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# REGULAR MAPS - COMBINATORIAL OBJECTS RELATING DIFFERENT FIELDS OF MATHEMATICS

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ABSTRACT. Regular maps and hypermaps are cellular decompositions of closed surfaces exhibiting the highest possible number of symmetries. The five Platonic solids present the most familiar examples of regular maps. The great dodecahedron, a 5-valent pentagonal regular map on the surface of genus 5 discovered by Kepler, is probably the first known non-spherical regular map. Modern history of regular maps goes back at least to Klein (1878) who described in [59] a regular map of type $(3, 7)$ on the orientable surface of genus 3. In its early times, the study of regular maps was closely connected with group theory as one can see in Burnside's famous monograph [19], and more recently in Coxeter's and Moser's book [25] (Chapter 8). The present-time interest in regular maps extends to their connection to Dyck's triangle groups, Riemann surfaces, algebraic curves, Galois groups and other areas. Many of these links are nicely surveyed in the recent papers of Jones [55] and Jones and Singerman [54].

The presented survey paper is based on the talk given by the author at the conference "Mathematics in the New Millenium" held in Seoul, October 2000. The idea was, on one hand side, to show the relationship of (regular) maps and hypermaps to the above mentioned fields of mathematics. On the other hand, we wanted to stress some ideas and results that are important for understanding of the nature of these interesting mathematical objects.

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## 1. Topological and combinatorial maps, permutation representation of maps

### Topological maps

A *map* on a surface is a cellular decomposition of a closed surface into 0-cells called *vertices*, 1-cells called *edges* and 2-cells called *faces*. The vertices and edges of a map form its *underlying graph*. A map is said to be *orientable* if the supporting surface is orientable, and is *oriented* if one of two possible orientations of the surface has been specified; otherwise, a map is *unoriented*.

Typically, a map on a surface is constructed by embedding a connected graph in the surface. Graphs considered in this paper are finite, non-trivial and connected unless the opposite follows from the immediate context. Edges of our graphs may belong to one of three kinds: links, loops and semiedges. Multiple adjacencies are allowed. A link is incident with two vertices while a loop or a semiedge is incident with a single vertex. A link or a loop gives rise to two oppositely directed darts that are reverse to each other. A semiedge incident with a vertex $u$ gives rise to a single dart initiating at $u$ that is reverse to itself. From the topological point of view, a semiedge is identical with a pendant edge except that its pendant end-point is not listed as a vertex. Summing up, a graph seen as a topological space is just a finite 1-dimensional cell complex.

The following result relating numerical invariants of maps with the Euler characteristic of the supporting surface is well-known.

THEOREM 1 (Euler formula). *Let $\mathcal{M}$ be a map on a closed surface $S$ of genus $g$ with $v$ vertices, $e$ edges, $s$ semiedges and $f$ faces. Then $v - e + s + f = 2 - 2g$, if $S$ is orientable; and $v - e + f + s = 2 - g$, if $S$ is nonorientable.*

### Graphs

For the sake of technical convenience we shall usually replace topological graphs and maps by their combinatorial counterparts. Formally, a (combinatorial) *graph* is a quadruple $\mathcal{G} = (D, V; I, L)$ where $D = D(\mathcal{G})$ and $V = V(\mathcal{G})$ are disjoint non-empty finite sets, $I\colon D \to V$ is a surjective mapping, and $L = L_{\mathcal{G}}$ is an involutory permutation on $D$. The elements of $D$ and $V$ are *darts* and *vertices*, respectively, $I$ is the incidence function assigning to every dart its *initial vertex*, and $L$ is the *dart-reversing involution*; the orbits of the group $\langle L\rangle$ on $D$ are *edges* of $\mathcal{G}$. If a dart $x$ is a fixed point of $L$, that is, $L(x) = x$, then the corresponding edge is a *semiedge*. If $IL(x) = I(x)$ but $L(x) \neq x$, then the edge is a *loop*. The remaining edges are *links*. Let us remark that the same type of graphs are considered in Jones and Singerman [51] (also [77, 71]).

The usual graph-theoretical concepts such as cycles, connectedness, etc., easily translate to our present formalism.

## Maps and oriented maps

As far as maps on surfaces are concerned, there are two essentially different approaches to their combinatorial description. The first approach, based on a rotation-involution pair acting on darts, involves the orientation of the supporting surface and so is suitable only for maps on orientable surfaces [38, 51]. The corresponding combinatorial structure is called a combinatorial (or, sometimes, algebraic) oriented map. The other approach, using three involutions acting on mutually incident (vertex, edge, face)-triples called flags, is orientation insensitive and thus allows us to represent maps on non-orientable surfaces as well [52]. The resulting combinatorial structure will be called a combinatorial unoriented map.

Accordingly, we shall usually employ the same notation for a topological map and for the corresponding combinatorial structure on it.

We start with necessary definitions concerning oriented maps. By a (combinatorial) *oriented map* we henceforth mean a triple $(D; R, L)$ where $D = D(\mathcal{M})$ is a non-empty finite set of *darts*, and $R$ and $L$ are two permutations of $D$ such that $L$ is an involution and the group $\mathrm{Mon}(\mathcal{M}) = \langle R, L\rangle$ acts transitively on $D$. The group $\mathrm{Mon}(\mathcal{M})$ is called the oriented *monodromy group* of $\mathcal{M}$. The permutation $R$ is called the *rotation* of $\mathcal{M}$. The orbits of the group $\langle R\rangle$ are the *vertices* of $\mathcal{M}$, and elements of an orbit $v$ of $\langle R\rangle$ are the darts *radiating* (or *emanating*) from $v$, that is, $v$ is their initial vertex. The cycle of $R$ permuting the darts emanating from $v$ is the *local rotation* $R_v$ at $v$. The permutation $L$ is the *dart-reversing involution* of $\mathcal{M}$, and the orbits of $\langle L\rangle$ are the *edges* of $\mathcal{M}$. The orbits of $\langle RL\rangle$ define the face-boundaries of $\mathcal{M}$. The incidence between vertices, edges and faces is given by nontrivial set intersection. The vertices, darts and the incidence function define the *underlying graph* $\mathcal{M}$, which is always connected due to the transitive action of the monodromy group.

An oriented map can be equivalently described as a pair $(\mathcal{G}; R)$ where $\mathcal{G} = (D, V; I, L)$ is a connected graph and $R$ is a permutation of the dart-set of $\mathcal{G}$ cyclically permuting darts with the same initial vertex, that is, $IR(x) = I(x)$ for every dart $x$ of $\mathcal{G}$.

Combinatorial unoriented maps are build from three involutions acting on a non-empty finite set $F$ of *flags* [52]. A (combinatorial) *unoriented map* is a quadruple $(F;\ \lambda, \rho, \tau)$ where $\lambda$, $\rho$ and $\tau$ are fixed-point free involutory permutations of $F = F(\mathcal{M})$ called the *longitudinal*, the *rotary* and the *transversal involution*, respectively, which satisfy the following conditions:

(i) $\lambda\tau = \tau\lambda$; and

(ii) the group $\langle \lambda, \rho, \tau \rangle$ acts transitively on $F$.

This group is the unoriented *monodromy group* $\mathrm{Mon}(\mathcal{M})$ of $\mathcal{M}$.

We define the *vertices* of $\mathcal{M}$ to be the orbits of the subgroup $\langle \rho, \tau \rangle$, the *edges* of $\mathcal{M}$ to be the orbits of $\langle \lambda, \tau \rangle$, and the *face-boundaries* to be the orbits of $\langle \rho, \lambda \rangle$ under the action on $F$, the incidence being given by nontrivial set intersection. Note that each orbit $z$ of $\langle \lambda, \tau \rangle$ has cardinality 2 or 4 according to whether $z$ is a semiedge or not.

Clearly, the even-word subgroup $\langle \rho\tau, \tau\lambda \rangle$ of $\mathrm{Mon}(\mathcal{M})$ has always index at most two. If the index is two, then $\mathcal{M}$ is said to be *orientable*.

With every oriented map $(D;\ R, L)$ we associate the *corresponding unoriented map* $\mathcal{M}^\natural = (F^\natural;\ \lambda^\natural, \rho^\natural, \tau^\natural)$ by setting $F^\natural = D \times \{1, -1\}$ and defining for a flag $(x, j) \in D \times \{1, -1\}$:

$$\lambda^\natural(x, j) = (L(x), -j), \quad \rho^\natural(x, j) = (R^j(x), -j), \quad \text{and} \quad \tau^\natural(x, j) = (x, -j).$$

Conversely, from an orientable unoriented map $\mathcal{M} = (F;\ \lambda, \rho, \tau)$ we can construct a pair of oriented maps $\mathcal{M}' = (D;\ R, L)$ and $\mathcal{M}'' = (D;\ R^{-1}, L)$ that are the *mirror image* of each other. We take $D$ to be the set $F/\tau$ of orbits of $\tau$ on $F$. Let us denote by $F^+ \subset F$ one of the two orbits induced by the action of the even-word subgroup of $\mathrm{Mon}(\mathcal{M})$. For a dart $\{z, \tau(z)\} = [z]$, where $z \in F^+$, we set $R([z]) = [\rho\tau(z)]$ and $L([z]) = [\lambda\tau(z)]$. Instead of $R$ we could have taken the rotation $R'([z]) = [\tau\rho(z)]$, but since $R' = R^{-1}$ we get nothing but the mirror image–as expected.

### Homomorphisms of maps

Let $\mathcal{M}_1 = (D_1;\ R_1, L_1)$ and $\mathcal{M}_2 = (D_2;\ R_2, L_2)$ be two oriented maps. A *homomorphism* $\varphi\colon \mathcal{M}_1 \to \mathcal{M}_2$ of oriented maps is a mapping

$\varphi\colon D_1 \to D_2$ such that

$$\varphi R_1 = R_2\varphi \quad \text{and} \quad \varphi L_1 = L_2\varphi.$$

Analogously, a *homomorphism* $\varphi\colon \mathcal{M}_1 \to \mathcal{M}_2$ of unoriented maps $\mathcal{M}_1 = (F_1;\, \lambda_1, \rho_1, \tau_1)$ and $\mathcal{M}_2 = (F_2;\, \lambda_2, \rho_2, \tau_2)$ is a mapping $\varphi\colon F_1 \to F_2$ such that

$$\varphi\lambda_1 = \lambda_2\varphi, \quad \varphi\rho_1 = \rho_2\varphi \quad \text{and} \quad \varphi\tau_1 = \tau_2\varphi.$$

The properties of homomorphisms of both varieties of maps are similar except that homomorphisms of unoriented maps ignore orientation. Every map homomorphism induces an epimorphism of the corresponding monodromy groups. Indeed, it is not difficult to see that if $\psi\colon (F_1;\, \lambda_1, \rho_1, \tau_1) \to (F_2;\, \lambda_2, \rho_2, \tau_2)$ is a map homomorphism then the assignment $\lambda_1 \mapsto \lambda_2$, $\rho_1 \mapsto \rho_2$, $\tau_1 \mapsto \tau_2$ extends to an epimorphism $\psi^*$ called the *induced epimorphism* of the corresponding monodromy groups. Furthermore, transitive actions of the monodromy groups ensure that every map homomorphism is surjective and that it also induces an epimorphism of the underlying graphs. Topologically speaking, a map homomorphism is a graph preserving branched covering projection of the supporting surfaces with branch points possibly at vertices, face centres or free ends of semiedges. Therefore we can say that a map $\tilde{\mathcal{M}}$ *covers* $\mathcal{M}$ if there is a homomorphism $\tilde{\mathcal{M}} \to \mathcal{M}$. A map homomorphism is *smooth* if it preserves the valency of vertices, the length of faces and does not send a link or a loop onto a semiedge.

With map homomorphisms we use also isomorphisms and automorphisms. The automorphism group $\mathrm{Aut}(\mathcal{M})$ of an oriented map $\mathcal{M} = (D;\, R, L)$ consists of all permutations in the full symmetry group $S(D)$ of $D$ which commute with both $R$ and $L$. Similarly, the automorphism group $\mathrm{Aut}(\mathcal{M})$ of an unoriented map $\mathcal{M} = (F;\, \lambda, \rho, \tau)$ is formed by all permutations in the symmetry group $\mathrm{Sym}(F)$ which commute with each of $\lambda$, $\rho$ and $\tau$. Hence, in both cases the automorphism group is nothing but the centralizer of the monodromy group in the full symmetry group of the supporting set of the map (cf. [51, Proposition 3.3(i)]).

