THE RANDERS CHANGES OF FINSLER SPACES
WITH $\langle \alpha, \beta \rangle$-METRICS OF DOUGLAS TYPE

HONG-SUH PARK AND IL-YONG LEE

ABSTRACT. A change of Finsler metric $L(x, y) \rightarrow \bar{L}(x, y)$ is called a Randers change of $L$, if $\bar{L}(x, y) = L(x, y) + \rho(x, y)$, where $\rho(x, y) = \rho_i(x)y^i$ is a 1-form on a smooth manifold $M^n$. Let us consider the special Randers change of Finsler metric $L \rightarrow \bar{L} = L + \beta$ by $\beta$. On the basis of this special Randers change, the purpose of the present paper is devoted to studying the conditions for Finsler space $\bar{F}^n$ which are transformed by a special Randers change of Finsler spaces $F^n$ with $\langle \alpha, \beta \rangle$-metrics of Douglas type to be also of Douglas type, and vice versa.

1. Introduction

An $n$-dimensional Finsler space $F^n$ is a Douglas space or of Douglas type if and only if the Douglas tensor vanishes identically. Recently R. Bácsó and M. Matsumoto ([2]) have introduced the notion of Douglas space as a generalization of Berwald space from the viewpoint of geodesic equations. The conditions for some Finsler spaces with an $\langle \alpha, \beta \rangle$-metric to be Douglas space are obtained by M. Matsumoto ([8]).

A change of Finsler metric $L(x, y) \rightarrow \bar{L}$ is called a Randers change of $L$, if $\bar{L}(x, y) = L(x, y) + \rho(x, y)$, where $\rho(x, y) = \rho_i(x)y^i$ is a 1-form on a smooth manifold $M^n$. The notion of a Randers change has been proposed by M. Matsumoto ([5]). If $L(x, y)$ is a Riemannian metric, then $\bar{L}(x, y)$ becomes the Randers metric.

The purpose of the present paper is to study the Randers change of the Finsler space which is Douglas type. After the section 4, we consider

2000 Mathematics Subject Classification: 53B40.
Key words and phrases: Douglas space, Finsler metric, homogeneous polynomials, Randers change, special Randers change.
This research was supported by the Kyungsung University Research Grants in 2001.
a special Randers change of certain Finsler spaces with an \((\alpha, \beta)\)-metric \(L\) by \(\beta\). The 1-form \(\beta\) of modification is coincided with 1-form \(\beta\) of \((\alpha, \beta)\)-metric \(L\). We are devoted to finding the conditions for Finsler spaces changed by a special Randers change to be of Douglas type.

2. Preliminaries

The geodesics of an \(n\)-dimensional Finsler space \(F^n = (M^n, L)\) are given by the system of the differential equations ([1]):

\[
\frac{d^2 x^i}{dt^2} y^j - \frac{d^2 x^j}{dt^2} y^i + 2\{G^i(x,y)y^j - G^j(x,y)y^i\} = 0, \quad y^i = \frac{dx^i}{dt}
\]

in a parameter \(t\). The function \(G^i(x,y)\) are given by

\[
2G^i(x,y) = g^{ij}(y^r \partial_j \partial_r F - \partial_j F),
\]

where \(\partial_i = \partial/\partial y^i\), \(\partial_j = \partial/\partial x^i\), \(F = L^2/2\) and \(g^{ij}(x,y)\) are the inverse of Finsler metric tensor \(g_{ij}(x,y)\). According to [2], \(F^n\) is of Douglas type if

\[
(2.1) \quad D^i = G^i(x,y)y^j - G^j(x,y)y^i
\]

are homogeneous polynomials in \((y^i)\) of degree three. We shall denote the homogeneous polynomials in \((y^i)\) of degree \(r\) by \(h^r\) for brevity.

Let \(L_i = \partial_i L\), \(L_{ij} = \partial_i \partial_j L\), \(L_{ijk} = \partial_i \partial_j \partial_k L\). Then we have

\[
L_i = l_i, \quad LL_{ij} = h_{ij}, \quad L^2 L_{ijk} = h_{ij}l_k + h_{jk}l_i + h_{ki}l_j.
\]

And we put

\[
(2.2) \quad 2E_{ij} = \rho_{ij} + \rho_{j|i}, \quad 2F_{ij} = \rho_{ij} - \rho_{j|i},
\]

where \((\cdot)\) denotes the \(h\)-covariant derivative with respect to the Cartan connection \(\Gamma_{kl} = (F^i_{k,j}, G^i_{j,k}, C^i_{k,j}).\)

