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MULTIPLICITY RESULTS FOR PERIODIC SOLUTIONS OF SEMILINEAR DISSIPATIVE HYPERBOLIC EQUATIONS WITH COERCIVE NONLINEAR TERM

Wan Se Kim

Abstract. Multiplicity for doubly-periodic solutions and Dirichlet-periodic solutions are treated.

1. Doubly-periodic problem in one space dimension

In this chapter we will discuss the multiplicity result for weak doubly-periodic problem for dissipative hyperbolic equations with coercive growth nonlinearity in one dimensional space.

1.1. Introduction

Let Z, R^+ and R be the set of all integers, non-negative real numbers and real numbers, respectively and let $\Omega = [0, 2\pi] \times [0, 2\pi]$.

Let $L^1(\Omega)$ be the space of measurable real-valued functions $u : \Omega \to R$ which are Lebesgue integrable over Ω with usual norm $\|\cdot\|_{L^1}$. Let $L^2(\Omega)$ be the space of measurable real-valued functions $u : \Omega \to R$ which are Lebesgue square integrable over Ω with usual inner product $(,)$ and usual norm $\|\cdot\|_{L^2}$ and let $L^\infty(\Omega)$ be the space of measurable real-valued functions $u : \Omega \to R$ which are essentially bounded with usual essential norm $\|\cdot\|_{L^\infty}$.

Let $C^k(\Omega)$ be the space of all continuous functions $u : \Omega \to R$ such that the partial derivatives up to order k with respect to both variables

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are continuous on Ω, while $C(\Omega)$ is used for $C^0(\Omega)$ with the usual norm $\|\cdot\|_\infty$ and we write $C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega)$.

Let $W^{k.2}(\Omega)$ be the Sobolev space of all functions $u : \Omega \to R$ in $L^2(\Omega)$ such that all partial distributional derivatives up to k belongs to $L^2(\Omega)$ with the usual Sobolev norm.

The purpose of this work is to investigate the multiplicity results for weak double-periodic solutions of the semilinear telegraph equations of the form

$$\beta u_t + u_{tt} - u_{xx} + g(t,x,u) = h(t,x) \quad \text{in} \quad \Omega \tag{1.1.1}$$

where $\beta(\neq 0) \in R$, $u = u(t,x)$, $h \in L^2(\Omega)$ and $g : \Omega \times R \to R$ is a continuous function.

A weak doubly-periodic solution of (1.1.1) will be $u \in L^2(\Omega)$ such that

$$(u, -\beta v_t + v_{tt} - v_{xx}) + (g(\cdot,\cdot,u), v) = (h, v) \tag{1.1.2}$$

for every $v \in C^2(\Omega)$ satisfying the boundary conditions

$$v(t,0) - v(t,2\pi) = v_x(t,0) - v_x(t,2\pi), t \in [0,2\pi]$$
$$v(0,x) - v(2\pi,x) = v_t(0,x) - v_t(2\pi,x), t \in [0,2\pi].$$

Let us remark that a necessary condition for (1.1.2) to have meaning is that g be such that $g(\cdot,\cdot,u) \in L^2(\Omega)$ when $u \in L^2(\Omega)$.

Besides, g is a continuous function on $\Omega \times R$, we assume the following.

$(H_{1.1})$ There exist $a \in L^\infty(\Omega)$ and $b \in L^2(\Omega)$ such that

$$|g(t,x,u)| \le a(t,x)|u| + b(t,x) \quad \text{a.e.} \quad \text{on} \quad \Omega.$$

The existence results for telegraph equations with sublinear growth nonlinearity are treated in [9, 17, 21, 24] and the existence result for superlinear case is treated in [15]. For more information for this equation, we can refer [27]. Our results are somewhat related to that of Ambrosetti-Prodi [1] who initiated so called Ambrosetti-Prodi type multiplicity in 1972 in the study of Dirichlet problem to elliptic equations and it has been developed in various directions by several authors to ordinary and partial differential equations [5, 6, 7, 12, 13, 14, 17, 19, 20]. In this note, we give new conditions on forcing term for the multiplicity of doubly-periodic solutions for telegraph equations having coercive growth nonlinearity. Our method of proof is based on continuation theorem to coincidence topological degree [10, 20]. We use the specific properties of the periodic problem for (1.1.1) in obtainment of the required a'priori bound.

1.2. Preliminary results

Now consider the equation

$$\beta u_t + u_{tt} - u_{xx} = h(t,x), \beta \neq 0 \quad \text{and write} \tag{1.2.1}$$

$$u(t,x) = \sum_{(l,m)\in Z\times Z} u_{lm} \exp[i(lt+mx)]$$
$$h(t,x) = \sum_{(l,m)\in Z\times Z} h_{lm} \exp[i(lt+mx)]$$

with $\bar{u}_{lm} = u_{-l-m}$ and $\bar{h}_{lm} = h_{-l-m}$ since u and h are real.

LEMMA 1.2.1. *$u \in L^2(\Omega)$ is a weak solution of (1.2.1) if and only if, for all $(l,m) \in Z \times Z$,*

$$[\beta li + (m^2 - l^2)]u_{lm} = h_{lm}.$$

Let

$$\text{Dom}L = \left\{ u \in L^2(\Omega) | \sum_{(l,m)\in Z\times Z} [\beta^2 l^2 + (m^2 - l^2)^2] |u_{lm}|^2 < \infty \right\}.$$

Define an operator $L : \text{Dom}L \subseteq L^2(\Omega) \to L^2(\Omega)$ by

$$(Lu)(t,x) = \sum_{(l,m)\in Z\times Z} [\beta li + (m^2 - l^2)] u_{lm} \exp[i(lt+mx)].$$

Then Dom$L$ is dense in $L^2(\Omega)$, Ker$L = R$

$$\text{Im}L = \{h \in L^2(\Omega) | \int\int_\Omega h(t,x)\,dt\,dx = 0\},$$

ImL is closed, and

$$[\text{Ker}L]^\perp = \text{Im}L.$$

Moreover, $L^2(\Omega) = \text{Ker}L \bigoplus \text{Im}L$. Consider a continuous projection

$$P : L^2(\Omega) \to L^2(\Omega) \quad \text{such that} \quad \text{Im}L = \text{Ker}P.$$

Then $L^2(\Omega) = \mathrm{Ker}L \bigoplus \mathrm{Ker}P$. We consider another continuous projection $Q : L^2(\Omega) \to L^2(\Omega)$ defined by

$$(Qh)(t,x) = \frac{1}{4\pi^2} \int\int_{\Omega} h(t,x)\,dt\,dx.$$

Then we have $L^2(\Omega) = \mathrm{Im}Q \bigoplus \mathrm{Im}L$, $\mathrm{Ker}Q = \mathrm{Im}L$, and $L^2(\Omega)/\mathrm{Im}L$ is isomorphic to $\mathrm{Im}Q$.

Since $\dim[L^2(\Omega)/\mathrm{Im}L] = \dim[\mathrm{Im}Q] = \dim[\mathrm{Ker}L] = 1$, we have an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$ and L is a Fredholm mapping of index 0. Moreover, we have easily the following lemma.

LEMMA 1.2.2. *$L : \mathrm{Dom}L \subseteq L^2(\Omega) \to L^2(\Omega)$ is a closed operator.*

If $h \in L^2(\Omega)$, then u is a weak solution of (1.2.1) if and only if $u \in \mathrm{Dom}L$, $Lu = h$. L is not bijective but the restriction

$$L|_{\mathrm{Dom}L\cap\mathrm{Im}L} : \mathrm{Im}L \cap \mathrm{Dom}L \to \mathrm{Im}L$$

is bijective, so we can define a right inverse

$$K^R = [L|_{\mathrm{Dom}L\cap\mathrm{Im}L}]^{-1} : \mathrm{Im}L \to \mathrm{Im}L \cap \mathrm{Dom}L$$

and

$$(K^Rh)(t,x) = \sum_{\substack{(l,m)\in Z\times Z \\ (l,m)\neq(0,0)}} [\beta li + (m^2 - l^2)]^{-1} h_{lm} \exp[i(lt + mx)].$$

We have the following lemma.

LEMMA 1.2.3. $\mathrm{Dom}L \cap \mathrm{Im}L = K^R[\mathrm{Im}L] \subseteq W^{1.2}(\Omega) \cap C(\Omega) \cap \mathrm{Im}L$ *and*

$$K^R[W^{k.2} \cap \mathrm{Im}L] \subseteq W^{k+1.2} \cap \mathrm{Im}L,\ k = 0, 1, 2, 3, \dots .$$

Moreover, if $h \in \mathrm{Im}L$, then $\|K^Rh\|_{W^{1.2}} \le C_1\|h\|_{L^2}$ for some $C_1 > O$ independent of h.