Since the action of the monodromy group $\mathrm{Mon}(\mathcal{M})$ is transitive, $|\mathrm{Mon}(\mathcal{M})| \geq |D(\mathcal{M})|$ for every oriented map $\mathcal{M}$, and $|\mathrm{Mon}(\mathcal{M})| \geq |F(\mathcal{M})|$ for every unoriented map $\mathcal{M}$. If the equality is attained, then the monodromy group acts regularly on the supporting set, and therefore the map is called *orientably-regular* or *regular*, respectively. As it will become to be clear in Section 3 the automorphism group of an orientably regular map $\mathcal{M}$ acts regularly on darts of $\mathcal{M}$, and similarly $\mathrm{Aut}(\mathcal{M})$ of a regular map $\mathcal{M}$ acts regularly on flags of $\mathcal{M}$. Our use of the term *regular map* thus agrees with that of Gardiner et al. [30] and Wilson [106], but is not yet standard. For instance, Jones and Thornton [52] uses the term "reflexible", and White [101] calls such maps "reflexible symmetrical". On the other hand, our orientably regular maps are called "regular" in Coxeter and Moser [25], "symmetrical" in [11] and [101], and "rotary" in Wilson [106].

For each homomorphism $\varphi\colon \mathcal{M}_1 \to \mathcal{M}_2$ of oriented maps there is the corresponding homomorphism $\varphi^\natural\colon \mathcal{M}_1^\natural \to \mathcal{M}_2^\natural$ defined by $\varphi^\natural(x,i) = (\varphi(x), i)$. If $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$, that is, $\varphi$ is an automorphism, then this definition and the assignment $\varphi \mapsto \varphi^\natural$ yield the isomorphic embedding of $\mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(\mathcal{M}^\natural)$. This allows us to treat $\mathrm{Aut}(\mathcal{M})$ as a subgroup of $\mathrm{Aut}^\natural(\mathcal{M}^\natural)$ and, consequently, speak that every orientable regular map is orientably-regular (but not necessarily vice versa). It is easy to see that the index $|\mathrm{Aut}^\natural(\mathcal{M}^\natural) : \mathrm{Aut}(\mathcal{M})|$ is at most two. If it is two, then the map $\mathcal{M}$ is said to be *reflexible*, otherwise it is *chiral*. In the former case, there is an isomorphism $\psi$ of the map $\mathcal{M} = (D;\, R, L)$ with its mirror image $(D;\, R^{-1}, L)$ called a *reflection* of $\mathcal{M}$. Clearly, $\psi^\natural$ is an automorphism that extends $\mathrm{Aut}(\mathcal{M})$ to $\mathrm{Aut}^\natural(\mathcal{M}^\natural)$. Topologically speaking, oriented map automorphisms preserve the chosen orientation of the supporting surface whereas reflections reverse it.

Transitivity of the action of the automorphism group of a regular (orientably regular) map forces all the vertices to have the same valency and all the faces to have same size (covalency). We say that a map $\mathcal{M}$ has a *type* $(p,q)$ if the covalency of every face is $p$ and the valency of every vertex of $\mathcal{M}$ is $q$ for some integers $p$, $q$. Generally, we can define the *type* of a map to be the couple $(p,q)$ of integers, where $p$ $(q)$ is the least common multiple of covalencies (valencies) of faces (vertices) of $\mathcal{M}$.

## 2. Generalization to hypermaps, Walsh map of a hypermap

A *topological hypermap* $\mathcal{H}$ is a cellular embedding of a connected trivalent graph $X$ into a closed surface $S$ such that the cells are 3-colored (say by black, grey and white colours) with adjacent cells having different colours. Numbering the colours 0, 1 and 2, and labeling the edges of $X$ with the missing adjacent cell number, we can define 3 fixed points free involutory permutations $r_i$, $i = 0,1,2$, on the set $F$ of vertices of $X$;

each $r_i$ switches the pairs of vertices connected by $i$-edges (edges labeled $i$). The elements of $F$ are called *flags* and the group $G$ generated by $r_0$, $r_1$ and $r_2$ will be called the *monodromy group*[1] $\mathrm{Mon}(\mathcal{H})$ of the hypermap $\mathcal{H}$. The cells of $\mathcal{H}$ colored 0, 1 and 2 are called the *hypervertices, hyperedges* and *hyperfaces*, respectively. Since the graph $X$ is connected, the monodromy group acts transitively on $F$ and orbits of $\langle r_0, r_1\rangle$, $\langle r_1, r_2\rangle$ or $\langle r_0, r_2\rangle$ on $F$ determine hyperfaces, hypervertices and hyperedges, respectively. The order of the element $k = \mathrm{ord}(r_0 r_1)$, $m = \mathrm{ord}(r_1 r_2)$ and $n = \mathrm{ord}(r_2 r_0)$ is the *valency* of a hyperface, hypervertex and hyperedge, respectively. The triple $(k, m, n)$ is called *type* of the hypermap.

Maps correspond to hypermaps satisfying condition $(r_0 r_2)^2 = 1$, or in other words, maps are hypermaps of type $(p, q, 2)$ or of type $(p, p, 1)$. Thus we can view the category of Maps as a subcategory of the category *of Hypermaps* which is formed by 4-tuples $(F;\, r_0, r_1, r_2)$, where $r_i$ $(i = 0, 1, 2)$ are fixed points free involutory permutations generating the monodromy group $\mathrm{Mon}(\mathcal{H})$ acting transitively on $F$. Similarly, the category of Oriented Hypermaps arises by relaxing the condition $L^2 = 1$ in the definition of an oriented map. More precisely, an *oriented hypermap* is a 3-tuple $(D;\, R, L)$, where $R$ and $L$ are permutations acting on $D$ such that the oriented monodromy group is transitive on $D$. The notions defined in the previous section extend from maps to hypermaps in an obvious way. For more information on hypermaps the reader is referred to [23].

The modified Euler formula for hypermaps reads as follows.

THEOREM 2 (Euler formula for hypermaps). *Let $\mathcal{H} = (F;\, r_0, r_1, r_2)$ be a hypermap on a closed surface $S$ of genus $g$ having $v$ hypervertices, $e$ hyperedges and $f$ hyperfaces. Then $v + e + f - |F|/2 = 2 - 2g$, if $S$ is orientable, and $v + e + f - |F|/2 = 2 - g$, if $S$ is nonorientable.*

**Walsh representation**

An important and convenient way to visualize hypermaps is by bipartite maps introduced by Walsh in [99]. Topologically, a map can be seen as a cellular embedding of a graph in a closed surface and a hypermap as a cellular embedding of a hypergraph in a closed surface. Since hypergraphs are in a sense bipartite graphs (with one monochromatic set of vertices representing the hypervertices and the other monochromatic set of vertices representing the hyperedges) a hypermap can be viewed

[1] This group has been called the monodromy group of $\mathcal{H}$ [55, 77], the connection group of $\mathcal{H}$ [106] and the $\Omega$-group of $\mathcal{H}$ [15].

as a bipartite map. In fact, given any topological hypermap $\mathcal{H}$ we can construct a topological bipartite map $W(\mathcal{H})$, called the Walsh bipartite map associated to $\mathcal{H}$ by taking first the dual of the underlying 3-valent map and then deleting the vertices (together with the edges attached to them) lying inside the hyperfaces of $\mathcal{H}$. The resulting map is bipartite with one monochromatic set of vertices lying on the faces colored black, representing the hypervertices of $\mathcal{H}$, and the other monochromatic set lying on the faces colored grey, representing the hyperedges. In Figure 1 the Walsh map of the Fano plane hypermap embedding in torus is drawn.

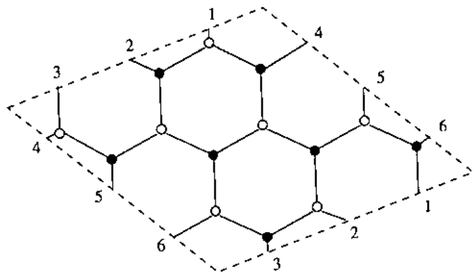


FIGURE 1. Walsh map of the Fano plane embedding in torus

This construction can be reversed: given any topological bipartite map $\mathcal{B}$, where the vertices are bipartitioned in black and grey, we construct an associated topological hypermap $W^{-1}(\mathcal{B})$=$\mathrm{Tr}(\mathcal{B}^*)$ by truncating the dual map $\mathcal{B}^*$; the faces of the resulting 3-valent map $\mathrm{Tr}(\mathcal{B}^*)$ contains the vertices and the face-centres of the original map and are henceforth 3-colorable black, grey and white, with all these colours meeting at each vertex of $\mathrm{Tr}(M^*)$. If $\mathcal{B} = W(\mathcal{H})$ is the Walsh bipartite map of an oriented hypermap $\mathcal{H} = (D; R, L)$ then $R$ and $L$ are the respective rotations on the two bipartition sets of the dart set of $\mathcal{B}$, so the rotation of $\mathcal{B}$ is $RL = LR$.

## 3. Maps, hypermaps and groups

### Schreier representations

In the previous Section we have seen that maps and hypermaps can be represented by means of two or three permutations satisfying some conditions. The aim of this Section is to show that one can study hypermaps as purely group theoretical objects. The idea emerges from the fact that every transitive permutation group is equivalent to a group acting on cosets by translation. Following [96, 97], we call these representations Schreier representations.

Schreier representations of oriented maps appear implicitly in Jones and Singerman [51]. Vince [96] developed a theory of Schreier representations of (hyper)maps on closed surfaces described by three involutions. Here we introduce Schreier representations of oriented hypermaps.

Let $G$ be a finite group generated by two elements $r$ and $\ell$. In other words, $G$ is a finite quotient of some triangle group $T^+(k,m,n) = \langle r,\ell;\, \ell^n = r^m = (r\ell)^k = 1\rangle$, $k$, $m$ and $n$ being some positive integers. Further let $S$ be a subgroup of $G$. Using the action of $G$ on the set $C = G/S$ of left cosets of $S$ in $G$ by the left translation, we construct a hypermap $A(G/S;\, r,\ell)$ whose monodromy group is a homomorphic image of $G$ and the local monodromy group is a homomorphic image of $S$. We take the cosets as darts of the hypermap and define the rotation $R$ and the dart reversing involution $L$ by setting

$$R(hS) = rhS,$$
$$L(hS) = \ell hS,$$

respectively, $hS$ being an arbitrary element of $C$. For the resulting hypermap $(C;\, R,L) = A(G/S;\, r,\ell)$ we easily check that the assignment $r \mapsto R$, $\ell \mapsto L$ extends to a homomorphism $T^+(k,m,n) \to \mathrm{Mon}(A(G/S;\, r,\ell))$.

A *Schreier representation* of a hypermap $\mathcal{H}$ is an isomorphism $\mathcal{H} \to A(G/S;\, r,\ell)$ for an appropriate group $G = \langle r,\ell\rangle$ and a subgroup $S \leq G$, or simply the hypermap $A(G/S;\, r,\ell)$ itself. Given any hypermap $\mathcal{H} = (D;\, R,L)$, it is not difficult to find a Schreier representation for $\mathcal{H}$. Indeed, we first fix any dart $a$ of $\mathcal{H}$ and set $G = \mathrm{Mon}(\mathcal{H}) = \langle R,L\rangle$ and $S = \mathrm{Mon}(\mathcal{H},a)$, to be the stabilizer of $a$. Then, for an arbitrary dart $x$ we take any element $h \in \mathcal{H}$ with $h(a) = x$ and label $x$ by the coset $hS \in C$, thereby obtaining a labeling $\alpha(x) = hS$. Observe that $\alpha$ is well-defined since for any two elements $h$ and $h'$ of $\mathrm{Mon}(\mathcal{M})$ with $h(a) = x = h'(a)$ we have $hS = h'S$. In fact, $\alpha$ is a bijection of $D(\mathcal{H})$ onto $C$. Clearly, $\alpha(Rx) = R\alpha(x)$ and $\alpha(Lx) = L\alpha(x)$ which means that $\alpha\colon \mathcal{H} \to A(G/S;\, R,L)$ is the required isomorphism.

If we start from a given hypermap $\mathcal{H}$, the Schreier representation we have just described is in some sense best possible because the monodromy group $\mathrm{Mon}(\mathcal{H})$ is not merely a homomorphic image of $G$ but is actually isomorphic to it. In this case we say that the Schreier representation is *effective*. In general, a Schreier representation $A(G/S;\, r,\ell)$

is effective if and only if $G$ acts faithfully on $C$, i.e., the translation by every non-identity element of $G$ is a non-identity permutation of $C$. Elementary theory of group actions or straightforward computations yield that the latter occurs precisely when the subgroup $\bigcap_{h\in G} hSh^{-1}$, the *core* of $S$ in $G$, is trivial (cf. Rotman [83]).