On the other hand, a Finsler metric \(L(x,y)\) is called an \((\alpha, \beta)\)-metric, when \(L\) is a positively homogeneous function \(L(\alpha, \beta)\) of degree one in two variables \(\alpha(x,y) = \sqrt{a_{ij}(x)y^i y^j}\) and \(\beta(x,y) = b_i(x)y^i\). The space \(R^n = (M^n, \alpha)\) is called the associated Riemannian space with \(F^n\) ([1], [7]).
We have the covariant differentiation (;) with respect to the Christoffel symbols $\gamma_{j}^{i}k(x)$ in $R^n$. We shall use the symbols as follows:

$$
\begin{align*}
    r_{ij} &= \frac{1}{2} (b_{i;j} + b_{j;i}), \\
    s_{ij} &= \frac{1}{2} (b_{i;j} - b_{j;i}), \\
    s_{i}^j &= a^{ir} r_{rj}, \\
    s_{j} &= b_{r}s_{r}^j.
\end{align*}
$$

Now we consider the functions $G^{i}(x,y)$ of $F^n$ with an $(\alpha, \beta)$-metric. According to [8], $G^{i}(x,y)$ are written in the form

$$
2G^{i} = \gamma^{i}{}_{0} + 2B^{i},
$$

$$
(2.3)
B^{i} = \frac{\alpha L^{\beta}_{\alpha} s^{0}{}_{0} + C^{*}}{L^{\alpha}} \left\{ \frac{\beta L^{\beta}_{\alpha} y^{i}}{\alpha L_{\alpha}} - \frac{\alpha L^{\alpha}_{\alpha}}{L_{\alpha}} \left( \frac{y^{i}}{\alpha} - \frac{\alpha b^{i}}{\beta} \right) \right\},
$$

where $L_{\alpha} = \partial L / \partial \alpha$, $L_{\beta} = \partial L / \partial \beta$, $L_{\alpha\alpha} = \partial^2 L / \partial \alpha \partial \alpha$, the subscript 0 means contraction by $y^{i}$ and

$$
C^{*} = \frac{\alpha \beta (r_{00} L_{\alpha} - 2 \alpha s_{0} L_{\beta})}{2 (\beta^2 L_{\alpha} + \alpha \gamma^2 L_{\alpha\alpha})},
$$

$$
\gamma^{2} = \frac{b^2}{\alpha^2 - \beta^2},
$$

$$
b^{i} = a^{ij} b_{j}, \quad b^{2} = a^{ij} b_{i} b_{j}.
$$

Since $\gamma^{i}{}_{0}(x)$ are $hp(2)$, $F^n$ with an $(\alpha, \beta)$-metric is Douglas space, if and only if $B^{ij} = B^{i}y^{j} - B^{j}y^{i}$ are $hp(3)$. Form (2.1) and (2.3) we have

$$
(2.4)
B^{ij} = \frac{\alpha L^{\beta}_{\alpha}}{L_{\alpha}} (s^{i}{}_{0} y^{j} - s^{j}{}_{0} y^{i}) + \frac{\alpha^2 L^{\alpha}_{\alpha}}{\beta L_{\alpha}} C^{*} (b^{i} y^{j} - b^{j} y^{i}).
$$

The following lemma ([9]) is used for latter:

**Lemma.** A system of linear equations $L_{ir} X^{r} = Y_{i}$, $(l_{r} + \rho_{r}) X^{r} = Y$ and $(Y_{i} y^{i} = \alpha^{2})$ in $X^{i}$ has the unique solution $X^{i} = Y^{i} + \frac{1}{r} (Y - LY^{r} \rho_{r}) l^{i}$, where $Y^{i} = g^{ir} Y_{r}$ and $\tau = L / L$.

3. Randers change of Douglas type

For a Randers change: $L \longrightarrow \bar{L} = L(x,y) + \rho(x,y)$, $\rho(x,y) = \rho(x)y^{i}$, we may put

$$
(3.1)
\bar{G}^{i} = G^{i} + D^{i}.
$$
Then $G^{i}_{j} = G^{i}_{j} + D^{i}_{j}$ and $G^{i}_{j} = G^{i}_{j} + D^{i}_{j}$, where $D^{i}_{j} = \partial_{j}D^{i}$ and $D^{i}_{j} = \partial_{j}D^{i}$. The tensors $D^{i}$, $D^{i}_{j}$ and $D^{i}_{j}$ are positively homogeneous in $y^{i}$ of degree two, one and zero respectively. In the following the explicit form of $D^{i}$ is necessary. To find this, we deal with equation $L_{ij|k} = 0$, where $L_{ij|k}$ is the h-covariant derivative of $L_{ij} = h_{ij}/L$ in $CT$. Then

$$\partial_{k}L_{ij} = L_{ijr}G^{r}_{k} + L_{rj}F^{r}_{i}|_{k} + L_{ir}F^{r}_{j}|_{k}.$$ 