Proof. See [15, 18, 24]. Since

$$(K^R h)(t,x) = \sum_{\substack{(l,m)\in Z\times Z \\ (l,m)\neq(0,0)}} [\beta li + (m^2 - l^2)]^{-1} h_{lm} \exp[i(lt + mx)],$$

we can represent K^R as a convolution product

$$(K^R h)(t,x) = (K * h)(t,x) = \iint_\Omega K(t-s, x-y) h(s,x) dsdy$$

where $K(t,x) : \frac{1}{4\pi^2} \sum_{\substack{(l,m)\in Z\times Z \\ (l,m)\neq(0,0)}} [\beta li + (m^2 - l^2)]^{-1} \exp[i(lt + mx)]$. □

We have the following lemma.

LEMMA 1.2.4. *The operator $K^R : \mathrm{Im}L \to C(\Omega)$ is compact. If $h \in \mathrm{Im}L$, then $\|K^R h\|_\infty \le C_2 \|h\|_{L^2}$ for some constant $C_2 > 0$ independent of h.*

Proof. See [15, 18]. Now we can extend K^R to $L^1(\Omega)$ by defining $\bar{K}^R : L^1(\Omega) \to L^2(\Omega)$ by the formula

$$(\bar{K}^R h)(t,x) = \iint_\Omega K(t-s, x-y) h(s,y) dsdy \quad \text{for} \quad h \in L^1(\Omega). \quad \square$$

Then, by Hölder's inequality and Fubini's theorem, we have the following lemma.

LEMMA 1.2.5. $\|\bar{K}^R h\|_{L^2} \le \|K\|_{L^2} \|h\|_{L^1}$.

Proof. See [15]. □

1.3. Multiplicity results

To treat our problem, let us consider the following doubly-periodic boundary value problem for a family of homotopy equation

(1.3.1$_\lambda$)

$$\beta u_t + u_{tt} - u_{xx} + \lambda g(t,x,u) = \lambda h(t,x), \quad \lambda \in [0,1],$$

where $g : \Omega \times R \to R$ is a continuous function and $h \in \mathrm{Im}L$. Let $L : \mathrm{Dom}L \subseteq L^2(\Omega) \to L^2(\Omega)$ be defined as before and define a substitution operator $N_\lambda : L^2(\Omega) \to L^2(\Omega)$

$$(N_\lambda)(t,x) = \lambda g(t,x,u) - \lambda h(t,x)$$

for $u \in L^2(\Omega)$ and $(t,x) \in \Omega$. By Krasnosel'skii's results, N_λ is continuous and bounded. Let G be any open bounded subset of $L^2(\Omega)$, then $QN : \bar{G} \to L^2(\Omega)$ is bounded and $K^R(I-Q) : \bar{G} \to L^2(\Omega)$ is compact and continuous. Thus, N_λ is L-compact on $\bar{G}$. The coincidence degree $D_L(L + N_\lambda, G)$ is well-defined and constant in λ if $Lu + N_\lambda u \neq 0$ for $\lambda \in [0,1]$ and $u \in \mathrm{Dom}L \cap \partial G$. It is easy to check that (u, λ) is a weak doubly-periodic solution of $(1.3.1_\lambda)$ if and only if $u \in \mathrm{Dom}L$ and

$$Lu + N_\lambda u = 0. \tag{1.3.2$_\lambda$}$$

Here we assume the following;

$$g(t,x,u) \geq 0 \quad \text{on} \quad \Omega \times R, \tag{$H_{1.2}$}$$

$$\lim_{|u| \to +\infty} g(t,x,u) = +\infty \quad \text{uniformly} \quad \text{on} \quad \Omega. \tag{$H_{1.3}$}$$

LEMMA 1.3.1. *If $(H_{1.2})$ and $(H_{1.3})$ are satisfied, then there exists $M > 0$ such that*

$$\|\tilde{u}\|_{L^2} \leq M$$

holds for each possible weak doubly-periodic solution $u = \bar{u} + \tilde{u}$, with $\bar{u} \in \mathrm{Ker}L$ and $\tilde{u} \in \mathrm{Im}L$, of $(1.3.1_\lambda)$ where $\lambda \in [0,1]$.

Proof. Let (u, λ) be any weak doubly-periodic solution of $(1.3.1_\lambda)$. Then (u, λ) is a solution of $(1.3.2_\lambda)$ where $u = \bar{u} + \tilde{u}$ with $\bar{u} \in \mathrm{Ker}L$ and $\tilde{u} \in \mathrm{Im}L$. By applying $\bar{K}^R$ on the both sides of equation $(1.3.2_\lambda)$, we have, since

$$\bar{K}^R_{|\mathrm{Im}L} = K^R,$$

$$\tilde{u} = -\lambda \bar{K}^R N_\lambda u = \lambda \bar{K}^R[-g(\cdot,\cdot,u) + h(\cdot,\cdot)].$$

Hence, by Lemma 1.2.5,

$$\|\tilde{u}\|_{L^2} \leq \|K\|_{L^2}[\|g(\cdot,\cdot,u)\|_{L^1} + \|h\|_{L^1}].$$

By taking the inner product with 1 on the both sides of $(1.3.2_\lambda)$, since $1 \in KerL$, we have

$$\iint_\Omega g(t,x,u(t,x))dtdx = \iint_\Omega h(t,x)dtdx.$$

Hence, by $(H_{1.2})$, we have $\|g(\cdot,\cdot,u)\|_{L^1} \le \|h\|_{L^1}$. Therefore, we have

$$\|\tilde{u}\|_{L^2} \le 2\|K\|_{L^2}\|h\|_{L^1} \equiv M.$$

The proof is complete. □

LEMMA 1.3.2. *If $(H_{1.1})$, $(H_{1.2})$, and $(H_{1.3})$ are satisfied, then there exists γ such that*

$$|\bar{u}| \le \gamma$$

holds for each possible weak doubly-periodic solution $u = \bar{u} + \tilde{u}$, with $\bar{u} \in \mathrm{Ker}L$ and $\tilde{u} \in \mathrm{Im}L$, of $(1.3.1_\lambda)$ where $\lambda \in [0,1]$.

Proof. Suppose there exist a sequence of weak doubly-periodic solutions $\{(u_n,\lambda_n)\}$ of $(1.3.1_{\lambda_n})$ with $\{|\bar{u}_n|\}$ is unbounded. Then (u_n,λ_n) is a solution of $(1.3.2_{\lambda_n})$ where $u_n = \bar{u}_n + \tilde{u}_n$ with $\bar{u}_n \in \mathrm{Ker}L$ and $\tilde{u}_n \in \mathrm{Im}L$. We may choose a subsequence, say again $\{\bar{u}_n\}$ such that $|\bar{u}_n| \to +\infty$ as $n \to +\infty$. Now suppose that $\bar{u}_n \to +\infty$ as $n \to +\infty$. Let $M_0 > 2\pi M$ where M is given in Lemma 1.3.1 and let

$$\Omega_n = \{(t,x)|\tilde{u}_n(t,x) \le -\frac{M_0}{4\pi^2}\}.$$

Then

$$\begin{aligned} 2\pi M &\ge \iint_\Omega |\tilde{u}_n(t,x)|dtdx \\ &\ge \iint_{\Omega_n} |\tilde{u}_n(t,x)|dtdx \\ &\ge \frac{M_0}{4\pi^2}|[\Omega_n]|. \end{aligned}$$

Therefore, $|[\Omega_n]| \le 4\pi^2\frac{2\pi M}{M_0}$ and hence $|[\Omega - \Omega_n]| = |\{(t,x)|u_n(t,x) > -\frac{M_0}{4\pi^2}\}| \ge 4\pi^2[1 - \frac{2\pi M}{M_0}] > 0$.

