \lambda_2 d_2^2$. This is rather surprising if we compare this with the case of convex polygons: If D_j $(j = 1, 2)$ are convex polygons and g is a Neumann data with index 0, then $\Lambda_{k_1,D_1}(g) = \Lambda_{k_2,D_2}(g)$ implies $D_1 = D_2$ and $k_1 = k_2$. This fact can be proved by the exactly same argument as in the proof of [14, 2.6 Theorem].

Proof of Corollary 3.2. Let D be a ball in $\mathbb{R}^n$ $(n = 2, 3)$ and g be a nonzero Neumann data on $\partial\Omega$. Let H be the corresponding harmonic function, namely, $H = -\mathcal{S}_\Omega(g) + \mathcal{D}_\Omega(\Lambda_{k,D}(g))$. If H is homogeneous with respect to a point in Ω, then H is harmonic in $\mathbb{R}^n$. By Lemma 3.3,

u^e is harmonic in $\mathbb{R}^n \setminus \overline{D}$. Therefore $g = \nabla u^e \cdot \nu$ must be continuous at every point of $\partial\Omega$ where $\partial\Omega$ is C^1. This completes the proof. □

4. A remark on stability

In this section we give examples of simply connected domains for which the stability fails. For simplicity, assume $k_1 = k_2 = 2$. We begin with examples of domains for which the uniqueness fails. See also [1]

EXAMPLE 4.1. For a positive integer n, let

$$D_1 = B_2(0) \setminus \overline{B_1(0)}, \qquad D_2 = B_{r_n}(0) \text{ with } r_n^{2n} = \frac{9(2^{2n}-1)}{9-2^{-2n}}.$$

Here $B_r(a)$ is the disk centered at a with radius r. Suppose that D_1 and D_2 are contained in $\Omega = B_5(0)$. (5 is of no significance.) Let the Neumann data be given by

$$g(\theta) = \cos n\theta \quad \text{on} \quad \partial\Omega.$$

We claim that

$$\Lambda_{D_1}(g) = \Lambda_{D_2}(g).$$

In fact, let

$$V_1(z) := \begin{cases} z^n & \text{if } 0 \le |z| < 1 \\ \dfrac{3}{4} z^n + \dfrac{1}{4}\dfrac{1}{z^n} & \text{if } 1 \le |z| < 2 \\ \dfrac{9-2^{-2n}}{8} z^n - 3\dfrac{2^{2n}-1}{8}\dfrac{1}{z^n} & \text{if } 2 \le |z| < 5, \end{cases}$$

$$V_2(z) := \begin{cases} \dfrac{9-2^{-2n}}{12} z^n & \text{if } |z| < r_n \\ \dfrac{9-2^{-2n}}{8} z^n - 3\dfrac{2^{2n}-1}{8}\dfrac{1}{z^n} & \text{if } r_n \le |z| < 5. \end{cases}$$

Then $V_j^e := V_j|_{\Omega\setminus D_j}$ and $V_j^i := V_j|_{D_j}$ satisfy

$$3V_j^e + \overline{V_j^e} = 4V_j^i \quad \text{on } \partial D_j \quad (j = 1, 2). \tag{4.1}$$

Thus $u_j := \frac{1}{\alpha_n}\Re V_j$ (the real part of V_j) with $\alpha_n = \frac{n}{8}[5^{n-1}(9-2^{-2n}) - 5^{-n-1}3(2^{2n}-1)]$ satisfies the transmission condition $\frac{\partial u_j^e}{\partial\nu} = 2\frac{\partial u_j^i}{\partial\nu}$ and $u_j^e = u_j^i$ on ∂D_j (see [2]) and hence satisfies

$$\nabla \cdot ((1 + \chi(D_j))\nabla u_j) = 0 \quad \text{in } \Omega.$$

Of course,

$$\frac{\partial u_1}{\partial\nu} = \frac{\partial u_2}{\partial\nu} = g \quad \text{on} \quad \partial\Omega.$$

For a general simply connected domain Ω, we let G be the conformal mapping from $B_5(0)$ onto Ω. Then it is easy to see that $\tilde{D}_1 = G(D_1)$ and $\tilde{D}_2 = G(D_2)$ can produce the same Cauchy data on $\partial\Omega$.

REMARK. In Example 4.1, even the size of two domains D_1 and D_2 are different even though $\Lambda_{D_1}(g) = \Lambda_{D_2}(g)$.