Since $\overline{L}_{ij} = L_{ij}$ and $\overline{L}_{ijk} = L_{ijk}$ hold,

$$\overline{L}_{ijk} = L_{ijr}(G^{r}_{k} + D^{r}_{k}) + L_{rj}(F^{r}_{i}|_{k} - D^{r}_{i}|_{k}) + L_{ir}(F^{r}_{j}|_{k} + D^{r}_{j}|_{k}),$$

which imply

$$L_{ijr}D^{r}_{k} + L_{rj}D^{r}_{i}|_{k} + L_{ir}D^{r}_{j}|_{k} = 0.$$ 

Thus transvection of this equation by $y^{k}$ yields

$$(3.2) \quad 2L_{ijr}D^{r} + L_{rj}D^{r}_{i} + L_{ir}D^{r}_{j} = 0.$$ 

Next, we deal with $L_{ij} = 0$, that is,

$$\partial_{j}L_{i} = L_{ir}G^{r}_{j} + L_{r}F^{r}_{i},$$

$$\partial_{j}\overline{L}_{i} = L_{ir}(G^{r}_{j} + D^{r}_{j}) + (L_{r} + \rho_{r})(F^{r}_{j} + ^{c}D^{r}_{j}),$$

where $^{c}D^{r}_{j} = F^{r}_{i} - F^{r}_{i}$. Substitution of the equations above in $\partial_{j}\overline{L}_{i} = \partial_{j}L_{i} + \partial_{j}\rho_{i}$ leads to

$$\partial_{j}\rho_{i} - \rho_{r}F^{r}_{i} = L_{ir}D^{r}_{j} + (L_{r} + \rho_{r})^{c}D^{r}_{i}.$$ 

Then we have

$$(3.3) \quad 2E_{ij} = L_{ir}D^{r}_{j} + L_{jr}D^{r}_{i} + 2(L_{r} + \rho_{r})^{c}D^{r}_{i},$$

$$(3.4) \quad 2F_{ij} = L_{ir}D^{r}_{j} - L_{jr}D^{r}_{i}.$$ 

Therefore (3.2) and (3.4) give

$$L_{ir}D^{r}_{j} = F_{ij} - L_{ijr}D^{r}$$

and transvection of (3.3) by $y^{i}$ shows

$$(3.5) \quad (L_{r} + \rho_{r})D^{r}_{j} = E_{ij}y^{i} - L_{ijr}D^{r}.$$
Furthermore transvection of (3.5) and (3.6) by \( y^j \) leads to

\[
(3.7) \quad (a) \quad L_{ir}D^r = F_{ij}y^j, \quad (b) \quad (l_r + \rho_r)D^r = \frac{1}{2}E_{ij}y^iy^j.
\]

The equations (3.7)(a)(b) constitute a system of linear equations respectively. Applying Lemma to (3.7), we have

\[
(3.8) \quad D^i = LF^i_0 + \frac{1}{L} \left( \frac{1}{2}E_{00} - LF_0 \right)y^i,
\]

where \( F^i_j = g^{ir}F_{rj} \) and \( F_j = \rho_rF^r_j \). Thus we have the following

**Proposition 3.1.** ([9]) The tensor \( D^i \) of (3.1) arising from a Randers change are given by (3.8).

From (3.1) and (3.8) we have

\[
\overline{G}^i_j y^j - \overline{G}^j_i y^i = G^i_j y^j - G^j_i y^i + L(F^i_0 y^j - F^j_0 y^i).
\]

Suppose \( F^n \) is a Douglas space, that is, \( G^i_j y^j - G^j_i y^i \) are hp (3). Thus we have

**Proposition 3.2.** Let \( F^n \) be a Douglas space and \( \overline{F}^n \) a Finsler space which is obtained by Randers change by \( \rho \). \( \overline{F}^n \) is also a Douglas space if and only if \( L(F^i_0 y^j - F^j_0 y^i) \) are hp (3).

The Randers changes is called *projective Randers changes* if all the geodesic curves are preserved under the Randers changes. According to Hashiguchi-Ichijyō ([4]), a Randers change is projective, if and only if \( \rho_i \) are gradient vector fields. In this case (3.8) is reduced to \( D^i = E_{00}y^i/2L \). Therefore \( D^i y^j - D^j y^i = 0 \). Thus we have \( \overline{G}^i_j y^j - \overline{G}^j_i y^i = G^i_j y^j - G^j_i y^i \).