Since $\lim_{|u|\to+\infty} g(t,x,u) = +\infty$ uniformly on Ω, there exists $C > 0$ such that

$$g(t,x,u) > \frac{1}{|[\Omega - \Omega_n]|} \iint_\Omega h(t,x)dtdx$$

for all n if $|u| \geq C$.
Since $\bar{u}_n \to +\infty$, there exists $N > 0$ such that

$$\bar{u}_n \geq \frac{M_o}{4\pi^2} + C \quad \text{if} \quad n \geq N.$$

Hence, for $(t,x) \in \Omega - \Omega_n$ and $n \geq N$, we have

$$u_n(t,x) = \bar{u}_n + \tilde{u}_n(t,x) \geq C.$$

Thus, for $n \geq N$, we have

$$\iint_{\Omega-\Omega_n} g(t,x,u_n(t,x))dtdx > \iint_\Omega h(t,x)dtdx.$$

On the other hand, by taking the inner product with 1 on the both sides of $(1.3.1_{\lambda_n})$, we have

$$\iint_\Omega g(t,x,u_n(t,x))dtdx \leq \iint_\Omega h(t,x)dtdx.$$

Therefore, for $n \geq N$, by $(H_{1.2})$,

$$\begin{aligned}\iint_\Omega h(t,x)dtdx &= \iint_\Omega g(t,x,u_n(t,x))dtdx \\ &\geq \iint_{\Omega-\Omega_n} g(t,x,u_n(t,x))dtdx \\ &> \iint_\Omega h(t,x)dtdx\end{aligned}$$

which is impossible.

Similarly, we can treat the case where $\bar{u}_n \to -\infty$. The proof is complete. □

THEOREM 1.3.1. *Assume* $(H_{1.1})$, $(H_{1.2})$, *then the doubly-periodic boundary value problem on* Ω *for the equation* (1.1.1) *has at least two weak solutions if there exists a constant* $r_0 \in R$ *such that*

(1.3.3)
$$\iint_\Omega g(t,x,r_0+\tilde{u}(t,x))dtdx < \iint_\Omega h(t,x)dtdx$$

for every $\tilde{u} \in L^2(\Omega)$ *having mean value zero on* Ω, *satisfying the doubly-periodic conditions and such that*

$$\|\tilde{u}\|_{L^2} \le 2\|K\|_{L^2}\|h\|_{L^1}. \tag{1.3.4}$$

Proof. To prove our multiplicity result, we construct two disjoint bounded open sets G_1 and G_2 on which the coincidence degree is well-defined and non-zero, respectively.

Since $\lim_{|u|\to\infty} g(t,x,u) = \infty$ uniformly on Ω, there exists $\delta > 0$ such that
$$g(t,x,u) > \iint_\Omega h(t,x)dtdx$$
for all $|u| > \delta$ and uniformly on Ω.

Let
$$G_1 = \{u \in L^2(\Omega) | r_0 < \bar{u} < \bar{r} + \bar{M}, \|\tilde{u}\|_{L^2} < \bar{M}\}$$
where $u = \bar{u} + \tilde{u}$ with $\bar{u} \in \text{Ker}L, \tilde{u} \in \text{Im}L$ and $\bar{M}, \tilde{u}$ and $\bar{r}$ are constant such that $\bar{M} > M, \bar{r} > \max\{r, \delta\}$.

If $u \in \partial G_1$, then necessary $\bar{u} = r_0$ or $\bar{u} = \bar{r} + \bar{M}$ and if (u, λ) satisfies the equation $(1.3.2_\lambda)$, then (u, λ) satisfies

$$\iint_\Omega g(t,x,u(t,x))dtdx = \iint_\Omega h(t,x)dtdx. \tag{1.3.5}$$

If $\bar{u} = r_0$, then, from (1.3.3), we have a contradiction. If $\bar{u} = \bar{r} + \bar{M}$, then $u = \bar{r} + \bar{M} + \tilde{u}$. let

$$\Omega_0 = \{(t,x) | |\tilde{u}(t,x)| \ge \bar{M}\}.$$

Then

$$\begin{aligned} 2\pi\bar{M} &\ge \iint_\Omega |\tilde{u}(t,x)|dtdx \\ &\ge \iint_{\Omega_0} |\tilde{u}(t,x)|dtdx \\ &\ge |[\Omega_0]|\bar{M}. \end{aligned}$$

Therefore $|[\Omega_0]| \leq 2\pi$ and hence $|[\Omega - \Omega_0]| = |[\{(t,x) | |\tilde{u}(t,x)| < \bar{M}\}]| > 1$. Thus we have $|u| > \delta$ on $\Omega - \Omega_0$ and hence

$$\begin{aligned}\iint_\Omega g(t,x,u(t,x))dtdx &\geq \iint_{\Omega-\Omega_0} g(t,x,u(t,x))dtdx \\ &\geq |[\Omega - \Omega_0]| \iint_\Omega h(t,x)dtdx \\ &> \iint_\Omega h(t,x)dtdx\end{aligned}$$

which leads another contradiction. Therefore the coincidence degree $D_L(L-N,G_1)$ is well defined on Ω.

Now, since, for $u \in \text{Ker}L \cap \partial G_1$, we have $u = r_0$ or $u = \bar{r} + \bar{M}$, we conclude

$$(QN)(r_0) = \frac{1}{|[\Omega]|} \iint_\Omega [h(t,x) - g(t,x,r_0)]dtdx > 0,$$

$$(QN)(\bar{r}+\bar{M}) = \frac{1}{|[\Omega]|} \iint_\Omega [h(t,x) - g(t,x,\bar{r}+\bar{M})]dtdx < 0.$$

Hence, the coincidence degree exists and the corresponding value

$$D_L(L-N,G_1) = d_B[JQN, \text{Ker}L \cap G_1, 0] = 1$$

where d_B is Brouwer degree. Therefore, the equation $(1.3.2_1)$ has at least one solution in $\text{Dom}L \cap Cl(G_1)$.

Similary, we can prove that the equation $(1.3.2_1)$ has at least one solution in $\text{Dom}L \cap Cl(G_2)$ where

$$G_2 = \{u \in L^2(\Omega) | -(\bar{r}+\bar{M}) < \bar{u} < r_0, \|\tilde{u}\|_{L^2} < \bar{M}\}.$$

Since, by (1.3.3), $u \equiv r_0$ is not solution to $(1.3.2_1)$ and $Cl(G_1) \cap Cl(G_2) = \{r_0\}$, the doubly-periodic boundary value problem to the equation (1.1.3) has at least two weak solutions. □

REMARK 1.3.1. If

$$\frac{1}{|[\Omega]|} \iint_\Omega h(t,x)dtdx < \inf_{\substack{(t,x)\in\Omega \\ u \in R}} g(t,x,u),$$

then the doubly-periodic boundary value problem for the equation (1.1.3) has no solution. Indeed, let

$$g(t_0, x_0, u_0) = \inf_{\substack{(t,x)\in\Omega \\ u\in R}} g(t, x, u).$$

Let E be any open bounded set in ImL such that

$$E \supseteq \{\tilde{u} \in \text{Im}L | \|\tilde{u}\|_{L^2} \leq M\}$$

and let, for any $\delta > 0$, $G = (u_0 - \delta, u_0 + \delta) \bigoplus Q$. Suppose $u \in \partial G$ and (u, λ) satisfies the equation ($1.3.2_\lambda$), then (u, λ) satisfies (1.3.5). But $u = \bar{u} + \tilde{u}$ and

$$\iint_\Omega g(t, x, \bar{u} + \tilde{u}) dtdx \geq |[\Omega]| \inf_{\substack{(t,x)\in\Omega \\ u\in R}} g(t, x, u) > \iint_\Omega h(t, x)\, dtdx$$

which contradicts to (1.3.5). Therefore the coincidence degree $D_L(L - N, G)$ is well-defined. But, for any $u \in \text{Ker}L \cap G$,

$$(QN)(u) = \frac{1}{|[\Omega]|} \iint_\Omega [h(t, x) - g(t, x, u)] dtdx < 0$$

$$D_L(L - N, G) = d_B(JQN, \text{Ker}L \cap G, 0) = 0.$$

Therefore the double-periodic boundary value problem to the equation (1.1.3) has no solution.

Next we consider multiplicity result for equation (1.1.1) when the nonlinear term $g(t, x, u)$ depends only on u, i.e.

(1.3.6) $$\beta u_t + u_{tt} - u_{xx} + g(u) = h(t, x).$$

To treat our problem, let us consider the following doubly-periodic boundary value problem for a family of homotopy equations

($1.3.7_\lambda$)

$$\beta u_t + u_{tt} - u_{xx} + \lambda g(u) = \lambda h(t, x), \lambda \in [0.1]$$

where $g : R \to R$ is continuous and $h \in \text{Im}L$.

$(H'_{1.1})$ There exist $a, b > 0$ such that $|g(u)| \leq a|u| + b$ for all $u \in \mathbf{R}$.