EXAMPLE 4.2. By perturbing D_1 in Example 4.1, one can see that the stability fails even within the class of simply connected domains. For $\epsilon > 0$ let

$$D_1^\epsilon := \{re^{i\theta} |\ 1 < r < 2,\ \epsilon < |\theta| \le \pi\}.$$

Then D_1^ϵ is simply connected for each ϵ. Let u_1^ϵ be the weak solution to

$$\nabla \cdot ((1 + \chi(D_1^\epsilon))\nabla u) = 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} = g \quad \text{on } \partial\Omega.$$

Then, one can see that

$$\begin{aligned} &\int_\Omega (1 + \chi(D_1^\epsilon))|\nabla(u_1^\epsilon - u_1)|^2 dx \\ &= \int_\Omega (\chi(D_1^\epsilon) - \chi(D_1))|\nabla u_1|^2 dx + \int_\Omega (\chi(D_1) - \chi(D_1^\epsilon))\nabla u_1 \nabla u_1^\epsilon dx. \end{aligned}$$

Since $D_1^\epsilon \subset D_1$, it follows that

$$\begin{aligned} \text{(4.2)} \quad \int_\Omega |\nabla(u_1^\epsilon - u_1)|^2 dx &\le C\sqrt{\epsilon}\|\nabla u_1\|_{L^\infty(\Omega)} \left(\int_\Omega |\nabla u_1^\epsilon|^2 dx\right)^{1/2} \\ &\le C\sqrt{\epsilon}\|\nabla u_1\|_{L^\infty(\Omega)}\|g\|_{L^2(\partial\Omega)}. \end{aligned}$$

Let

$$|(u_1^\epsilon - u_1)(x_0)| = \max_{\overline{B_5(0)\setminus B_4(0)}} |(u_1^\epsilon - u_1)(x)|.$$

Since $\frac{\partial(u_1^\epsilon - u_1)}{\partial \nu} = 0$ on $\partial\Omega$, by Hopf lemma, $x_0 \in \partial B_4(0)$. By the mean value property and (4.2), we have

$$|(u_1^\epsilon - u_1)(x_0)| \le C\sqrt{\epsilon}.$$

In particular, we have

$$\Lambda_{D_2}(g) = \Lambda_{D_1}(g) = \lim_{\epsilon \to 0} \Lambda_{D_1^\epsilon}(g) \qquad \text{in } L^\infty(\partial\Omega).$$

However, the Hausdorff distance between D_2 and D_1^ϵ is larger than 1 for all ϵ. This gives the desired instability.

References

[1] G. Alessandrini, *Remark on a paper of Bellout and Friedman*, Boll. Un. Mat. Ital. A **23** (1989).

[2] G. Alessandrini, V. Isakov, and J. Powell, *Local uniqueness in the inverse problem with one measurement*, Trans. Amer. Math. Soc. **347** (1995), 3031–3041.

[3] H. Bellout, A. Friedman, and V. Isakov, *Inverse problem in potential theory*, Trans. Amer. Math. Soc. **332** (1992), 271–296.

[4] B. Barcelo, E. Fabes, and J. K. Seo, *The inverse conductivity problem with one measurement: uniqueness for convex polyhedra*, Proc. Amer. Math. Soc. **122** (1994), 183–189.

[5] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Appl. Math. Sci. 93, Springer-Verlag, Berlin.

[6] R. R. Coifman, A. McIntosh, and Y. Meyer, *L'intégrale de Cauchy definit un opérateur bournée sur L^2 pour courbes lipschitziennes*, Ann. of Math. **116** (1982), 361–387.

[7] A. Friedman and V. Isakov, *On the uniqueness in the inverse conductivity problem with one measurement*, Indiana Univ. Math. J. **38** (1989), 553–580.

[8] V. Isakov, *On uniqueness of recovery of discontinuous conductivity coefficient*, Comm. Pure Appl. Math. **41** (1988), 865–877.

[9] ———, *Inverse Source Problems*, Math. Surveys and Monographs, Amer. Math. Soc. **34**.

[10] V. Isakov and J. Powell, *On the inverse conductivity problem with one measurement*, Inverse Problems **6** (1990), 311–318.

[11] H. Kang and J. K. Seo, *Layer potential technique for the inverse conductivity problem*, Inverse Problems **12** (1996), 267–278.

[12] ———, *Inverse conductivity problem with one measurement: uniqueness for balls in $\mathbb{R}^3$*, SIAM J. Appl. Math. **59** (1999), no. 3, 1533–1539.

[13] H. Kang, J. K. Seo, and D. Sheen, *Numerical identification of discontinuous conductivity coefficients*, Inverse Problems **13** (1997), 113–123.

[14] J. K. Seo, *A uniqueness result on inverse conductivity problem with two measurements*, J. Fourier Anal. Appl. **2** (1996), no. 3, 227–235.

[15] G. C. Verchota, *Layer potentials and boundary value problems for Laplace's equation in Lipschitz domains*, J. Funct. Anal. **59** (1984), 572–611.

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