For an arbitrary Schreier representation $A(G/S;\, r,\ell)$ of a hypermap $\mathcal{H}$ we have $\mathrm{Aut}(\mathcal{H}) \cong N_G(S)/S$, where $N_G(S)$ is the normalizer of $S$ in $G$ (Proposition 3.7 in [24]). In particular, if $\mathcal{H}$ is orientably regular we can take $G = \mathrm{Mon}(\mathcal{H})$ and $S = 1$. Then $\mathrm{Aut}(A(G/1;\, r,\ell)) \cong G \cong \mathrm{Mon}(A(G/1;\, r,\ell))$, implying that $\mathrm{Aut}(\mathcal{H}) \cong \mathrm{Mon}(\mathcal{H})$. Let us remark that the above isomorphism assigns the left translation by an element $h \in G$ (representing a monodromy of $\mathcal{H}$) to the right translation $\xi_h$ (representing an automorphism of $\mathcal{H}$). Summing up we get the following theorem.

THEOREM 3. *Let $\mathcal{H} = (D; R, L)$ be an oriented hypermap. Then* $|\,\mathrm{Aut}(\mathcal{H})| \leq |D| \leq |\,\mathrm{Mon}(\mathcal{H})|$ *and the following conditions are equivalent:*

(1) *$\mathcal{H}$ is orientably regular,*
(2) $\mathrm{Mon}(\mathcal{H}) \cong \mathrm{Aut}(\mathcal{H})$,
(3) *the action of* $\mathrm{Aut}(\mathcal{H})$ *on $D$ is regular.*

In order to get a similar characterization of regular (unoriented) hypermaps it suffices to replace, in the above statement, an oriented hypermap by a hypermap and darts by flags.

Schreier representations provide a convenient tool to deal not only with automorphisms but also with homomorphisms between hypermaps. If

$$G = \langle r, \ell;\ \ell^n = r^m = (r\ell)^k = 1, \dots \rangle$$

is a finite quotient of the triangle group $T^+(k,m,n)$ and $S \leq S' \leq G$ are two subgroups then the natural projection $\pi\colon G/S \to G/S',\quad hS \mapsto hS'$ $(h \in G)$, is a homomorphism $A(G/S;\, r,\ell) \to A(G/S';\, r,\ell)$. In fact, every hypermap homomorphism $\varphi\colon \mathcal{H}_1 \to \mathcal{H}_2$, where $\mathcal{H}_i = (D_i;\, R_i, L_i)$, is in the usual sense equivalent to an appropriate natural projection.

**Generic hypermap**

One consequence of these considerations is that every oriented hypermap is a quotient of a (finite) oriented regular hypermap. In fact, for every oriented hypermap $\mathcal{H} = (D;\, R, L)$ there exists a regular hypermap

$\mathcal{H}^{\#}$ and a homomorphism $\pi\colon \mathcal{H}^{\#} \to \mathcal{H}$ with the following universal property: for every regular hypermap $\tilde{\mathcal{H}}$ and a homomorphism $\varphi\colon \tilde{\mathcal{H}} \to \mathcal{H}$ there is a homomorphism $\varphi'\colon \tilde{\mathcal{H}} \to \mathcal{H}^{\#}$ such that $\varphi = \pi\varphi'$. In terms of Schreier representations, the homomorphism $\pi$ is equivalent to the natural projection $A(G/1;\, R, L) \to A(G/S; R, L) \cong \mathcal{H}$ where $G = \mathrm{Mon}(\mathcal{H})$ and $S = \mathrm{Mon}(\mathcal{H}, a)$, is the stabilizer of some dart $a \in D(\mathcal{H})$. We shall call the hypermap $\mathcal{H}^{\#}$ the *generic regular hypermap* over $\mathcal{H}$ and $\pi\colon \mathcal{H}^{\#} \to \mathcal{H}$ the *generic homomorphism*. It is obvious that the induced homomorphism $\pi^*\colon \mathrm{Mon}(\mathcal{H}^{\#}) \to \mathrm{Mon}(\mathcal{H})$ is an isomorphism and that $\mathcal{H}^{\#}$ and $\mathcal{H}$ have the same type.

**Construction of generic hypermap**

To construct the generic hypermap $\mathcal{H}^{\#} = (D^{\#};\, R^{\#}, L^{\#})$ for an oriented hypermap $\mathcal{H} = (D;\, R, L)$ it is sufficient to set $D^{\#} = \mathrm{Mon}(\mathcal{H})$, $R^{\#}(x) = Rx$, and $L^{\#}(x) = Lx$ for any $x \in D^{\#}$. Observe that the automorphisms of $\mathcal{H}^{\#}$ are just the right translations of $D^{\#} = \mathrm{Mon}(\mathcal{H})$ by the elements of $\mathrm{Mon}(\mathcal{H})$, and so $\mathcal{H}^{\#}$ is indeed an orientably-regular hypermap. Similarly, if $\mathcal{H} = (F;\, \lambda, \rho, \tau)$ is an unoriented hypermap, then the generic regular hypermap $\mathcal{H}^{+} = (F^{+};\, \lambda^{+}, \rho^{+}, \tau^{+})$ over $\mathcal{H}$ can be constructed by setting $F^{+} = \mathrm{Mon}(\mathcal{H})$, $\lambda^{+}(x) = \lambda x$, $\rho^{+}(x) = \rho x$ and $\tau^{+}(x) = \tau x$, for any $x \in F^{+}$. Again, the hypermap automorphisms are given by the right translations of $F^{+}$ by the elements of $\mathrm{Mon}(\mathcal{H}) = F^{+}$.

It is obvious that if $\mathcal{H}$ is orientable, then so is $\mathcal{H}^{+}$. Moreover, the hypermap $\mathcal{H}^{+}$, as a topological hypermap, smoothly covers $(\mathcal{H}^{\#})^{\natural}$.

In the following sections we shall see that maps or Walsh bipartite hypermap representations combined with the generic hypermap construction provide a convenient tool for construction regular hypermaps satisfying certain constrains (for instance [50, 78]). Let us note that there is an interesting relationship between hypermaps and algebraic curves (Section 5). Via this relationship, actions of Galois groups of algebraic number fields on maps on surfaces are investigated ([39, 55, 54, 85]). In this context the maps are (following Grothendieck) called *dessigns d'enfants*.

EXAMPLE. Figure 2 shows spherical maps $\mathcal{M}_1, \ldots \mathcal{M}_5$ which generic maps are the five Platonic solids. The respective (oriented) monodromy groups are $\mathbb{A}_4$, $\mathbb{S}_4$, $\mathbb{S}_4$, $\mathbb{A}_5$ and $\mathbb{A}_5$. "Dessigns d'enfants" $M_6$ and $M_7$ represent regular maps on surfaces with higher genera. The associated monodromy groups are the projective linear group $PSL(2, 7)$ and Mathieu group $\mathrm{M}_{12}$, respectively.

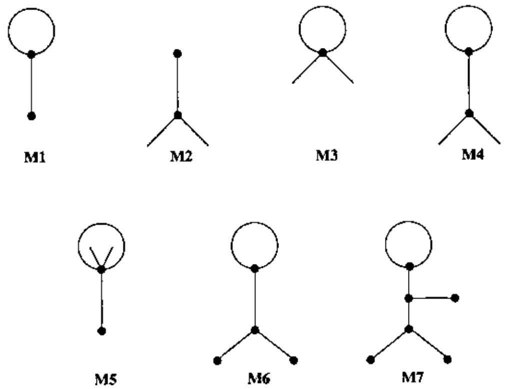


FIGURE 2.

The generic map $M_6^{\#}$ is known as the dual of the Klein's triangulation of the surface of genus 3 and this map is the smallest Hurwitz map (Section 4).

**Maps and hypermaps from triangle groups**

The above theory of Schreier representations apply without any problem to infinite hypermaps as well. It follows that oriented maps and hypermaps of given type $(k, m, n)$ can be described as quotients of the *universal oriented hypermap* of type $(k, m, n)$ which (oriented) monodromy group is $T^+(k, m, n)$. This is the even-word subgroup of the triangle group

$$T(k, m, n) = \langle r_0, r_1, r_2;\ r_0^2 = r_1^2 = r_2^2 = (r_0 r_1)^k = (r_1 r_2)^m = (r_2 r_0)^n = 1\rangle,$$

which is the monodromy group of the *universal hypermap* $A$ $(T(k,\ m,\ n)$ $;r_0,\ r_1,\ r_2)$ for the category of (unoriented) hypermaps of type $(k, m, n)$. Note that the universal maps of type $(k, m)$ with the monodromy group $T(k, m, 2)$ are the well known tessellations of the sphere, plane or hyperbolic plane by $k$-gons ($m$ of them meeting at each vertex) provided the expression $\frac{1}{k} + \frac{1}{m}$ is greater, equal or less than $\frac{1}{2}$, respectively.

**Hypermap subgroups**

We can go even one step further. Let us denote by

$$G = T(\infty, \infty, \infty) = \langle r_0, r_1, r_2;\ r_0^2 = r_1^2 = r_2^2 = 1\rangle,$$

the free product of three two-element groups. Since the monodromy group of any hypermap $\mathcal{H}$ is a finite quotient of $G$ we can identify every hypermap with the algebraic hypermap $A(G/S; r_0, r_1, r_2)$ for some

$S \leq G$ of finite index. The subgroup $S$ is called the *hypermap subgroup*. Consequently, one can study hypermaps via the subgroups of $G$ of finite index. The facts listed in the following statement are well-known between map- and hypermap experts ([23, 24]).

THEOREM 4. *Let $\mathcal{H}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ be hypermaps, and let*

$$G = \langle r_0, r_1, r_2;\, r_0^2 = r_1^2 = r_2^2 = 1\rangle.$$

(1) *$\mathcal{H}_1$ covers $\mathcal{H}_2$ if and only if there are $S_1 \leq S_2 \leq G$ such that $\mathcal{H}_1 \cong A(G/S_1'; r_0, r_1, r_2)$ and $\mathcal{H}_2 \cong A(G/S_2; r_0, r_1, r_2)$, where $S_1$ and $S_1'$ are conjugate in $G$,*
(2) *$\mathcal{H}_1 \cong \mathcal{H}_2$ if and only if the corresponding hypermap subgroups are conjugate in $G$,*
(3) *$\mathcal{H}$ is orientable if and only if its hypermap subgroup is contained in the even-word subgroup $G^+ \leq G$,*
(4) *the hypermap subgroup of the unoriented generic hypermap for a hypermap given by hypermap subgroup $S \leq G$ is the largest normal subgroup contained in $S$. In particular, regular hypermaps correspond to normal subgroups of $G$.*

Using the algebraic representation via hypermap subgroups one can handle many problems. For instance, it is straightforward that given two hypermaps $\mathcal{H}_1$, $\mathcal{H}_2$ with the respective subgroups $S_1$, $S_2$, the intersection $S_1 \cap S_2$ defines the *smallest common cover* for both $\mathcal{H}_1$ and $\mathcal{H}_2$. Because of many advantages the investigation of maps and hypermaps via the corresponding hypermap subgroups posses, sometimes a hypermap itself is identified with its hypermap subgroup ([6, 29]).

**Product of hypermaps**

The smallest common cover of given hypermaps $\mathcal{H}_1 = (F_1; r_0, r_1, r_2)$ and $\mathcal{H}_2 = (F_2; q_0, q_1, q_2)$ can be viewed as a *product of two hypermaps.* The question arises whether we can construct the product explicitly, or in other words, can we derive the monodromy group of the product in terms of the monodromy groups of factors? The most natural approach is to set $F = F_1 \times F_2$ and $p_i(x, y) = (r_i x, q_i y)$ for any $(x, y) \in F$ and $i = 0, 1, 2$. Unfortunately, the hypermap $\mathcal{H} = (F; p_0, p_1, p_2)$ is, in general, not correctly defined since the action of $\mathrm{Mon}(\mathcal{H}_1) \times \mathrm{Mon}(\mathcal{H}_2)$ may not be transitive on $F$. Necessary and sufficient condition ([16, 14]) follows.

THEOREM 5 [16]. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be hypermaps with the hypermap subgroups $S_1$ and $S_2$. Then the monodromy group of the smallest common cover for $\mathcal{H}_1$, $\mathcal{H}_2$ is the direct product* $\mathrm{Mon}(\mathcal{H}_1) \times \mathrm{Mon}(\mathcal{H}_2)$ *if and only if* $T(\infty, \infty, \infty) = \langle S_1, S_2 \rangle$.