Let $L : \text{Dom}L \subseteq L^2(\Omega) \to L^2(\Omega)$ be defined us before and define the substitution operator by

$$(N_\lambda)(t,x) = \lambda g(u(t,x)) - \lambda h(t,x)$$

for $u \in L^2(\Omega)$ and $(t,x) \in \Omega$. Then N_λ is L-compact on $\bar{G}$ for any open bounded subset G of $L^2(\Omega)$. Thus the coincidence degree $D_L(L = N_\lambda, G)$ is well defined and constant in λ if $Lu + N_\lambda u \neq 0$ for $\lambda \in [0,1]$ and $u \in \text{Dom}L \cap \partial G$. It is easy to check that (u, λ) is a weak doubly-periodic solution to $(1.3.7_\lambda)$ if and only if $u \in \text{Dom}L$ and

$$(1.3.8_\lambda) \qquad Lu + N_\lambda u = 0.$$

Here we assume the following.

$(H'_{1.2})$ $\lim_{|u| \to +\infty} g(u) = +\infty$,

$(H'_{1.3})$ there exists $0 < \alpha < 1$ such that

$$|g(u) - g(v)| \leq \frac{\alpha}{2\pi C_2}|u - v| \quad \text{for all} \quad u, v \in \mathbf{R},$$

where C_2 is a constant defined in Lemma 1.2.4.

LEMMA 1.3.3. *If $(H_{1.1})$ is satisfied, then there exists $M > 0$ such that*

$$\|\tilde{u}\|_{L^2} \leq M$$

holds for each possible weak solution $u = \bar{u} + \tilde{u}$, with $\bar{u} \in \text{Ker}L$ and $\tilde{u} \in \text{Im}L$, of $(1.3.7_\lambda)$ where $\lambda \in [0.1]$.

Proof. Let (u, λ) be any weak solution of $(1.3.7_\lambda)$ where $u = \bar{u} + \tilde{u}$ with $\bar{u} \in \text{Ker}L$ and $\tilde{u} \in \text{Im}L$.

By taking the inner product with $\tilde{u}_t$ on the both sides of $(1.3.7_\lambda)$, we have

$$(L\tilde{u}, \tilde{u}_t) + \lambda \iint_\Omega g(u)\tilde{u}_t dtdx = \lambda \iint_\Omega h(t,x)\tilde{u}_t dtdx.$$

Since $L\tilde{u} \in L^2(\Omega)$, there exists a sequence $\{\tilde{y}_n\}$ in $C^\infty(\Omega) \cap \text{Im}L$ such that $\tilde{y}_n \to L\tilde{u}$ in $L^2(\Omega)$ as $n \to +\infty$.

Let $\tilde{u}_n = K^R \tilde{y}_n$. By Lemma 1.2.3 and the Sobolev embedding theorem $W^{j+2.2}(\Omega) \hookrightarrow C^j(\Omega)$, $(j = 0, 1, 2, \dots)$, $\tilde{u}_n \in C^\infty(\Omega) \cap \text{Im}L$. Since K^R is continuous from $L^2(\Omega)$ into each of $W^{1.2}(\Omega)$ and $C(\Omega)$, we have that $\tilde{u}_n \to K^R(L\tilde{u})$ in each of those spaces as $n \to +\infty$.

Thus $\tilde{u}_{n_t} \to \tilde{u}_t$ in $L^2(\Omega)$. Integration of these smooth functions, using the boundary conditions, show that for each $n = 1, 2, 3, \ldots$,

$$(L\tilde{u}_n, \tilde{u}_{n_t}) = \beta \|\tilde{u}_{n_t}\|_{L^2}^2.$$

Letting $n \to +\infty$, we have $(L\tilde{u}_t, \tilde{u}_t) = \beta\|\tilde{u}_t\|_{L^2}^2$. Moreover, since, for each n, the periodicity of $\tilde{u}_n(t,x)$ in t implies $(g(u_n), \tilde{u}_{n_t}) = 0$, we have $(g(u), \tilde{u}_t) = 0$.

Hence, we have

$$\beta\|\tilde{u}_t\|_{L^2}^2 = \lambda(h, \tilde{u}_t)$$

and

$$\|\tilde{u}_t\|_{L^2}^2 \le \frac{1}{|\beta|}\|h\|_{L^2}.$$

But since $\|\tilde{u}\|_{L^2} \le \|\tilde{u}_t\|_{L^2}^2$ for all $\tilde{u} \in \mathrm{Dom}L \cap \mathrm{Im}L$, we have

$$\|\tilde{u}\|_{L^2} \le \frac{1}{|\beta|}\|h\|_{L^2}.$$

The proof is complete. □

THEOREM 1.3.2. *Assume* $(H_{1.1})$, $(H'_{1.2})$, *and* $(H'_{1.3})$. *Then the doubly-periodic boundary value problem on* Ω *for the equation* (1.3.6) *has at least two solutions if*

$$\inf_{\bar{u}\in R} \iint_\Omega g(\bar{u} + \tilde{u}(t,x))dtdx < \frac{1}{|[\Omega]|}\iint h(t,x)dtdx \tag{1.3.9}$$

for every $\tilde{u} \in L^2(\Omega)$ *having mean value zero on* Ω, *satisfying the doubly-periodic conditions such that*

$$\|\tilde{u}\|_{L^2} \le \|h\|_{L^2}.$$

Proof. It is easy to see that $(1.3.7_1)$ is equivalent to

$$L\tilde{u} + (I - Q)g(\bar{u} + \tilde{u}) = h - \bar{h} \tag{1.3.10}$$

$$Qg(\bar{u} + \tilde{u}) = \bar{h} \tag{1.3.11}$$

where $u = \bar{u} + \tilde{u}$ with $\bar{u} \in \mathrm{Ker}L$ and $\tilde{u} \in \mathrm{Im}L$, $\bar{h} = \frac{1}{\|\Omega\|} \iint_\Omega h(t,x)dtdx$ and Q is the continuous projection defined in Section 1.2. For fixed $\bar{u} \in R$, consider the equation (1.3.11). Define an operator $N : L^2(\Omega) \to \mathrm{Im}L$ by

$$(Nu)(t,x) = -(I-Q)g(\bar{u} + \tilde{u}(t,x)) + h - \bar{h}(t,x).$$

Then N is continuous and maps bounded sets into bounded sets. Since the inclusion mapping $i : C(\Omega) \to L^2(\Omega)$ is continuous, the right inverse $K^R : \mathrm{Im}L \to L^2(\Omega)$ is compact. Hence $K^R N : L^2(\Omega) \to L^2(\Omega)$ is completely continuous and (1.3.11) is equivalent to

$$\tilde{u} = K^R N \tilde{u}.$$

By Lemma 1.3.3, all possible solutions to the family of equations

$$\tilde{u} = \lambda K^R N \tilde{u}, \quad \lambda \in [0,1]$$

are bounded in $L^2(\Omega)$ independently of $\lambda \in [0,1]$.

Thus, by Leray-schauder's theory, (1.3.11) has at least one solution $\tilde{u}$ for each $\bar{u} \in \mathbf{R}$. Such a solution is unique. Indeed, if $\tilde{u}_1$ and $\tilde{u}_2$ are two different solutions with $\bar{u}$, then

$$L(\tilde{u}_1 - \tilde{u}_2) + (I-Q)[g(\bar{u} + \tilde{u}_1) - g(\bar{u} + \tilde{u}_2)] = 0.$$

Applying K^R on the both sides of the above equation, we have, by Lemma 1.2.4 and $(H'_{1.3})$,

$$\|\tilde{u}_1 - \tilde{u}_2\|_\infty \le \alpha \|\tilde{u}_1 - \tilde{u}_2\|_\infty$$

which is impossible since $0 < \alpha < 1$. Thus $\tilde{u}_1 = \tilde{u}_2$.

Denote this unique solution of (1.3.11) by $V(\bar{u})$, by Lemma 1.2.3, then $V : R \to C(\Omega) \cap \mathrm{Im}L$ is a continuous function.

If $\bar{u}, \bar{u}_0 \in R$, then

$$L[V(\bar{u}) - V(\bar{u}_0)] + (I-Q)[g(\bar{u} + V(\bar{u})) - g(\bar{u}_0 + V(\bar{u}_0))] = 0.$$

By Lemma 1.2.4 and $(H'_{1.3})$, we have

$$\|V(\bar{u}) - V(\bar{u}_0)\|_\infty \le \frac{\alpha}{1-\alpha} |\bar{u} - \bar{u}_0|.$$

Thus V is continuous.

By Lemma 1.3.3, $\|V(\bar{u})\|_{L^2} \leq M$ for all $\bar{u} \in R$. Let

$$\Omega_0 = \{(t,x) | |V(\bar{u})(t,x)| \geq \frac{1+M}{2\pi}\}.$$

Then

$$M^2 \geq \iint_\Omega |V(\bar{u})(t,x)|^2 dtdx \geq \left[\frac{1+M}{2\pi}\right]^2 |[\Omega_0]|.$$

Thus

$$|[\Omega_0]| \leq 4\pi^2 \left[\frac{1+M}{2\pi}\right]^2.$$

Let $\Omega_1 = \Omega - \Omega_0 = \{(t,x) | |V(\bar{u})(t,x)| \leq \frac{1+M}{2\pi}\}$, then $|[\Omega_1]| \geq 4\pi^2[1 - \frac{M}{1+M}]^2 > 0$.
Thus

$$\begin{aligned}\iint_\Omega g(\bar{u} + V(\bar{u})(t,x))dtdx &\geq \iint_\Omega [g(\bar{u} + V(\bar{u})(t,x)) - \alpha]dtdx + 4\pi^2\alpha \\ &\geq \iint_{\Omega_1} [g(\bar{u} + V(\bar{u})(t,x)) - \beta]dtdx + 4\pi^2\beta\end{aligned}$$

where $\alpha = \min_{u \in R} g(u)$.
Therefore, by $(H'_{1.2})$,

$$\iint_\Omega g(\bar{u} + V(\bar{u})(t,x))dtdx \to +\infty \quad \text{as} \quad |\bar{u}| \to +\infty.$$

Define $G : R \to R$ by

$$G(\bar{u}) = Qg(\bar{u} + V(\bar{u})) = \frac{1}{4\pi^2} \iint_\Omega g(\bar{u} + V(\bar{u})(t,x))dtdx,$$

G is continuous by the continuity of V and, by $(H'_{1.2})$, $G(\bar{u}) \to +\infty$ as $|\bar{u}| \to +\infty$.
Equation $(1.3.5_\lambda)$ is then reduced to the scalar equation in $\bar{u}$;

$$G(\bar{u}) = Qg(\bar{u} + V(\bar{u})) = \bar{h}. \tag{1.3.12}$$

Let $h_1 = \inf_{u \in R} G(\bar{u})$, then $\text{Im}G = [h_1, +\infty[$.

If $G(u_0) = h_1$, then from (1.3.9), we may easily prove (1.3.6) has one solution in $]-\infty, \bar{u}_0[$ and one in $]\bar{u}_0, +\infty[$ by intermediate value theorem. This completes the proof. □

REMARK 1.3.2. We may see easly that if $\bar{h} < h_1$, clearly (1.3.6) has no solution.

2. Dirichlet-periodic problem in n-space dimension

In this chapter we will discuss the multiplicity result for weak Dirichlet-periodic problem for dissipative hyperbolic equations with coercive growth nonlinearity in n-dimensional space. Here we have no restrictions on the dimension of domain.

2.1. Introduction

Let R be the set of all reals and $\Omega \subseteq R^n$, $n \geq 1$, be a bounded domain with smooth boundary $\partial\Omega$ which is assumed to be of class C^2.

Let $Q = (0, 2\pi) \times \Omega$ and $L^2(Q)$ be the space of measurable and Lebesgue square integrable real-valued functions on Q with usual inner product $< \cdot, \cdot >$ and corresponding norm $\| \cdot \|_2$.

By $H_0^1(\Omega)$ we mean the completion of $C_0^1(\Omega)$ with respect to the norm $\| \cdot \|_1$ defined by

$$\|\phi\|_1^2 = \int_\Omega \sum_{|\alpha| \leq 1} |D^\alpha \phi(x)|^2 dx.$$

$H^2(\Omega)$ stands for the usual Sobolev space; i.e., the completion of $C^2(\bar{\Omega})$ with respect to the norm $\| \cdot \|_2$ defined by

$$\|\phi\|_2^2 = \int_\Omega \sum_{|\alpha| \leq 2} |D^\alpha \phi(x)|^2 dx.$$

Let $g : R \to R$ be a continuous function. Moreover, we assume that there exist constants a_0 and b_0 such that

$$(H_{2.1}) \qquad |g(u)| \leq a_0|u| + b_0 \quad \text{for all } u \in R.$$

The purpose of this work is to investigate the multiplicity for periodic solutions of the semilinear hyperbolic equations

$$(\mathrm{E}) \qquad \beta\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} - \triangle_x u - \lambda_1 u + g(u) = h(t, x) \ \text{ in } \ Q,$$

$$(B_1) \qquad u(t, x) = 0 \ \text{ on } \ (0, 2\pi) \times \partial\Omega,$$

$$(B_2) \qquad u(0, x) = u(2\pi, x) \ \text{ on } \ \Omega$$

where λ_1 and λ_2 denotes the first and second eigenvalues of $-\Delta$ with zero Dirichlet boundary data and ϕ_1 is the positive normalized eigenfunction corresponding to λ_1 and $h \in L^2(Q)$.

The purpose of this paper is to give a multiplicity result for semilinear dissipative hyperbolic equations. Originally, the linear dissipative hyperbolic equations are derived from physical principle(see [4]). The existence and asymptotic theory of dissipative hyperbolic equations have been developed by several authors for initial value problems, boundary value problems, or mixed problems. For information on dissipative hyperbolic equations, we refer to [27]. On the existence of doubly-periodic solutions of semilinear dissipative hyperbolic equations have been done by Mawhin [24], Fucik and Mawhin [9]. Mawhin treat the existence of double-periodic solutions for semilinear dissipative hyperbolic equations of several types of $g(u)$ with at most linear growth in connection with the set $\Sigma = \{k^2 - j^2 | k, j \text{ integers}\}$. Fucik and Mawhin consider also the existence double-periodic solutions of semilinear dissipative hyperbolic equations with nonlinear term of the form $g(u) = \mu u^+ - \nu u^- - \phi(u)$, where ϕ is a continuous and bounded function, and μ, ν are real numbers related to the set Σ. In [11, 16], the existence of solutions for Dirichlet-periodic problem for semilinear dissipative hyperbolic equations at resonance, in [15, 21], the existence of Dirichlet-periodic solutions for semilinear dissipative hyperbolic problems with superlinear growth, in [17], the existence of double-periodic solutions for semilinear dissipative hyperbolic equations with non-decreasing type of non-linear term, in [20, 22], the multiple existence of double-periodic and Dirichlet-periodic problem, respectivelly, for semilinear dissipative hyperbolic equations and, in [18], the asymptotic behavior of Dirichlet-initial problem of semi-linear dissipative hyperbolic equations are discussed. Our result is related to the results in [20, 22] which are so called the Ambrosetti-Prodi type multiplicity result which has been initiated by Ambrosetti-Prodi [1] in the study of a Dirichlet problem to elliptic equations and developed in various directions by several authors to ordinary and partial differential equations. For more information on this problem for semilinear elliptic, parabolic and ordinary equations, we refer to [3, 5, 7, 12, 13, 14, 19, 23] and their references.

In our result, we will treat a multiplicity result for Dirichlet-periodic solutions of semilinear dissipative hyperbolic equations in n-dimensional space. We assume the coercive growth on g with restriction on the left-hand and our proof based on Mawhin's continuation theorem in [10].

2.2. Preliminary results

Let's define the linear operator

$$L : \mathrm{Dom}L \subseteq L^2(Q) \to L^2(Q)$$

by

$$\mathrm{Dom}L = \{u \in L^2((0,2\pi), H^2(\Omega) \cap H_0^1(\Omega)) | \frac{\partial u}{\partial t} \in L^2(Q),$$
$$\frac{\partial^2 u}{\partial t^2} \in L^2(Q), u(0,x) = u(2\pi, x), x \in \Omega\}$$

and

$$Lu = \beta \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} - \Delta u - \lambda_1 u.$$

Using Fourier series and Parseval inequality, we get easily

$$\langle Lu, \frac{\partial u}{\partial t} \rangle = \beta \| \frac{\partial u}{\partial t} \|_{L^2}^2 \quad \text{for all} \quad u \in \mathrm{Dom}L.$$

Hence $\mathrm{Ker}L = \mathrm{Ker}(\Delta + \lambda_1 I) = \mathrm{Ker}L^*$ since $\Delta + \lambda_1 I$ is self-adjoint and $\mathrm{Ker}(\Delta + \lambda_1 I)$ is one space dimension generated by the eigenfunction ϕ_1. Therefore L is a closed, densely defined linear operator and $\mathrm{Im}(L) = [\mathrm{Ker}L]^\perp$; i.e., $L^2(Q) = \mathrm{Ker}L \bigoplus \mathrm{Im}L$. Let's consider a continuous projection $P_1 : L^2(Q) \to L^2(Q)$ such that $\mathrm{Ker}P_1 = \mathrm{Im}L$. Then $L^2(Q) = \mathrm{Ker}L \bigoplus \mathrm{Ker}P_1$. We consider another continuous projection $P_2 : L^2(Q) \to L^2(Q)$ defined by

$$(P_2 h)(t,x) = \frac{1}{2\pi} \iint_Q h(t,x)\phi(x) dt dx \phi(x).$$

Then we have $L^2(Q) = \mathrm{Im}P_1 \bigoplus \mathrm{Im}L$, $\mathrm{Ker}P_2 = \mathrm{Im}L$, and $L^2(Q)/\mathrm{Im}L$ is isomorphism to $\mathrm{Im}P_2$.