**Chirality index**

Orientable hypermaps split into two families: family of reflexible and family of chiral hypermaps. Reflexible hypermap is *mirror symmetric* which means that the two associated oriented hypermaps $(D; R, L)$ and $(D; R^{-1}, L^{-1})$ are isomorphic. Topologically speaking, a reflexible hypermap admits an orientation reversing self-homeomorphism of the supporting surface preserving the embedded graph and the colours of faces. Note that S. Wilson [103] and some other authors use the term reflexible to denote (reflexible) regular hypermap. An orientable hypermap which is not reflexible is *chiral* (or *mirror asymmetric*). From the first point of view the "chirality" seems to be a binary invariant for the category of orientable hypermaps. Surprisingly, it turns out [17] that one can measure by a group, called the chirality group of a hypermap, of how much given orientable hypermap deviates from being mirror symmetric. For the simplicity let us restrict to orientably regular hypermaps now. Let us denote by $\mathcal{H}^{\Delta}$ the largest (reflexible) regular hypermap covered by $\mathcal{H}$ and by $\mathcal{H}_{\Delta}$ the smallest (reflexible) regular hypermap covering $\mathcal{H}$. With the help of hypermap subgroup representation the following result is proved in [17].

THEOREM 6 [17]. *Let $\mathcal{H}$ be an orientably regular hypermap. Then there exists a finite group $G = X(\mathcal{H})$ of size $\kappa$ such that $\mathcal{H}_{\Delta}$ is a smooth $\kappa$-fold cover of $\mathcal{H}$, and $\mathcal{H}$ is a smooth $\kappa$-fold cover of $\mathcal{H}^{\Delta}$. Moreover, both coverings are regular, with groups of covering transformations isomorphic to $G$.*

The above defined group $X(\mathcal{H})$ is called the *chirality group* while its size $\kappa$ is the *chirality index*. It follows that $X(\mathcal{H})$ is trivial if and only if $\mathcal{H}$ is reflexible. Structure of chirality groups is studied in [17] in a more detail. It is proved there that every abelian group is the chirality group of a hypermap. On the other hand, many non-abelian groups including symmetric groups and dihedral groups cannot serve as chirality groups.

**Two-generator groups**

Finite groups generated by two involutions coincide with dihedral groups. Any finite 2-generator group can be interpreted as a monodromy group of a regular oriented hypermap, while groups generated by three involutions give rise to (unoriented) regular hypermaps. A lot of finite groups belong to one or to the other above mentioned class of groups. In this context, it is worth to mention a recent result of Malle, Saxl and Weigel [67] showing that every non-abelian simple group can be generated by two elements, one of them being of order two. In other words, every non-abelian simple group is a monodromy group of some oriented regular map. Interesting problem of classification of all possible couples $R, L$ ($L^2 = 1$) of generators for a given group arises. Having in mind the above result, the solution of the problem for the class of (finite) simple groups means to prove a refinement of the classification of simple groups. Although, in general this problem seems to be intractable, at least for some groups it can be solved. Moreover, using the Hall's counting principle [42] one can calculate the number of nonisomorphic pairs of generators in terms of group characters ([27, 53]).

## 4. Regular maps of large planar width and residual finiteness of triangle groups

While regularity (in either arithmetical or group-theoretical sense) is an obvious property of any generalization of the Platonic solids, it is somewhat less obvious whether or how their planarity should be generalized. The idea is to replace the global planarity of the Platonic solids by a certain local variant of this notion. This should guarantee that a sufficiently "large" neighbourhood of each face is simply connected. Recent works in topological graph theory (cf. [73, 81, 82, 95]) suggest the following concept as a convenient measure of local planarity. A map $\mathcal{M}$ on a closed surface $S$ other than the 2-sphere is said to have *planar width* at least $k$, $w(\mathcal{M}) \geq k$, if every non-contractible simple closed curve on $S$ intersects the underlying graph of $\mathcal{M}$ in at least $k$ points. Planar width (most often called "face-width" or "representativity") has recently received a considerable attention as an important tool for the study of graph embeddings on surfaces. The following theorem presents the main result of [78]. Its proof consists in construction of a certain planar map $\mathcal{M}_w(p,q)$ of type $(p,q)$ for which the generic map construction applies. Another proof of the theorem can be found in [91].

THEOREM 7 [78]. *For every pair of integers $p \geq 3$ and $q \geq 3$ such that $1/p + 1/q \leq 1/2$ and for every integer $k \geq 2$ there exists an orientable regular map $\mathcal{M}$ of type $(p,q)$ with planar width $w(\mathcal{M}) \geq k$. Moreover, we can require the map $\mathcal{M}$ to be reflexible.*

This theorem has several predecessors in the literature.

### Grünbaum's problem

In 1976, Grünbaum [40] asked if for every pair of positive integers $p$ and $q$ with $1/p + 1/q < 1/2$ (i.e., in the hyperbolic case) there are infinitely many finite regular maps of type $(p, q)$. He also remarked, however, that it was not even known whether for such $p$ and $q$ there was at least one map of that type. The question was answered in the affirmative by Vince [97] (1983) within a more general framework of higher-dimensional analogues of regular maps. His proof, based on a theorem of Mal'cev saying that every finitely generated matrix group is residually finite (e.g., Kaplansky [58]), was non-elementary and non-constructive. Constructive proofs of Vince's theorem were subsequently given by Gray and Wilson [31] and Wilson [105, 107] along with some refinements. Further constructions of regular maps of each type $(p, q)$ have recently been given by Jendrol' et al. [50] and Archdeacon et al. [3].

Parallel to this development there is another line of research which is closely related to our main theorem. The bridge between the two streams is the observation that an orientable regular map of type $(p, q)$ exists if and only if there is a finite group with presentation $G = \langle x, y;\, x^q = y^2 = (xy)^p = 1, \dots \rangle$ forming the oriented monodromy group of the oriented regular map $A(G; x, y)$. With this relationship in mind, the solution of the above Grünbaum's $(p, q)$-problem can be derived from an old result (1902) of Miller [72] (rediscovered by Fox [28] in 1952) which states:

THEOREM 8 (Miller [72]). *For any three integers $p$, $q$, and $n$, all greater than* 1, *there exist infinitely many pairs of permutations $\alpha, \beta$ such that $\alpha$ has order $p$, $\beta$ has order $q$, and $\alpha\beta$ has order $n$, except that the three numbers are* $2, 3, 3$ *or* $2, 3, 4$ *or* $2, 3, 5$ (*ordered arbitrarily*) *or two of the numbers are* 2.

In the latter cases, the triples determine the groups uniquely: they are the tetrahedral, the octahedral and the dodecahedral group, and the dihedral groups.

### Hurwitz maps

The special case where $p = 7$, $q = 3$ and $n = 2$ (or its dual, with the roles of $p$ and $q$ interchanged) was an object of an extensive research in the area of Fuchsian groups, hyperbolic geometry and Riemann surfaces originating from the famous theorem of Hurwitz (1893):

THEOREM 9 (Hurwitz's Theorem). *For any Riemann surface $S$ of genus $g \geq 2$, the number of its automorphisms (that is, conformal homeomorphisms) does not exceed* $84(g-1)$.

The Hurwitz bound is attained precisely when $\mathrm{Aut}(S)$ is a *Hurwitz group*, a finite group $G$ generated by an element $x$ of order 3 and an element $y$ of order 2 whose product has order 7. As mentioned above, such a group gives rise to a regular map of type $(7,3)$ on $S$, a (trivalent) *Hurwitz map*, whose automorphism group is isomorphic to $G$. In this context, the fact that there are infinitely many Hurwitz groups was first established by MacBeath in 1969 [65]. More recent results on Hurwitz maps one can find in [20] where all Hurwitz groups giving rise to reflexible regular maps of type $(7,3)$ and of genus $< 11,905$ are classified.

Let us note that MacBeath's theorem immediately follows from Vince's theorem which in turn is a consequence of Theorem 7. Indeed, it is sufficient to take an infinite sequence of regular maps of any type $(p,q)$ (in particular, $(7,3)$) with increasing planar width. Thus Theorem 7 has the following two corollaries:

THEOREM 10. (Vince [97]) *For any pair of integers with $p \geq 2$, $q \geq 2$ and $1/p + 1/q \leq 1/2$ there exist infinitely many orientable regular maps of type $(p,q)$.*

THEOREM 11. (MacBeath [65]) *There exist infinitely many Hurwitz groups.*

### Residual finiteness of triangle groups

Another notable consequence of Theorem 7 is of group-theoretic nature. Let $T^+(p,q,2)$ be the oriented triangle group with presentation $\langle x,y;\, x^q = y^2 = (xy)^p = 1\rangle$. Then for any integer $w \geq 1$ there exist infinitely many finite quotients $H$ of $T^+(p,q,2)$ such that in any presentation of $H$ in terms of $x$ and $y$ all reduced identities that are not identities of $T^+(p,q,2)$ have length greater than $w$. The latter sentence is nothing but a reformulation of the well-known fact that triangle groups $T^+(p,q,2)$ are residually finite [58]. A group $G$ is called *residually finite* if for each element $g \in G$ there exists a finite quotient $H$ of $G$ such that the epimorphism $G \to H$ does not send $g$ onto the identity. Furthermore, as it was already mentioned the group $T^+(p,q,2)$ is isomorphic to the even-word subgroup of the full triangle group $T(p,q,2) = \langle x,y,z;\, x^2 = y^2 = z^2 = (xy)^p = (yz)^2 = (xz)^q\rangle$. Hence, by using Theorem 7 (as well as its unoriented version) we deduce the following corollary (also [91, 92].)

COROLLARY 12. *For each pair of integers $p$ and $q$ such that $1/p + 1/q \leq 1/2$ both the oriented and the full triangle group $T^+(p, q, 2)$ and $T(p, q, 2)$ are residually finite.*

As was noted, Vince [97] proved Grünbaum's conjecture by employing the residual finiteness of triangle groups. In contrast, we have just shown that Theorem 7 implies both Vince's theorem and the residual finiteness of triangle groups. In fact, the residual finiteness of triangle groups is equivalent to Theorem 7.

It is well known ([44, 64, 86]) that the fundamental groups of closed surfaces are residually finite. Alternatively, this can be proved by using Corollary 12 and observing that every fundamental group of a closed surface embeds into some triangle group $T(p, q, 2)$. A natural way to make see that the fundamental group of a given surface embeds into a triangle group $T(p, q, 2)$ is to try to construct a map of type $(p, q)$ supported by $S$. For instance, if the surface is orientable of genus $g$ one can take $p = 4g = q$. In [50, 78, 91, 92] one can find further applications of Theorem 7.

## 5. Maps, hypermaps and Riemann surfaces

The aim of this section is to explain briefly the relationship between hypermaps and Riemann surfaces. Following the approach of Jones [55] we firstly show that any hypermap can be endowed with the structure of Riemann surface. A natural question arises: What kind of Riemann surfaces are associated with hypermaps? Surprisingly beautiful answer [55] is a consequence of the theorem of Belyĭ [5]. In what follows we have extracted some ideas from [55] where one can find more detailed information as well as an exhaustive list of relevant references.

A *Riemann surface* is a surface with locally-defined complex coordinates, such that the changes of coordinates between intersecting neighbourhoods are conformal. The the upper half-plane $U = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$, and the Riemann sphere (or complex projective line) $\Sigma = P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ are examples of simply-connected Riemann surfaces.

The upper half-plane is a model of hyperbolic geometry, the geodesics being the euclidean lines and semi-circles which meet the real line $\mathbb{R}$ at right-angles. The *modular group* $\Gamma = PSL_2(\mathbb{Z})$, consisting of the Möbius transformations

$$T : z \mapsto \frac{az + b}{cz + d} \qquad (a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1),$$

acts on $U$ as a group of orientation-preserving hyperbolic isometries. It is well-known (for instance [55]) that modular group is a free group of rank 2. Hence it can serve as a universal covering group for the set of monodromy groups of oriented hypermaps which are two-generator groups. In order to get a representation of a hypermap by means of a Riemann surface one has to embed the underlying graph of the universal Walsh bipartite map into $U$ in the "right way". This can be done provided we extend $U$ as follows. Observe that $\Gamma$ acts (transitively) on the rational projective line $P^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$, and hence it acts on the extended hyperbolic plane

$$\overline{U} = U \cup \mathbb{Q} \cup \{\infty\}.$$

Let us denote by

$$[0] = \{\, \frac{a}{b} \in \mathbb{Q} \cup \{\infty\} \mid a \text{ is even and } b \text{ is odd} \,\},$$

$$[1] = \{\, \frac{a}{b} \in \mathbb{Q} \cup \{\infty\} \mid a \text{ and } b \text{ are both odd} \,\}.$$

The subgroup stabilizing both [0] and [1] is

$$\Gamma(2) = \{\, T \in \Gamma \mid b \equiv c \equiv 0 \mod (2) \,\}.$$

The *universal* (*Walsh*) *bipartite map* $\hat{\mathcal{B}}$ on $\overline{U}$ has [0] and [1] as its sets of black and white vertices, and its edges are the hyperbolic geodesics between vertices $a/b$ and $c/d$ where $ad - bc = \pm 1$; this implies that $a$ and $c$ have opposite parity, so the map is indeed bipartite.