Since $\dim[L^2(Q)/\mathrm{Im}L] = \dim[\mathrm{Im}P_2] = \dim[\mathrm{Ker}L] = 1$, we have an isomorphism $J : \mathrm{Im}P_2 \to \mathrm{Ker}L$.

By the closed graph theorem, the generalized right inverse of L defined by

$$K = [L|_{\mathrm{Dom}L \cap \mathrm{Im}L}]^{-1} : \mathrm{Im}L \to \mathrm{Im}L$$

is continuous. If we equip the space $\mathrm{Dom}L$ with the norm

$$\|u\|_{\mathrm{Dom}L} = \iint_Q \left[u^2 + (\frac{\partial u}{\partial t})^2 + (\frac{\partial^2 u}{\partial t^2})^2 + \sum_{|\beta| \le 2} (D_x^\beta u)^2 \right] dt dx.$$

Then there exist a constant $c > 0$ independently of $h \in \mathrm{Im}L$, $u = Kh$ such that

$$\|Kh\|_{\mathrm{Dom}L} \leq c\|h\|_{L^2}.$$

Therefore $K : \mathrm{Im}L \to \mathrm{Im}L$ is continuous and by the compact imbedding of $\mathrm{Dom}L$ in $L^2(Q)$, we have that $K : \mathrm{Im}L \to \mathrm{Im}L$ is compact.

LEMMA 2.2.1. *L is closed, densely defined linear operator such that $\mathrm{Ker}L = [\mathrm{Im}L]^{\perp}$ and such that the right inverse $K : \mathrm{Im}L \to \mathrm{Im}L$ is completely continuous.*

Proof. See [2, 26]. □

2.3. Multiplicity result

Let us consider the following

$$(E_h^\mu) \qquad \beta\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} - \Delta_x u - \lambda_1 u + \mu g(u) = \mu h(t,x) \text{ in } Q,$$

$$(B_1) \qquad u(t,x) = 0 \text{ on } (0, 2\pi) \times \partial\Omega,$$

$$(B_2) \qquad u(0,x) = u(2\pi, x) \text{ on } \Omega$$

where $\mu \in [0,1]$.

Let $L : \mathrm{Dom}L \subseteq L^2(Q) \to L^2(Q)$ be defined as before. If we define a substitution operator $N_h^\mu : L^2(Q) \to L^2(Q)$ by

$$(N_h^\mu)(t,x) = \mu g(u) - \mu h(t,x)$$

for $u \in L^2(Q)$ and $(t,x) \in Q$, then N_h^μ maps continuously into itself and take bounded sets into bounded set. Let G be any open bounded subset of $L^2(Q)$, then $P_2 N_h^\mu : \bar{G} \to L^2(Q)$ is bounded and $K(I - P_2) : \bar{G} \to L^2(Q)$ is compact and continuous. Thus N_h^μ is L-compact on $\bar{G}$.

The coincidence degree $D_L(L + N_h^\mu, G)$ is well defined and constant in μ if $Lu + N_h^\mu u \neq 0$ for $\mu \in [0,1]$ and $u \in \mathrm{Dom}L \cap \partial G$. It is easy to check that (u, μ) is a weak solution of (E_h^μ) if and only if $u \in \mathrm{Dom}L$ and

$$(2.3.1_h^\mu) \qquad Lu + N_h^\mu u = 0.$$

Here, we assume the following

$$(H_{2.2}) \qquad \lim_{|u|\to\infty} \inf g(u) = +\infty,$$

$$(H_{2.3}) \qquad \lim_{u\to-\infty} \sup \left|\frac{g(u)}{u}\right| < \lambda_2 - \lambda_1.$$

From $(H_{2.2})$ and $(H_{2.3})$, we may assume that

$$m = \inf_{u\in R} g(u) > 0$$

and there exist $a \in (0, \lambda_2 - \lambda_1)$ and $b \geq 0$ such that

$$|g(u)| \leq a|u| + b \text{ for all } \ u \leq 0.$$

For $h \in L^2(Q)$, we write $\wedge h = \iint_Q h(t,x)\phi(x)dtdx$.

LEMMA 2.3.1. *If $(H_{2.1})$, $(H_{2.2})$, and $(H_{2.3})$ are satisfied, then, for each $h^* \in L^2(Q)$, there exists $M(h^*) > 0$ independently of μ such that*

$$\|\tilde{u}\|_{L^2} \leq M$$

holds for each possible weak solution $u = \alpha\phi + \tilde{u}$, with $\alpha \in R$ and $\tilde{u} \in \mathrm{Im}L$, of (E_h^{μ}) with $\mu \in [0.1]$, and with $\wedge h \leq \wedge h^*$ and $\|h\|_{L^2} \leq \|h^*\|_{L^2}$.

Proof. Suppose there exists $h \in L^2(Q)$ with $\wedge h \leq \wedge h^*$ and $\|h\|_{L^2} \leq \|h^*\|_{L^2}$ and the corresponding sequence of solutions $\{(u_n, \mu_n)\}$, with $\mu \in [0,1]$, of $(2.3.1_h^{\mu_n})$ such that

$$\lim_{n\to\infty} \|\tilde{u}_n\|_{L^2} = \infty,$$

then clearly

$$\lim_{n\to\infty} \|u_n\|_{L^2} = \infty.$$

For each $n \geq 1$, we put $u_n(t,x) = \alpha_n\phi(x) + \tilde{u}_n(t,x)$.

First, we are going to prove that

$$\lim_{n\to\infty} \frac{|\alpha_n|}{\|\tilde{u}_n\|_{L^2}} = c < \infty.$$

If it is not the case, we may assume that, by extracting subsequence if it is necessary,

$$\lim_{n\to\infty} \frac{\|\tilde{u}_n\|_{L^2}}{|\alpha_n|} = 0.$$

Therefore, we may have a subsequence, say again, $\{\tilde{u}_n\}$ such that we have easily

$$\lim_{n\to\infty} |u_n(t,x)| = \infty \text{ a.e. on } Q.$$

By taking the inner product with ϕ on both sides of $(2.3.1^{\mu}_{h})$, we have

$$\iint_Q g(u_n(t,x))\phi(x)dtdx = \iint_Q h\phi(x)dtdx \leq \wedge h^*.$$

On the other hand, by $(H_{2.2})$ and Fatou's lemma, we have

$$\lim_{n\to\infty} \iint_Q g(u_n(t,x))\phi(x)dtdx = \infty$$

which leads to a contradiction. First, we assume that $0 < c < \infty$, then there exist $n_0 \in N$ such that

$$(c/2)\|\tilde{u}_n\|_{L^2} \leq |\alpha_n| \leq (3c/2)\|\tilde{u}_n\|_{L^2} \text{ for all } n \geq n_0.$$

For given $\epsilon > 0$, we may choose $\delta > 0$ such that

$$\iint_A |\phi|^2 dtdx < \epsilon \|\phi\|^2_{L^2}$$

for any measurable set $A \subset \bar{Q}$ with $|A| \leq \delta$.

Let $0 < \gamma < \|\phi\|_\infty$ and $\Omega_0 = \{x \in \Omega : \phi(x) \geq \gamma\}$. Choose $M_0 > 0$ such that

$$\delta M_0 - |m| \iint_Q \phi dtdx > \iint_Q h^* \phi(x)dtdx.$$

Then, since $\lim_{u\to\infty} g(u) = \infty$, we have that

$$m_0 = \sup\{|u| : \gamma g(u) < M_0\} < \infty.$$

We put

$$Q_n = \{(t,x) \in [0,2\pi] \times \Omega_0 : |u_n(t,x)| \geq m_0\}.$$

Then we have $|Q_n| \leq \delta$. In fact, if $|Q_n| > \delta$, then from the definition of m_0 we have

$$\begin{aligned}
&\iint_Q g(u_n(t,x))\phi(x)dtdx \\
&= \iint_{Q_n} g(u_n)\phi(x)dtdx + \iint_{Q\setminus Q_n} g(u_n)\phi(x)dtdx \\
&> \delta M_0 - m \iint_Q \phi(x)dtdx \\
&> \iint_Q h^* \phi(x)dtdx
\end{aligned}$$

and this leads to a contradiction. Therefore, we have

$$\iint_{Q\setminus Q_n} |\alpha_n \phi|^2 \geq (1-\epsilon) \iint_Q |\alpha_n \phi|^2.$$

On the other hand,

$$\begin{aligned}
0 &= \iint_Q \alpha_n \phi \tilde{u}_n \\
&= \iint_{Q\setminus Q_n} \alpha_n \phi \tilde{u}_n + \iint_{Q_n} \alpha_n \phi \tilde{u}_n \\
&\leq (1/2) \iint_{Q\setminus Q_n} (|\alpha_n \phi + \tilde{u}_n|^2 - |\alpha_n \phi|^2 - |\tilde{u}_n|^2) + \iint_{Q_n} |\alpha_n \phi||\tilde{u}_n|.
\end{aligned}$$

From the definition of m_0 and the above facts, we have, for all $n \geq n_0$,

$$\begin{aligned}
0 &\leq (1/2)m_0^2 - (1/2)(1-\epsilon)(c/2)\|\tilde{u}_n\|_{L^2}^2 + \epsilon(3c/2)\|\tilde{u}_n\|_{L^2}^2 \\
&= (1/2)m_0^2 - (c/4)(1+5\epsilon c)\|\tilde{u}_n\|_{L^2}^2.
\end{aligned}$$

Therefore, $\{\|\tilde{u}_n\|_{L^2}\}$ is bounded which leads to a contraction.

Next, we assume $c = 0$, then $\lim_{n\to\infty} \frac{\|\tilde{u}_n\|}{\|u_n\|_{L^2}} = 1$.
Multiplying $(2.3.1_h^\mu)$ by $\frac{\partial u}{\partial t}$ and integrate over Q, we find from the periodicity of u that

$$\|\frac{\partial u}{\partial t}\|_{L^2} \leq \frac{1}{|\beta|}\|h\|_{L^2}.$$

Again, taking the inner product with u_n on both sides of $(2.3.1^{\mu}_{h})$, we have

$$(\lambda_2 - \lambda_1)\|\tilde{u}_n\|^2_{L^2} - \|\frac{\partial u_n}{\partial t}\|^2_{L^2} + < g(u_n), u_n > \leq \|h\|_{L^2}\|\tilde{u}_n\|_{L^2}$$

and hence

$$\lim_{n\to\infty} \sup(\lambda_2 - \lambda_1 - a)\|\tilde{u}_n\|^2_{L^2} \leq \left[\max\{m, b\}|Q|^{1/2} + \frac{1}{|\beta|^2}\|h^*\|^2_{L^2} + \|h^*\|_{L^2}\right]$$

Thus $\{\|\tilde{u}_n\|_{L^2}\}$ is bounded which leads to another contradiction. □

LEMMA 2.3.2. *If $(H_{2.1})$, $(H_{2.2})$, and $(H_{2.3})$ are satisfied, then, for each $h^* \in L^2(Q)$, there exists $r = r(h^*) > 0$ independently of μ such that*

$$|\bar{u}| \leq r$$

holds for each possible weak solution $u = \bar{u} + \tilde{u}$, with $\bar{u} = \alpha\phi(x)$, $\alpha \in R$ and $\tilde{u} \in \mathrm{Im}L$, of $(2.3.1^{\mu}_{h})$ where $\mu \in [0,1]$, and with $\wedge h \leq \wedge h^*$ and $\|h\|_{L^2} \leq \|h^*\|_{L^2}$.

Proof. Suppose there exists $h \in L^2(Q)$ with $\wedge h \leq \wedge h^*$ and $\|h\|_{L^2} \leq \|h^*\|_{L^2}$, and the corresponding sequence of weak solutions $\{(u_n, \mu_n)\}$ of $(2.3.1^{\mu_n}_{h})$ with $\{|\bar{u}_n|\}$ is unbounded. Then (u_n, μ_n) is a solution of $(2.3.1^{\mu_n}_{h})$ where $u_n = \bar{u}_n + \tilde{u}_n$ with $\bar{u}_n = \alpha_n\phi(x)$ and $\tilde{u}_n \in \mathrm{Im}L$. We may choose a subsequence, say again $\{\bar{u}_n\}$ with $\bar{u}_n = \alpha_n\phi(x)$ such that $|\alpha_n| \to +\infty$ as $n \to +\infty$. Now, let $\tilde{M} > M$ which is given in Lemma 2.3.1. Let

$$Q_n = \left\{(t,x) \in Q |\, |\tilde{u}_n(t,x)| \geq \frac{1+\tilde{M}}{|Q|}\right\}.$$

Then

$$\begin{aligned}\tilde{M}^2 &\geq \iint_Q |\tilde{u}_n(t,x)|^2 dtdx \\ &\geq \iint_{Q_n} |\tilde{u}_n(t,x)|^2 dtdx \\ &\geq |Q_n| \left[\frac{1+\tilde{M}}{|Q|}\right]^2.\end{aligned}$$

Therefore $|Q_n| \le [\frac{\tilde{M}}{1+\tilde{M}}]^2|Q|$ and hence $|Q \setminus Q_n| = |\{(t,x) \in Q||\tilde{u}(t,x)| \le \frac{1+\tilde{M}}{|Q|}\}| \ge [1 - \frac{\tilde{M}}{1+\tilde{M}}]^2|Q| > 0$.

Let $W = (0.2\pi) \times \Omega_0$. Then we have $|\alpha_n\phi(x)| \to \infty$ for each $x \in \Omega_0$ as $n \to \infty$. Hence, by Fatou's lemma and $(H_{2.2})$, we have

$$\begin{aligned}
&2\liminf_{n\to\infty} \iint_Q g(\alpha_n\phi(x) + \tilde{u}(t,x))\phi(x)dtdx \\
&= \liminf_{n\to\infty} \iint_Q g(\alpha_n\phi(x) + \tilde{u}(t,x))\phi(x)dtdx \\
&\ge \iint_{W\cap(Q\setminus Q_n)} \liminf_{n\to\infty} g(\alpha_n\phi(x) + \tilde{u}(t,x))\phi(x)dtdx \\
&= \infty.
\end{aligned}$$

Hence, there exists $r_0(h^*) > 0$ such that, for $|\alpha_n| > r_0$, we have

$$\iint_Q g(\alpha_n\phi(x) + \tilde{u}_n(t,x))\phi(x)dtdx > \iint_Q h^*\phi(x)dtdx. \tag{2.3.1}$$

On the other hand, by taking the inner product with $\phi(x)$ on the both sides of $(2.3.1_h^{\mu_n})$, we have

$$\iint_Q g(\alpha_n\phi(x) + \tilde{u}_n(t,x))\phi(x)dtdx = \iint_Q h\phi(x)dtdx \le \wedge h^*$$

which is impossible. The proof is complete. □

LEMMA 2.3.3. *If $(H_{2.1})$, $(H_{2.2})$, and $(H_{2.3})$ are satisfied, then, for each $h^* \in L^2(Q)$, we can find an open bounded set $G(h^*)$ in $L^2(Q)$ such that, for each open bounded set G in $L^2(Q)$ such that $G \supseteq G(h^*)$, we have*

$$D_L(L + N_h^1, G) = 0 \quad \textit{for all} \quad h \in L^2(Q)$$

with $\wedge h \le \wedge h^*$ and $\|h\|_{L^2} \le \|h^*\|_{L^2}$.