The automorphism group of $\hat{\mathcal{B}}$ (preserving orientation and colours) is $\Gamma(2)$. This is a free group of rank 2, freely generated by

$$\hat{R} : z \mapsto \frac{z}{-2z+1} \quad \text{and} \quad \hat{L} : z \mapsto \frac{z-2}{2z-3}.$$

It follows that if $\mathcal{B}$ is any bipartite map, representing an oriented hypermap $\mathcal{H}$ with monodromy group $G = \langle R, L \rangle$, then there is an epimorphism

$$\Gamma(2) \to G, \quad \hat{R} \mapsto R, \quad \hat{L} \mapsto L,$$

giving a transitive action of $\Gamma(2)$ on the set $E$ of edges of $\mathcal{B}$. The stabilizer of an edge in this action is a subgroup $B$ of index $N = |E|$ in

$\Gamma(2)$, called the *map subgroup* corresponding to $\mathcal{B}$ (different choices of an edge lead to conjugate subgroups). Since $B \leq \Gamma(2) = \text{Aut}(\hat{\mathcal{B}})$, one can form the quotient map $\hat{\mathcal{B}}/B$ isomorphic to $\mathcal{B}$. In particular, the oriented hypermap $\mathcal{H}$ is regular if and only if $B$ is normal in $\Gamma(2)$, in which case $\text{Aut}(\mathcal{H}) \cong \Gamma(2)/B \cong G$. Note that the above map subgroup $B$ can be viewed as an oriented version of the hypermap subgroup introduced above.

Using the above factorization process we end with an isomorphic copy of our original Walsh bipartite map $\mathcal{B}$, endowed with some extra structure. The underlying surface is now a compact Riemann surface $X = \overline{\mathcal{U}}/B$, in which the underlying graph is very rigidly embedded: the edges are all geodesics, the angles between successive edges around a vertex are all equal, and the automorphisms of $\mathcal{B}$ are all conformal automorphisms of $\overline{\mathcal{U}}/B$ (induced by the action of $N_{\Gamma(2)}(B)$ on $\hat{\mathcal{B}}$).

Clearly, we can represent the trivial bipartite map on the Riemann surface $\Sigma$ with one black vertex at 0, one white vertex at 1, and the unique edge joining them as the interval [0,1]. Let $\mathcal{B}_1$ denotes this representation. Since any bipartite map covers $\mathcal{B}_1$, alternatively, one can derive the structure of Riemann surface associated with $\mathcal{B}$ by considering the branched regular covering $\mathcal{B} \cong \hat{\mathcal{B}}/B \to \mathcal{B}_1 \cong \hat{\mathcal{B}}/\Gamma(2)$ given by the inclusion $B \leq \Gamma(2)$.

### Riemann surfaces and algebraic curves

If $A(x, y) \in \mathbb{C}[x, y]$ is a polynomial in $x$ and $y$ with complex coefficients, then the equation $A(x, y) = 0$ defines the complex variable $y$ as an $N$-valued function of the complex variable $x$, where $N$ is the degree of $A$ in $y$. Consequently, the Riemann surface $X_A$ of this equation can be constructed by taking $N$ copies of the Riemann sphere $\Sigma$ (one for each branch of the function), cutting them between the branch-points, and then rejoining the sheets across these cuts.

A Riemann surface is called *algebraic* if it is isomorphic to $X_A$ for such a polynomial $A$. The following major result was known to Riemann:

THEOREM 13 (Riemann). *A Riemann surface is compact if and only if it is algebraic.*

If $K$ is a subfield of $\mathbb{C}$, then we say that a compact Riemann surface $X$ is *defined over* $K$ if $X \cong X_A$ for some polynomial $A(x, y) \in K[x, y]$. Let $\overline{\mathbb{Q}}$ denotes the field of algebraic numbers.

A Belyĭ function is a meromorphic function $X \to \Sigma$ with no critical values outside $\{0, 1, \infty\}$. The following powerful result is due to Belyĭ

[5]:

THEOREM 14 (Belyĭ). *A compact Riemann surface $X$ is defined over $\overline{\mathbb{Q}}$ if and only if there is a Belyĭ function $\beta : X \to \Sigma$.*

Belyĭ theorem implies the following theorem (Jones [55]) giving a correspondence between hypermaps, Riemann surfaces and algebraic curves. It shows that the Riemann surfaces associated with hypermaps are precisely those defined over the field of algebraic numbers.

THEOREM 15 [55]. *If $X$ is a compact Riemann surface then the following are equivalent:*

(1) *$X$ is defined over $\overline{\mathbb{Q}}$;*
(2) *$X \cong \overline{\mathcal{U}}/M$ for some subgroup $M$ of finite index in the modular group $\Gamma$;*
(3) *$X \cong \overline{\mathcal{U}}/B$ for some subgroup $B$ of finite index in $\Gamma(2)$;*
(4) *$X \cong \mathcal{U}/H$ for some subgroup $H$ of finite index in a hyperbolic triangle group $T(k,m,n)$ where the integers $k$, $m$, $n$ satisfy $1/k + 1/m + 1/n < 1$.*

## 6. Regular hypermaps on fixed surface

### Orientably regular hypermaps of genus at most two

By the Hurwitz bound (Theorem 9) the size of the automorphism group, and consequently, the size of the oriented monodromy group $G = \langle R, L\rangle$ of an orientably regular hypermap $\mathcal{H} = (D;\ R, L)$ is bounded by $84(g-1)$, where $g > 1$ is the genus. Hence, a surface of genus at least two admits only finitely many orientably regular hypermaps. One of the central problems in the theory of maps and hypermaps is the problem of classification of all orientably regular hypermaps on a fixed underlying surface.

Recall that orientable regular hypermaps of genus $g$ form a subset of the orientably regular hypermaps of genus $g$. Also the classification of regular hypermaps on a non-orientable surface of genus $\tilde{g}$ can be derived from the classification of orientably regular hypermaps of genus $g = \tilde{g} - 1$ by using the construction of the antipodal cover. Hence the classification problem for regular hypermaps on non-orientable surface of genus $\tilde{g}$ can be solved by checking the list of all orientably regular hypermaps of genus $g = \tilde{g} - 1$ ([77] for more detailed explanation).

Using the Euler formula (Theorem 2) it is not difficult to see that spherical orientably regular hypermaps consist of the five Platonic solids, and of two infinite families of types $(1, n, n)$, $(2, 2, n)$ and their duals. Orientably regular maps on torus were classified by Coxeter and Moser [25], and the generalization to hypermaps was done by Corn and Singerman [24]. Up to duality, there are three infinite families of toroidal orientably regular hypermaps which types are $(2, 4, 4)$, $(2, 3, 6)$ and $(3, 3, 3)$. In [15] the classification problem for double torus is settled. As concerns surfaces of higher genera only partial results for hypermaps are known. For instance, in [18] mirror asymmetric orientably regular hypermaps up to genus 4 are classified.

**Regular maps with prescribed genus**

For regular maps we have the following results. Sherk in [84] classified oriented regular maps of genus 3. Grek in [32, 33, 34, 35] derived characterization of unoriented and oriented regular maps on surfaces (orientable and unorientable) with Euler characteristic $\geq -4$. Regular maps and oriented regular maps on surfaces of genus less than 8 and non-orientable surfaces of genus less than 9 have been completely classified by Garbe ([29], [6]). Recently, Conder and Dobcsányi [22] with the help of a computer programme gave a list of all orientably regular maps up to genus 15.

**Existence problem**

Following MacLachlan [68] (also [1]) let us denote by $\mu(g)$ the order of largest group of conformal self-mappings (automorphisms) of a compact Riemann surface of genus $g$. Hurwitz bound gives $\mu(g) \leq 84(g-1)$. The following result was proved independently by Accola [1] and MacLachlan [68].

THEOREM 16 [68, 1]. *With the above notation we have*

(1) $8(g+1) \leq \mu(g) \leq 84(g-1)$ *provided* $g \geq 2$,
(2) *There are infinitely many integers* $g \geq 2$ *for which the equality* $\mu(g) = 8(g+1)$ *holds.*

Having in mind the correspondence between hypermaps and Riemann surfaces we get that there are infinitely many surfaces $S$ such that the number of darts of any orientably regular hypermap on $S$ is at most $8(g+1)$. On the other hand, one needs at least $4(g+1)$ darts to cut the surface into cells. Thus the size of the monodromy group of an orientably regular hypermap on such a surface ranges between $4(g+1)$ and $8(g+1)$.

While in the orientable case, for any $g \geq 1$, the regular embedding of the bouquet of $2g$ cycles gives a regular map of genus $g$, search for regular hypermaps on nonorientable surfaces shows that there are surfaces which do not support any regular hypermap. More precisely, Breda and Wilson prove in [108] the following result.

THEOREM 17 [108]. *Let $g$ be an integer $1 \leq g \leq 52$. Then there is a regular non-orientable hypermap of genus $g$ if and only if $g$ is different from* 2, 3, 18, 24, 27, 39, *or* 48.

On the other hand, Conder and Everitt [21] cover by constructions about 75 percent of non-orientable genera. The problem, whether there are infinitely many non-orientable surfaces supporting no regular hypermaps remains open.

Modifications of the classification and existence problem can be considered. For instance, in [18] the classification of chiral hypermaps with genus at most four is carried. The main result of the paper implies that there is no chiral (orientably regular) hypermap on surface of genus 2, while each of the surfaces of genus 3 and 4 supports (up to duality) exactly one chiral hypermap.

Examining the list of orientably regular maps up to genus 15 [22] one can see that surfaces of genera 2, 3, 4, 5, 6, 9, and 13 support no chiral maps.

## 7. Operations on maps and hypermaps, external symmetries of hypermaps

Generally, an operation $\Phi$ is a function associating a given (hyper)map $\mathcal{M}$ another (hyper)map $\Phi(\mathcal{M})$. Typically, we require that $\Phi$ preserve some important properties of maps such as the underlying surface, or the underlying graph, or the monodromy group, or the hypermap subgroup etc. With each operation $\Phi$ and a fixed map $\mathcal{M}$ a family of external symmetries, defined by the relation $\mathcal{M} \cong \Phi(\mathcal{M})$, is associated. Depending on the class of operations we consider, the external symmetries, called exomorphisms in [77] form a group $\mathrm{Exo}(\mathcal{M})$ containing the automorphism group $\mathrm{Aut}(\mathcal{M})$ as a (normal) subgroup. The factor group $\mathrm{Exo}(\mathcal{M})/\mathrm{Aut}(\mathcal{M})$ then can be interpreted as a group of outer symmetries leaving $\mathcal{M}$ invariant.

### Functors

Important class of operations is formed by *functors* between categories of hypermaps, i.e., it is required that morphisms between hypermaps are preserved. Perhaps the most familiar functors in the category of unoriented maps is the duality operation defined by $(F;\ \lambda,\rho,\tau) \mapsto (F;\ \tau,\rho,\lambda)$, and the Petrie operation defined by $(F;\ \lambda,\rho,\tau) \mapsto (F;\ \lambda',\rho,\lambda)$, where $\lambda'(x) = \tau(x)$ or $\lambda'(x) = \lambda\tau(x)$ depending on whether a flag $x$ is, or is not, associated with a semiedge, respectively. These two functors have a nice geometric description also. The above two functors generate a group of functors isomorphic to $S_3$ (Wilson [104] and Lins [63]). Jones and Thornton showed that the above group of six operations is induced by the action of the outer automorphism group on conjugacy classes of subgroups of the group $T(\infty,\infty,2) = \langle \lambda,\rho,\tau;\ \rho^2 = \tau^2 = \lambda^2 = (\tau\lambda)^2 = 1\rangle$, which is the automorphism group of the universal map for the category of unoriented maps.