Proof. By similar fashion as we did in the proof of Lemma 2.3.2 to get (2.3.1), there exists $\bar{r}(h^*) > 0$ such that, for $|\alpha| > \bar{r}$, we have

$$\iint_Q g(\alpha\phi(x))\phi(x)dtdx > \iint_Q h^*\phi(x)dtdx.$$

Let

$$G(h^*) = \{u \in L^2(Q) | -\tilde{r}\phi(x) < \alpha\phi(x) < \tilde{r}\phi(x) \text{ for } x \in \Omega, \|\tilde{u}\|_{L^2} < \tilde{M}\}$$

where $u = \alpha\phi(x) + \tilde{u}$ with $\tilde{r}(h^*) > \max\{r(h^*), r_0(h^*), \bar{r}(h^*)\}$ and $\tilde{M} > M$ which are given in Lemma 2.3.1 and Lemma 2.3.2. If $(2.3.1^{\mu}_{\bar{h}})$ has a solution u for some $\bar{h} \in L^2(Q)$ such that $\wedge\bar{h} < 2\pi m \int_\Omega \phi(x)dx$ and $\mu \in [0,1]$, then by taking the inner product with ϕ on the both sides of the equation $(2.3.1^{\mu}_{\bar{h}})$, we have

$$2\pi m \int_\Omega \phi(x)dx \leq \iint_Q g(u(t,x))\phi(x)dtdx = \iint_Q \bar{h}\phi(x)dtdx.$$

Thus $(2.3.1^{\mu}_{\bar{h}})$ has no solution for $\bar{h} \in L^2(Q)$ such that $\wedge\bar{h}<2\pi m \int_\Omega \phi(x)dx$.
Hence, for each open bounded set $G \supseteq G(h^*)$, we have

$$D_L(L + N^1_{\bar{h}}, G) = 0 \quad \text{for} \quad \bar{h} \in L^2(Q)$$

such that $\wedge\bar{h} < 2\pi m \int_\Omega \phi(x)dx$. Choose $\bar{h} \in L^2(Q)$ with

$$\wedge\bar{h} \leq 2\pi m \int_\Omega \phi(x)dx \quad \text{and} \quad \|\bar{h}\|_{L^2} \leq \|h^*\|_{L^2},$$

and define

$$F : (D(L) \cap G) \times [0,1] \to L^2(Q) \text{ by}$$
$$F(u,\lambda) = Lu + N_{(1-\lambda)\bar{h}+\lambda h}(u) \quad \text{for} \quad h \in L^2(Q)$$

with $\wedge h \leq \wedge h^*$ and $\|h\|_{L^2} \leq \|h^*\|_{L^2}$. Then by Lemma 2.3.1 and Lemma 2.3.2, we have

$$0 \notin F(D(L) \cap \partial G) \times [0,1] \quad \text{for} \quad h \in L^2(Q)$$

with $\wedge h \leq \wedge h^*$ and $\|h\|_{L^2} \leq \|h^*\|_{L^2}$. By the homotopy invariance of degree, we have, for all $h \in L^2(Q)$ with $\wedge h \leq \wedge h^*$ and $\|h\|_{L^2} \leq \|h^*\|_{L^2}$,

$$\begin{aligned} D_L(L + N^1_h, G) &= D_L(F(\cdot,1), G) \\ &= D_L(F(\cdot,0), G) \\ &= D_L(L + N^1_{\bar{h}}, G) \\ &= 0 \end{aligned}$$

and the proof is completed. □

THEOREM 2.3.1. *Assume* $(H_{2.1})$, $(H_{2.2})$, *and* $(H_{2.3})$. *Then there exists a constant* α_0 *such that the boundary value problem* (E), (B_1), (B_2) *has at least two solutions for* h *such that*

(2.3.2)
$$\iint_Q g(\alpha_0\phi(x)+\tilde{u}(t,x))\phi(x)dtdx < \iint_Q h\phi(x)dtdx$$

for every $\tilde{u}\in L^2(\Omega)$ *having mean value zero on* Ω, *satisfying the conditions* (B_1) *and* (B_2) *such that*

$$\|\tilde{u}\|_{L^2} < \tilde{M}. \tag{2.3.3}$$

where $\tilde{M}$ *is given Lemma 2.3.3.*

Proof. Let

$$g(\alpha_0\phi(x_0)+\tilde{u}_0) = \min_{\substack{x\in\bar{\Omega}\\ |\alpha|\le\tilde{r}\\ |\tilde{u}|\le\tilde{M}}} g(\alpha\phi(x)+\tilde{u}).$$

Define

$$\Delta(G(h)) = \{u\in L^2(Q) | \alpha_0\phi(x) < \alpha\phi(x) < \tilde{r}_0\phi(x) \text{ for } x\in\Omega, \|\tilde{u}\|_{L^2} < \tilde{M}\}$$

where $\tilde{r}_0(h) > \tilde{r}$ which is given in Lemma 2.3.3.

If $u\in\partial\Delta G(h)$, then necessary $u=\alpha_0\phi(x)+\tilde{u}$ or $u=\tilde{r}_0\phi(x)+\tilde{u}$. If $u=\alpha_0\phi(x)+\tilde{u}$ with $\|\tilde{u}\|_{L^2}<\tilde{M}$, then, by taking inner product with ϕ on the both sides of $(2.3.1^{\mu}_{h})$, we have

$$\iint_Q g(\alpha_0\phi(x)+\tilde{u}(t,x))\phi(x)dtdx = \iint_\Omega h\phi(x)dtdx$$

which, from (2.3.2) and (2.3.3), is impossible. If $u=\tilde{r}_0\phi(x)+\tilde{u}$ with $\|\tilde{u}\|_{L^2}<\tilde{M}$, then, by the choice of $\tilde{r}_0>0$, we have

$$\iint_Q g(\tilde{r}_0\phi(x)+\tilde{u})\phi(x)dtdx > \iint_\Omega h\phi(x)dtdx$$

which is also impossible. Thus for $\mu\in[0,1]$, $D_L(L+N^{\mu}_{h},\Delta G(h))$ is well defined and

$$D_L(L+N^{\mu}_{h},\Delta G(h)) = D_B(JP_2N^{\mu}_{h},\Delta G(h)\cap \operatorname{Ker}L, 0)$$

where D_B is Brouwer degree and $P_2N_h^\mu : L^2(Q) \to \mathrm{Ker}L$ is an operator defined by

$$(P_2N_h^\mu u)(t,x) = \mu \left[\iint_Q g(u(t,x))\phi(x)dtdx - \iint_\Omega hdtdx \right] \phi(x).$$

Now let $T : \mathrm{Ker}L \to R$ be defined by

$$T(\alpha\phi(x)) = \alpha.$$

Then, for $\mu = 1$,

$$\begin{aligned} D_L(L + N_h^1, \Delta G(h)) &= D_B(JP_2N_h^1, \Delta G(h) \cap \mathrm{Ker}L, 0) \\ &= D_B(T(JP_2N_h^1)T^{-1}, T(\Delta G(h)) \cap \mathrm{Ker}L), 0). \end{aligned}$$

If we let $J : \mathrm{Im}P_2 \to \mathrm{Ker}L$ be the identity map, then the operator $\Phi = T(JP_2N_h^1)T^{-1}$ will be defined by

$$\Phi(\alpha) = \iint_Q g(\alpha\phi(x))\phi(x)dtdx - \iint_Q h\phi(x)dtdx.$$

Thus, we have

$$\Phi(\alpha_0) = \iint_Q g(\alpha_0\phi(x))\phi(x)dtdx - \iint_Q h\phi(x)dtdx < 0$$

and by the choice of $\tilde{r}_0$, we have

$$\Phi(\tilde{r}_0) = \iint_Q g(\tilde{r}_0\phi(x))\phi(x)dtdx - \iint_Q h\phi(x)dtdx > 0.$$

Hence, the coincidence degree exists and the corresponding value

$$|D_L(L - N, \Delta G(h))| = |D_B[JP_2N_h^1, \Delta \cap \mathrm{Ker}L, 0]| = 1.$$

Therefore, the equation $(2.3.1_h^1)$ has at least one solution in $\Delta G(h)$.

Choose $G \supsetneq \Delta G(h)$, where G is defined in Lemma 2.3.3. By the additivity of degree, we have

$$0 = D_L(L + N_h^1, G) = D_L(L + N_h^1, \Delta G(h))) + D_L(L + N_h^1, G - \overline{\Delta G(h)})$$

and hence

$$|D_L(L + N_h^1, G - \overline{\Delta G(h)})| = 1.$$

Therefore $(2.3.1_h^1)$ has another solution in $G - \overline{\Delta G(h)}$. This proves our assertion. □

REMARK. If

$$\frac{1}{|\Omega|} \iint_Q h(t,x)\phi_1(x)dtdx < \inf_{u \in R} g(u),$$

then the boundary value problem $(E), (B_1), (B_2)$ has no solution.

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Department of Mathematics
College of Natural Sciences
Hanyang University
Seoul 133-791, Korea
E-mail: wanskim@email.hanyang.ac.kr