THEOREM 18 [52]. *The set of operations acting on the family of (unoriented) maps induced by the action of outer automorphism group of $T(\infty,\infty,2)$ consists of six operations forming a group $G$ generated by the duality and Petrie operations, and $G$ is isomorphic to the symmetric group $\mathbb{S}_3$.*

Moreover, Léger and Terrasson [62] proved that the outer automorphism group of the category of unoriented maps is $\mathbb{S}_3$, that is, they proved that the quotient of the group of invertible elements from the category of maps onto itself, modulo the normal subgroup induced by inner automorphisms of $T(\infty,\infty,2)$ is $\mathbb{S}_3$. As it was noted by Jones (personal communication), the above result is nothing but a restatement of Theorem 18 in the language of categories.

The above results generalize to hypermaps as follows. Given a hypermap $\mathcal{H} = (F;\ r_0, r_1, r_2)$ we can derive virtually six hypermaps by permuting the three colours 0, 1, 2 of its cells; in fact, for each permutation $\sigma \in S_3 = S_{\{0,1,2\}}$ we define the $\sigma$-dual $D_\sigma\mathcal{H}$ to be the hypermap $\mathcal{H} = (F;\ r_{\sigma 0}, r_{\sigma 1}, r_{\sigma 2})$. As expected this six operations form a group (Machi [66]). L. James [49] proved that the outer automorphism group of $T(\infty,\infty,\infty)$ is isomorphic to $PGL(2,\mathbb{Z})$. This infinite group induces a set of functors generated by $\sigma$-duals and one other twisting operator.

Similar results can be derived for other categories of hypermaps [57]. As concerns functors between distinguished categories of hypermaps, Singerman's list of triangle group inclusions [87] gives rise to a set of functors between different categories of hypermaps [57]. Let us note that the representation of a hypermap by the 3-valent 3-coloured map as well as the Walsh bipartite representation are examples of such functors. In the context of the correspondence between oriented maps and Riemann surfaces (Section 5) we would like to mention the following result.

**Regular maps and the associated Riemann surfaces**

The monodromy group of an oriented regular map $\mathcal{M}$ of type $(p,q)$ is an epimorphic image of $T^+(p,q,2)$. Let $K$ be the kernel of this epimorphism. By Theorem 15 $U/K$ is a Riemann surface which we denote by $S(\mathcal{M})$, here $K$ is considered to be a group of isometries of $U$. In [88, 89] Singerman and Syddall consider the problem whether the assignment $\mathcal{M} \mapsto S(\mathcal{M})$ is injective, or in other words, whether the same Riemann surface can underlie different regular maps. The late situation certainly happens for genus 0 regular maps, since the only Riemann surface of genus 0 is the Riemann sphere. As concerns genus 1, up to duality, there are two infinite families of regular maps (maps of type $(3,6)$ and of type $(4,4)$) and there are two Riemann surfaces associated with these two families. To continue our discussion we need to define the following two functorial operations. By the *truncation* of $\mathcal{M}$ we mean the cubic map whose vertices are darts of $\mathcal{M}$ and two are joined by an edge if they form an angle of $\mathcal{M}$, or they underlie the same edge. By the *medial map* we mean the map whose vertices are the middle points of edges of $\mathcal{M}$, two being adjacent if the respective edges form an angle of $\mathcal{M}$. Finally, let $e(\mathcal{M})$ denotes the number of edges of $\mathcal{M}$. Using the above defined operations one can describe regular maps sharing the same Riemann surface in almost all cases.

THEOREM 19 [88, 89]. *Let $\mathcal{M}$ and $\mathcal{N}$ be two orientably regular maps of genus $> 1$ with $S(\mathcal{M}) = S(\mathcal{N})$ and $e(\mathcal{M}) \leq e(\mathcal{N})$. Then one of the following statements holds:*

(1) *$\mathcal{N}$ is the medial map of $\mathcal{M}$,*
(2) *$\mathcal{N}$ is the truncation of $\mathcal{M}$,*
(3) *$\mathcal{N}$ has type $(7,3)$ and $\mathcal{M}$ has type $(7,7)$,*
(4) *$\mathcal{N}$ has type $(8,3)$ and $\mathcal{M}$ has type $(8,8)$.*
(5) *$\mathcal{N} \cong \mathcal{M}$ or to its dual map.*

**Exponents**

In order to explain what an exponent of a map is, consider an oriented map $\mathcal{M} = (D;\ R, L)$. Recall that the map $\mathcal{M}$ is reflexible (mirror symmetric) if and only if $\mathcal{M} \cong (D;\ R^{-1}, L)$. This suggests the following definition: an integer $e$ will be called an exponent of $\mathcal{M}$ if $\mathcal{M} \cong \mathcal{M}_e = (D;\ R^e, L)$. The mapping which realizes this isomorphism is an exomorphism of $\mathcal{M}$. In general, the map $\mathcal{M}_e$ is correctly defined provided $e$ is coprime with the valency $n$ of a map, i.e., with the least common multiple of lengths of cycles of $R$. If this is the case, the underlying graphs as well as the monodromy groups of both $\mathcal{M}$ and $\mathcal{M}_e$ are identical. This is the most important property of the *congruence operation* associated with an exponent $e \in \mathbb{Z}_n^*$. For regular maps, the converse statement is proved in [77].

THEOREM 20 [77]. *If $\mathcal{M}$ and $\mathcal{M}'$ are two oriented regular maps with the same underlying graph of valency $n$ and with identical monodromy groups. Then they are congruent, i.e., $\mathcal{M}' = \mathcal{M}_e$ for some $e \in \mathbb{Z}_n^*$.*

The exponents of a map $\mathcal{M}$ reduced modulo its valency form an Abelian group, the exponent group $\mathrm{Ex}(\mathcal{M})$ of $\mathcal{M}$. Similarly, the exomorphisms of $\mathcal{M}$ form a group $\mathrm{Exo}(\mathcal{M})$, and this group is an extension of the automorphism group of $\mathcal{M}$ by the exponent group.

The exponent group of a map $\mathcal{M}$ is naturally embedded in the multiplicative group $\mathbb{Z}_n^*$ of invertible elements of the ring $\mathbb{Z}_n$ of integers modulo $n$, $n$ being the valency of $\mathcal{M}$. Thus the order of $\mathrm{Ex}(\mathcal{M})$ divides $\phi(n)$ ($\phi$ is Euler's function).

In general, exponent groups are not functorial, but in [77] we investigate conditions under which exponents of a map can be transferred along a map homomorphism to another map.

**Inner exponents**

Among the exponents of a map $\mathcal{M}$, so called inner exponents play a special role. The corresponding exomorphisms act on darts as inner automorphisms of the monodromy group. These give rise to a subgroup of the exponent group, the inner exponent group $\mathrm{IEx}(\mathcal{M})$ of $\mathcal{M}$. In contrast to the general exponent groups, the inner exponents are functorial, which means that a map homomorphism induces a homomorphism between the inner exponent groups of the corresponding maps. The importance of inner exponents is emphasized by the fact that, apart from a short list of exceptions, inner exponent $-1$ implies that $\mathcal{M}^\natural$ covers a map on a non-orientable surface. On the other hand, the fact that a map antipodally covers a map on a non-orientable surface forces $-1$ to be inner exponent of the associated oriented map. More details about antipodality and exponents can be find in [77].

## 8. Lifting automorphisms of maps

Let $\pi\colon \tilde{X} \to X$ be a covering of topological spaces. Assume we have a homeomorphism $\psi\colon X \to X$. The question whether there is homeomorphism $\tilde{\psi} : \tilde{X} \to \tilde{X}$ such that $\psi\pi = \pi\tilde{\psi}$ is called the lifting automorphism problem. This problem is studied in [4, 12, 69, 109]. Particularly, the lifting automorphism problem for graphs is investigated in [9, 26, 11, 45, 60, 61, 71].

We show how a fruitful concept of voltage assignment ([36, 37, 38]), used to describe coverings of graphs, can be modified in order to describe homomorphisms between oriented regular maps. Voltage assignments can be used to build up regular maps from their regular quotients. The definitions and results presented in this section are taken from [70] (also [41, 2, 3, 75, 76, 94, 56]).

Let $\mathcal{M} = (D;\, R, L)$ be an oriented map. An (oriented) *angle* of $\mathcal{M}$ is an ordered pair $\boldsymbol{\alpha} = (x, y) = \overrightarrow{xy}$, where $x$ and $y$ are darts of $\mathcal{M}$ such that $y \in \{R(x), R^{-1}(x), L(x)\}$. The angle $\boldsymbol{\alpha}^{-1} = (y, x)$ is the inverse of $\boldsymbol{\alpha} = (x, y)$. Let us denote by $A(\mathcal{M})$ the set of all angles of $\mathcal{M}$. An *angle walk* is a sequence $\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2\ldots\boldsymbol{\alpha}_n$ of angles such that the initial dart of $\boldsymbol{\alpha}_i$ coincides with the terminal dart of $\boldsymbol{\alpha}_{i-1}$ for $i = 2, 3, \ldots, n$.

Let $G$ be a finite group. A voltage assignment on $\mathcal{M}$ valued in $G$ is a function $\alpha\colon A(\mathcal{M}) \to G$ such that for any angle $\alpha \in A(\mathcal{M})$ one has $\alpha(\boldsymbol{\alpha}^{-1}) = \alpha^{-1}(\boldsymbol{\alpha})$. A voltage assignment extends from angles to walks in an obvious way by setting $\alpha(W) = \alpha(\boldsymbol{\alpha}_1)\alpha(\boldsymbol{\alpha}_2)\ldots\alpha(\boldsymbol{\alpha}_n)$. The group generated by voltages of closed walks based at a fixed dart $x$ is called the *local voltage group* $G^x$. Since $\mathcal{M}$ is connected all the local groups are conjugate subgroups of $G$.

Given a voltage assignment $\alpha$ on $\mathcal{M} = (D;\, R, L)$ valued in $G$ set $D^\alpha = D \times G$ and define permutations on $D^\alpha$ by

$$R^\alpha(x, h) = (R(x), h\alpha(\overrightarrow{xRx})),$$

$$L^\alpha(x, h) = (L(x), h\alpha(\overrightarrow{xLx})).$$

One can prove that $\langle R^\alpha, L^\alpha\rangle$ is transitive on $D^\alpha$ if and only if $G^x = G$. If this is the case the map $\mathcal{M}^\alpha = (D^\alpha;\, R^\alpha, L^\alpha)$ is a correctly defined map called the *derived map*. In what follows we shall always assume that a considered voltage assignment on $\mathcal{M}$ valued in $G$ satisfies the condition $G^x = G$.

As one can expect, a regular homomorphism $\tilde{\mathcal{M}} \to \mathcal{M}$ is equivalent to a natural projection $\mathcal{M}^\alpha \to \mathcal{M}$ erasing the second coordinates ([70]). It is important to stress that using angle voltage assignments we can handle also coverings of maps (map homomorphisms) which may not induce coverings between the respective underlying graphs (a disadvantage of the classical approach based on associating ordinary voltages to darts of the quotient map [37, 38]). Maybe a bit surprising is the fact that an arbitrary homomorphism defined on an orientably regular map is necessarily regular. This can be easily seen from the Schreier representations of (hyper)maps introduced in Section 3 (also [70]). It follows that angle voltage assignments provide a convenient tool for study of orientably regular maps. A natural question arises: Under what condition a voltage assignment $\alpha$ defined on an orientably regular map $\mathcal{M}$ determines an orientably regular derived map $\mathcal{M}^\alpha$?

We say that a voltage assignment $\alpha\colon A(\mathcal{M}) \to G$ is *locally invariant* if for any $\tau \in \mathrm{Aut}(\mathcal{M})$ and any closed walk $W$ based at a dart $x$

$$\alpha(W) = 1 \Rightarrow \alpha(\tau W) = 1.$$

The following characterization of homomorphisms between regular maps is proved in [70].

THEOREM 21 [70]. *Let $\mathcal{M}$ and $\tilde{\mathcal{M}}$ be orientably regular maps. Then*

(1) *if $\alpha$ is a locally invariant assignment on $\mathcal{M}$, then the derived map $\mathcal{M}^\alpha$ is orientably regular.*
(2) *if $\varphi\colon \tilde{\mathcal{M}} \to \mathcal{M}$ is a homomorphism, then there exists a locally invariant voltage assignment $\alpha$ on $\mathcal{M}$ such that the natural projection $\mathcal{M}^\alpha \to \mathcal{M}$ is equivalent to $\varphi$.*

Let us note also that the lifting part of Theorem 21 is, in a weaker form, proved in [41]. Weaker form of the above theorem can be found also in [71].

By Theorem 21 coverings between regular maps are always regular with the group of covering transformations isomorphic to the voltage group. With some effort one can show that every oriented regular map with a non-solvable monodromy group covers regularly either a regular map with a non-abelian simple group monodromy group, or it covers regularly a bipartite oriented regular map which partition stabilizer is either simple non-abelian, or a direct product of two isomorphic simple non-abelian groups ([79]).

A global version of the invariance condition was used a long time ago in the proof that there are infinitely many 5-arc-transitive cubic graphs ([9]). We say that a voltage assignment $\alpha\colon A(\mathcal{M}) \to G$ is *invariant* if for any $\tau \in \mathrm{Aut}(\mathcal{M})$ and any walk $W$

$$\alpha(W) = 1 \Rightarrow \alpha(\tau W) = 1.$$

Of course, an invariant voltage assignment is locally invariant as well. If the assignment $\alpha\colon A(\mathcal{M}) \to G$ is invariant, then $\mathrm{Aut}(\mathcal{M}^\alpha)$ is a split extension of $G$ by $\mathrm{Aut}(\mathcal{M})$ ([70]). Invariant voltage assignments defined on graphs and valued in products of cyclic groups are used in [9, 11, 94, 71].

## 9. Regular embeddings of graphs

### Cayley maps

A *Cayley map* is an oriented map $\mathcal{M}$, which underlying graph is a Cayley graph $C(\Gamma, X)$ such that the local rotation of generators at any vertex is induced by a cyclic permutation $\pi$ of the generating set $X$. The above condition implies that the colour group $\Gamma \in \mathrm{Aut}(\mathcal{M})$ acts as a group of map automorphisms. Surprisingly large number of significant constructions in topological graph theory (explicitly or implicitly) include a construction of Cayley maps. Cayley maps received recently a lot of attention (for instance [8, 10, 46, 80, 90, 93, 102]). The following theorem is proved in [80].

THEOREM 22 [80]. *Let $\mathcal{M} = (D;\, R, L)$ be a regular map of valency $r$. Then $\mathcal{M}$ is a Cayley map if and only if there is a homomorphism $\varphi\colon \mathrm{Mon}(\mathcal{M}) \to Sym(r)$ sending $R \mapsto (1, 2, \ldots, r)$.*

The above theorem suggests that regular Cayley maps of the same valency $r$ split into finitely many classes, Cayley maps belonging to the same class cover the same embedding of a one-vertex graph. Particularly, if the one-vertex map is regular then the Cayley map $\mathcal{M}$ is *balanced*, i.e., the distribution $\pi$ of generators satisfies $(\pi x)^{-1} = \pi(x^{-1})$ for all $x \in X$. In this particular case, the covering is regular and it induces a locally invariant voltage assignment in $\Gamma$. The local invariance is in this case equivalent to the statement: "the local rotation of generators $\pi$ of $\Gamma$ extends to a group automorphism of $\Gamma$". The latter condition was known already for Biggs and White ([8, 100]). Regularity and other related properties of balanced Cayley maps are studied in [93].

### Classification of regular embeddings of given graph

The classification of regular maps by the underlying graph was initiated by Heffter [43] who constructed regular maps by embedding the complete graph $K_p$, $p$ a prime, into the surface of genus $(p-1)(p-4)/4$ or $(p^2 - 7p + 4)/4$ according to whether $p \equiv 1$ or $p \equiv 3 \pmod 4$. The classification of regular embeddings of the complete graph has been completed by work of several authors, see [7, 48] and [47, 106]. Besides the complete graphs, not much is known so far (although regular embeddings of several infinite classes of graphs have been constructed).

In [77] the following approach to classification of (orientably) regular maps with a given underlying graph divided into three stages is suggested.

The first stage consists in characterizing all the finite graphs that underlie some regular map. At the theoretical level, this stage is completed: necessary and sufficient conditions for a graph $\mathcal{G}$ to underlie a regular map (oriented or unoriented, reflexible or irreflexible) have been given by Gardiner, Nedela, Širáň and Škoviera [30]. The conditions are expressed in terms of the automorphism group of a graph. In the oriented case the existence of a group $G \leq \mathrm{Aut}(\mathcal{G})$ such that the action on darts of $\mathcal{G}$ is regular with cyclic stabilizer of a vertex is required. For the undirected case the statement follows.

THEOREM 23 [30]. *A connected graph $\mathcal{G}$ of valency at least 3 is the underlying graph of a regular map if and only if its automorphism group contains a subgroup $G$ which acts transitively on the set of darts of $\mathcal{G}$ such that the edge stabilizer $G_e$ of any edge is dihedral of order 4 and such that the stabilizer $G_v$ of a vertex $v$ is dihedral with cyclic group of index two acting regularly on darts incident to $v$.*

At the second stage, one has to analyze the automorphism group of a graph $\mathcal{G}$ satisfying one of the above mentioned necessary and sufficient conditions and to specify the appropriate subgroups in $\mathrm{Aut}(\mathcal{G})$ to be the map automorphism groups. Of course, it is sufficient to consider one representative of every conjugacy class.

Finally, at the third stage, one is dealing with one of the subgroups of $\mathrm{Aut}(\mathcal{G})$ chosen at the previous stage, say $G$. One then determines all the regular maps whose automorphism group is $G$. As shown in Section 7, this step in the classification process can be successfully accomplished. Indeed, if $(\mathcal{G}; R)$ and $(\mathcal{G}; R')$ are two regular maps (with the same underlying graph $\mathcal{G}$) whose automorphism group is $G$, then there is an integer $e$ coprime with the valency of $\mathcal{G}$, say $n$, such that $R' = R^e$ (Theorem 20). It follows that the isomorphism classes of maps $\mathcal{M}$ with $\mathrm{Aut}(\mathcal{M}) = G$ correspond to the cosets of the exponent group $\mathrm{Ex}(\mathcal{G}, R)$ in $\mathbb{Z}_n^*$, and their number is $|\mathbb{Z}_n^* : \mathrm{Ex}(\mathcal{M})|$.

**Complete maps**

The graph-theoretical approach to the classification problem can be nicely illustrated by the example of complete graphs. It follows from the general characterization that an automorphism group $G$ of an orientably regular embedding of the complete graph on $n$ vertices $K_n$ acts regularly on the set of darts of $K_n$ with cyclic vertex stabilizer of order $n-1$. Since the set of darts of $K_n$ can be identified with all pairs $(a,b) \in \mathbb{Z}_n \times \mathbb{Z}_n$, the action of $G$ on the set of vertices is sharply transitive. Such groups are well-known, it follows that $n = p^k$ is a power of prime and $G \leq \mathbb{S}_n \cong \mathrm{Aut}(K_n)$ is the group of affine transformations of $GF(p^k)$. It was proved by James and Jones [48] that every oriented regular map of $K_n$ is isomorphic to the map $\mathcal{M}_t = (D; R, L)$ for some primitive element $t$ of the field, where $D = \mathbb{Z}_n \times \mathbb{Z}_n$, $L(a,b) = (b,a)$ and $R(a,b) = (a, a+(b-a)t)$. Moreover, two such maps are isomorphic if and only if the corresponding primitive elements are conjugate under the Galois group $\mathrm{Gal}(p^k)$ ([77]). The latter is a consequence of the fact that the exponent group $Ex(\mathcal{M}_t)$ is isomorphic in a very natural way with $\mathrm{Gal}(p^k)$. Thus there are $\frac{\phi(p^k-1)}{k}$ nonisomorphic orientably regular embeddings of $K_n$, where $\phi(n)$ is the Euler function.

Note that the above maps $\mathcal{M}_t$ are balanced Cayley maps with the distribution of generators in $\mathbb{Z}_n^*$ determined by the multiplication by $t$. Of course this multiplication determines an automorphism of $\mathbb{Z}_n$ and hence all the maps $\mathcal{M}_t$ are orientably regular.

It is a bit surprising that not all complete graphs admit oriented regular embeddings. As concerns regular embeddings the condition on $n$ is even more restrictive. It is proved in [47, 106] that $K_n$ admits such an embedding if and only if $n$ is 2, 3, 4, or 6. Thus $\mathcal{M}_t$ form a family of chiral maps provided $n = p^k \geq 5$.

**Lifting of classification**

Sometimes, we are able to "lift" the classification of regular embeddings of a graph $K$ to the classification of regular embeddings of $\tilde{K}$, where $\tilde{K}$ covers $K$. This approach is used in [75, 76] to classify oriented regular embeddings of the tensor product $K_n \otimes K_2$ and of all oriented regular maps with two faces. The maps were previously dealt with by Brahana [13], Coxeter and Moser [25], Vince [98] and Garbe [29]. Surprisingly, the latter classification result can be employed in deriving a simple arithmetic condition for a generalized Petersen graph to be a Cayley graph [74].

The method of lifting regular maps wrapped by exponents from the base graph to its canonical double covering can be generalized to other coverings. In a more general approach developed in [76], the wrapping is controlled by a homomorphism from the group of covering transformations to the exponent group of the base map which is to be lifted with wraps. The technique of wrapped lifts produces many new regular maps and seems to be very fruitful and promising for the construction and classification of regular maps. Particularly, using this technique a lot of new oriented regular embeddings of $n$-dimensional cubes was constructed in [76].

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## References

[1] R. D. M. Accola, *On the number of automorphisms of a closed Riemann surface*, Trans. Amer. Math. Soc. **131** (1968), 398–408.

[2] D. Archdeacon, R. B. Richter, J. Širáň, and M. Škoviera, *Branched coverings of maps and lifts of map homomorphisms*, Australas J. Combin. **9** (1994), 109–121.

[3] D. Archdeacon, P. Gvozdjak, and J. Širáň, *Constructing and forbidding automorphisms in lifted maps*, Math. Slovaca **47** (1997), 113–129.

[4] M. A. Armstrong, *Lifting group actions to covering spaces*, Discrete groups and geometry (W. J. Harwey, C. Maclachlan, eds.), London Math. Soc. Lecture Notes, vol. 173, Cambridge Univ. Press, Cambridge, 1992, pp. 10–15.

[5] G. V. Belyĭ, *On Galois extensions of a maximal cyclotomic field*, (Russian) *Math. USSR Izvestiya* **14** (1980), 247–256 (English translation), Izv. Akad. Nauk SSSR **43** (1979), 269–276.

[6] P. Bergau and D. Garbe, *Non-orientable and orientable regular maps*, Proc. Groups–Korea 1988 (Pusan, August 1988), Lecture Notes in Math. 1398, Springer, Berlin–New York, 1989, pp. 29–42.

[7] N. L. Biggs, *Classification of complete maps on orientable surfaces*, Rend. Mat. **4** (1971), no. 6, 132–138.

[8] ______, *Cayley maps and symmetrical maps*, Proc. Cambridge Philos. Soc. **72** (1972), 381–386.

[9] ______, *Algebraic Graph Theory*, Cambridge Univ. Press, Cambridge, 1974.

[10] N. L. Biggs and A. T. White, *Permutation Groups and Combinatorial Structures*, London Math. Soc. Lecture Notes, vol. 33, Cambridge University Press, Cambridge, 1979.

[11] N. L. Biggs, *Homological coverings of graphs*, J. London Math. Soc. **30** (1984), 1–14.

[12] J. S. Birman and H. M. Hilden, *Lifting and projecting homeomorphisms*, Arch. Math. **23** (1972), 428–434.

[13] H. R. Brahana, *Regular maps and their groups*, Amer. J. Math. **48** (1927), 268–284.

[14] A. Breda d'Azevedo, *Hypermaps and symmetry*, Thesis, Univ. Southampton, Southampton, 1991.

[15] ———, *The reflexible hypermaps of characteristic-2*, Math. Slovaca **47** (1997), 131–153.

[16] A. Breda d'Azevedo and G. Jones, *Double coverings and reflexible hypermaps* (submitted).

[17] A. Breda d'Azevedo, G. Jones, R. Nedela, and M. Škoviera, *Chirality index of maps and hypermaps, in preparation.*

[18] A. Breda d'Azevedo and R. Nedela, *Chiral hypermaps of small genus*, submitted.

[19] W. Burnside, *Theory of Groups of Finite Order* (reprinted, 2nd edition), Dover, New York, 1955.

[20] M. Conder, *Maximal automorphism groups of symmetric Riemann surfaces with small genus*, J. Algebra **114** (1988), 16–28.

[21] M. Conder and B. Everitt, *Regular maps on non-orientable surfaces*, Geom. Dedicata **56** (1995), 209–219.

[22] M. Conder and P. Dobcsányi, *Determination of all regular maps of small genus*, J. Combin. Theory Ser. B **81** (2001), 224–242.

[23] R. Cori and A. Machì, *Maps, hypermaps and their automorphisms: a survey I, II, III'*, Expo. Math. **10** (1992), 403–427, 429–447, 449–467.

[24] D. Corn and D. Singerman, *Regular hypermaps*, European J. Combin. **9** (1988), 337–351.

[25] H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*, Fourth Edition, Springer-Verlag, Berlin, 1984.

[26] D. Ž. Djoković, *Automorphisms of graphs and coverings*, J. Combin. Theory Ser. B **16** (1974), 243–247.

[27] M. L. N. Downs and G. A. Jones, *Enumerating regular objects with a given automorphism group*, Discrete Math. **64** (1987), 134–151.

[28] R. H. Fox, *On Fenchel's conjecture about F-groups*, Mat. Tidskrift B (1952), 61–65.

[29] D. Garbe, *Über die regulären Zerlegungen orientierbarer Flächen*, J. Reine Angew. Math. **237** (1969), 39–55.

[30] A. Gardiner, R. Nedela, J. Širáň, and M. Škoviera, *Characterization of graphs which underlie regular maps on closed surfaces*, J. London Math. Soc. **59** (1999), 100–108.

[31] A. Gray and S. Wilson, *A more elementary proof of Grünbaum's conjecture*, Congr. Numer. **72** (1990), 25–32.

[32] A. S. Grek, *Regular polyhedra on a closed surface with Euler characteristic* $\chi = -1$, Trudy Tbiliss. Mat. Inst. **27** (1960), 103–112. (Russian)

[33] ———, *Regular polyhedra of the simplest hyperbolic types*, Ivanov Gos. Ped. Inst. Učen. Zap. **34** (1963), 27–30. (Russian)

[34] ———, *Polyhedra on surfaces with Euler characteristic* $\chi = -4$, Soobšč. Akad. Nauk Gruzin. SSR **42** (1966), 11–16. (Russian)

[35] ———, *Polyhedra on a closed surface whose Euler characteristic is* $\chi = -3$, Izv. Vyšš. Učeb. Zaved. **55** (1966), no. 6, 50–53 (Russian), English translation: *AMS Translations* **78** (1968), 127–131.

[36] J. L. Gross, *Voltage graphs*, Discrete Math. **9** (1974), 239–246.

[37] J. L. Gross and T. W. Tucker, *Generating all graph coverings by permutation voltage assignments*, Discrete Math. **18** (1977), 273–283.

[38] ———, *Topological Graph Theory*, Wiley, New York, 1987.

[39] A. Grothendieck, *'Esquisse d'un programme'*, Montpellier, preprint (1984).

[40] B. Grünbaum, *Regularity of graphs, complexes and designs*, Problèmes combinatoires et théorie des graphes, Colloques internationaux C. N. R. S. 260, Paris, 1978, pp. 191–197.

[41] P. Gvozdjak and J. Širáň, *Regular maps from voltage assignments*, Graph Structure Theory" (N. Robertson, P. Seymour, Eds., eds.), Contemporary Mathematics (AMS Series), vol. 147, 1993, pp. 441–454.

[42] P. Hall, *The Eulerian functions of a group*, Quart. J. Math. Oxford **7** (1936), 134–151.

[43] L. Heffter, *Über metacyklische Gruppen und Nachbarconfigurationen*, Math. Ann. **50** (1898), 261–268.

[44] J. Hempel, *Residual finiteness of surface groups*, Proc. Amer. Math. Soc. **32** (1972), 323.

[45] M. Hofmeister, *Isomorphisms and automorphisms of graph coverings*, Discrete Math. **98** (1991), 175–183.

[46] R. Jajcay, *Automorphis groups of Cayley maps*, J. Combin. Theory Ser. B **59** (1993), 297–310.

[47] L. D. James, *Imbeddings of the complete graph*, Ars Combin. **16-B** (1983), 57–72.

[48] L. D. James and G. A. Jones, *Regular orientable imbeddings of complete graphs*, J. Combin. Theory Ser. B **39** (1985), 353–367.

[49] L. D. James, *Operations on hypermaps and outer automorphism*, European J. Combin. **9** (1988), 551–560.

[50] S. Jendroľ, R. Nedela, and M. Škoviera, *Constructing regular maps and graphs from planar quotients*, Math. Slovaca **47** (1997), 155–170.

[51] G. A. Jones and D. Singerman, *Theory of maps on orientable surfaces*, Proc. London Math. Soc. **37** (1978), no. 3, 273–307.

[52] G. A. Jones and J. S. Thornton, *Operations on maps, and outer automorphisms*, J. Combin. Theory Ser. B **35** (1983), 93–103.

[53] G. A. Jones and S. A. Silver, *Suzuki groups and surfaces*, J. London Math. Soc. **48** (1993), no. 2, 117–125.

[54] G. A. Jones and D. Singerman, *Belyĭ functions, hypermaps and Galois groups*, Bull. London Math. Soc. **28** (1996), 561-590.

[55] G. A. Jones, *Maps on surfaces and Galois groups*, Math. Slovaca **47** (1997), 1–33.

[56] G. A. Jones and D. B. Surowski, *Regular cyclic coverings of the Platonic maps*, European J. Combin. **21** (2000), 407–418.

[57] G. A. Jones, *Maps, hypermaps and quotients of triangle groups*, Symmetry in graphs, maps and complexes (S. Wilson, ed.), Abstracts of the workshop SIGMAC'98, Northern Arizona Univ., Flagstaff, 1998, pp. 11–12.

[58] I. Kaplansky, *Commutative Rings*, University of Chicago Press, Chicago, 1974.

[59] F. Klein, *Über die Transform siebenter Ordnung der elliptischen Funktionen*, Math. Ann. **14** (1878-79), 428–471.

[60] J. H. Kwak and J. Lee, *Isomorphism classes of graph bundles*, Canad. J. Math. **4** (1990), 747–761.

[61] J. H. Kwak, J.-H. Chun, and J. Lee, *Enumeration of regular graph coverings having finite abelian covering transformation group*, Siam J. Discrete Math. **11** (1998), 273–285.

[62] Ch. Léger and J.-C. Terrasson, *The action of the symmetric group $S_3$ on the set of isomorphism classes of tesselations of closed surfaces*, Comptes Rendues Acad. Sci. Paris, Sér. I Math. **302** (1986), 39–42, French.

[63] S. Lins, *Graph encoded maps*, J. Combin. Theory Ser. B **32** (1982), 171–181.

[64] M. L. Lopez, *Residual finiteness of surface groups via tesselations*, Discrete Comput. Geom. **11** (1994), 201–211.

[65] A. M. MacBeath, *Generators of linear fractional groups*, Proc. Sympos. Pure Math., vol. XII, Amer. Math. Soc., Providence, R. I., 1969, pp. 14–32.

[66] A. Machi, *On the complexity of a hypermap*, Discrete Math. **42** (1982), 221–226.

[67] G. Malle, J. Saxl, and T. Weigel, *Generators of classical groups*, Geom. Dedicata **49** (1994), 85–116.

[68] C. MacLachlan, *A bound for the number of automorphisms of a compact Riemann surface*, J. London Math. Soc. **44** (1969), 265–272.

[69] A. Malnič, *Group actions, coverings and lifts of automorphisms*, Discrete Math. **182** (1998), 203–218.

[70] A. Malnič, R. Nedela, and M. Škoviera, *Regular homomorphism and regular maps*, No. 697, Preprint series of IMFM Univ. Ljubljana **38** (2000).

[71] ———, *Lifting graph automorphisms by voltage assignments*, European J. Combin. **21** (2000), 927–947.

[72] G. A. Miller, *Groups defined by the orders of two generators and the order of their product*, Amer. J. Math. **24** (1902), 96–106.

[73] B. Mohar, *Face-width of embedded graphs*, Math. Slovaca **47** (1997), 35–63.

[74] R. Nedela and M. Škoviera, *Which generalized Petersen graphs are Cayley graphs?*, J. Graph Theory **19** (1995), 1–11.

[75] ———, *Regular maps of canonical double coverings of graphs*, J. Combin. Theory Ser. B **67** (1996), 249–277.

[76] ———, *Regular maps from voltage assignments and exponent groups*, European J. Combin. **18** (1997), 807–823.

[77] ———, *Exponents of orientable maps*, Proc. London Math. Soc. **75** (1997), no. 3, 1–31.

[78] ———, *Regular maps on surfaces with large planar width*, European J. Combin. **22** (2001), 243–261.

[79] ———, *Cayley snarks and almost simple groups*, submitted.

[80] R. B. Richter, J. Širáň, R. Jajcay, T. W. Tucker, and M. C. Watkins, *Cayley maps* (submitted).

[81] N. Robertson and P. D. Seymour, *Graph minors. VII. Disjoint paths on a surface*, J. Combin. Theory Ser. B **45** (1988), 212–254.

[82] N. Robertson and R. P. Vitray, *Representativity of surface embeddings*, Paths, Flows, and VLSI-Layout (B. Korte, L. Lovász, H. J. Prömel, and A. Schrijver, eds.), Springer-Verlag, Berlin, 1990, pp. 293–328.

[83] J. J. Rotman, *An Introduction to the Theory of Groups*, 4th Edition, Springer V., Berlin, 1995.

[84] F. A. Sherk, *The regular maps on a surface of genus three*, Canad. J. Math. **11** (1959), 452–480.

[85] L. Schneps (ed.), *The Grothendieck Theory of Dessins d'Enfants*, London Math. Soc. Lecture Note Series 200, Cambridge University Press, Cambridge, 1994.

[86] P. G. Scott, *Subgroups of surface groups are almost geometric*, J. London Math. Soc. **17** (1978), no. 2, 555–565.

[87] D. Singerman, *Finitely Maximal Fuchsian groups*, J. London Math. Soc. **6** (1972), no. 2, 29–38.

[88] D. Singerman and R. Syddall, *Platonic surfaces*, Symmetry in graphs, maps and complexes (S. Wilson, ed.), Abstracts of the workshop SIGMAC'98, Northern Arizona Univ., Flagstaff, 1998, pp. 29–30.

[89] D. Singerman, *Belyi functions and hypermaps* (to appear in Topics in Riemann surfaces and Fuchsian groups (Conference to celebrate the 25th anniversary of UNED)).

[90] J. Širáň and M. Škoviera, *Regular maps from Cayley graphs. II Antibalanced Cayley maps*, Discrete Math. **124** (1994), 179–191.

[91] J. Širáň, *Triangle group representations and their applications to graphs and maps*, Discrete Math. **229** (2001), 341–358.

[92] ———, *Triangle group representations and constructions of regular maps*, Proc. London Math. Soc. **82** (2001), 1–.

[93] M. Škoviera and J. Širáň, *Regular maps from Cayley graphs. I: Balanced Cayley maps*, Discrete Math. **109** (1992), 265–276.

[94] D. B. Surowski, *Lifting map automorphisms and MacBeath's theorem*, J. Combin. Theory Ser. B **50** (1988), 135–149.

[95] C. Thomassen, *Embeddings of graphs with no short noncontractible cycles*, J. Combin. Theory Ser. B **48** (1990), 155–177.

[96] A. Vince, *Combinatorial maps*, J. Combin. Theory Ser. B **34** (1983), 1–21.

[97] ———, *Regular combinatorial maps*, J. Combin. Theory Ser. B **35** (1983), 256–277.

[98] ———, *Flag transitive maps*, Congr. Numer. **45** (1984), 235–250.

[99] T. R. S. Walsh, *Hypermaps versus bipartite maps*, J. Combinatorial Theory (B) **18** (1975), 155–163.

[100] A. T. White, *Strongly symmetric maps*, Graph Theory and Combinatorics (R. J. Wilson, ed.), Research Notes in Math., vol. 34, Pitman, London, 1979, pp. 106–132.

[101] ———, *Graphs, Groups and Surfaces*, Second Edition, North-Holland, Amsterdam, 1984.

[102] ———, *Cayley maps*, Congr. Numer. **115** (1996), 105–130.

[103] S. E. Wilson, *The smallest nontoroidal chiral maps*, J. Graph Theory **2** (1978), 315–318.

[104] ______, *Operators over regular maps*, Pacific J. Math. **81** (1979), 559–568.
[105] ______, *A construction from an existence proof*, Congr. Numer. **50** (1985), 25–30.
[106] ______, *Cantankerous maps and rotary embeddings of $K_n$*, J. Combin. Theory Ser. B **47** (1989), 262–273.
[107] S. Wilson, *Applications and refinements of Vince's construction*, Geom. Dedicada **48** (1993), 231–242.
[108] S. Wilson and A. Breda d'Azevedo, *Surfaces with no regular hypermaps*, submitted.
[109] H. Zieschang, *Lifting and projecting homeomorphisms*, Arch. Math. **24** (1973), 416–421.

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