On the other hand, it is well-known that the Douglas tensor is projectively invariant. Hence, if a Finsler space is projectively related to a Douglas space, then it is also a Douglas space. Thus, from Hashiguchi-Ichijyō's theorem, we have the following

**Theorem 3.3.** Let \( F^n(M^n, L) \rightarrow \overline{F}^n(M^n, L + \rho_i) \) be a projective Randers change. If \( F^n \) is a Douglas space, then \( \overline{F}^n \) is also a Douglas space, and vice versa.
4. Generalized Kropina spaces

Hereafter we consider a special Randers change of certain \((\alpha, \beta)\)-metric as follows: \(L(\alpha, \beta) \rightarrow \vec{L} = L(\alpha, \beta) + \beta\), that is, the 1-form \(\beta\) of modification coincides with 1-form \(\beta\) of \((\alpha, \beta)\)-metric. In this section we deal with a Finsler space \(F^n (n > 2)\) with a generalized Kropina metric. The metric of \(F^n\) is \(L = \alpha^{1+m}\beta^{-m}\), where \(m\) is a constant \(\neq 0, -1\). We consider the condition for a Finsler space \(\vec{F}^n = (\vec{M}^n, L + \beta)\) which is obtained by a special Randers change of a generalized Kropina space \(\vec{F}^n = (\vec{M}^n, \vec{L} = \alpha^{1+m}\beta^{-m})\) to be of Douglas type. It has been known ([8]) that a generalized Kropina space is of Douglas space, where \(\alpha^2 \not\equiv 0 \pmod{\beta}\), if and only if \(b_{i;j}\) are given by

\[
(4.1) \quad s_{ij} = \frac{1}{b^2} (b^i s^j_j - b_j s_i) ,
\]

\[
(4.2) \quad r_{ij} = \frac{k}{m(1 + m)} \{(1 - m)b_i b_j + mb^2 a_{ij}\} + \frac{1 - m}{(1 + m)b^2} (s_i b_j - s_j b_i).
\]

For \(\vec{F}^n\), (2.3) gives

\[
2\{(1-m)\beta^2 + mb^2 \alpha^2\} \{(1 + m)\beta \vec{B}^{ij}\}
\]

\[+(ma^2 - \alpha^{1-m}\beta^{m+1})(s^i_0 y^j - s^j_0 y^i)\} - m\alpha^2 \{(1 + m)r_{00} \beta
\]

\[+ 2s_0 (ma^2 - \alpha^{1-m} \beta^{m+1}) \}(b^i y^j - b^j y^i) = 0,
\]

which are equivalent to

\[
(4.4)
\]

\[2\{(1-m)\beta^2 + mb^2 \alpha^2\} \{(1 + m)\beta \vec{B}^{ij}\}
\]

\[+ mx^2(s^i_0 y^j - s^j_0 y^i)\} - m\alpha^2 \{(1 + m)r_{00} \beta + 2ms_0 \alpha^2\}(b^i y^j - b^j y^i)
\]

\[- 2\alpha^{1-m} \beta^{m+1} \{(1 - m)\beta^2 + mb^2 \alpha^2\}(s^i_0 y^j - s^j_0 y^i)
\]

\[- ms_0 \alpha^2(b^i y^j - b^j y^i)\} = 0.
\]

Then it will be better to divide our consideration into two cases as follows:

(I) \(\alpha^{1-m} \beta^{m+1}\) : rational in \((y^i)\), that is, \(m\) : odd integer,

(II) \(\alpha^{1-m} \beta^{m+1}\) : irrational in \((y^i)\), that is, \(m\) : the others.
The case (I): First we are concerned with $m \leq 1$, where $m$ is an odd integer. Multiplication of $(4.1)$ by $\beta^{-m-1}$ leads to
\[
2\{(1 - m^2)\beta^2 + mb^2\alpha^2\}((1 + m)\beta^{-m}B^{ij} + (m\alpha^2\beta^{-1-m}) \alpha^{1-m}(s^i_0y^j - s^j_0y^i) - m\alpha^2((1 + m)r_00\beta^{-m} + 2s_0(m\alpha^2\beta^{-1-m} - \alpha^{1-m})\{b^iy^j - b^jy^i\}) = 0.
\]

(4.5)

Since $B^{ij}$ are supposed to be $hp(3)$, the term in (4.5) which seemingly does not contain $\alpha^2$ is $2(1 - m^2)\beta^{-m}B^{ij}$ only, and hence we must have $hp(3 - m)u^{ij}_{3-m}$ such that
\[
2(1 - m^2)\beta^{-m}B^{ij} = \alpha^2u^{ij}_{3-m}.
\]

(4.6)

We treat of the general case $\alpha^2 \neq 0$ (mod. $\beta$). (4.6) shows that there exist $hp(1)u^{ij}$ satisfying $u^{ij}_{3-m} = \beta^{-m}u^{ij}$. Then (3.4) is reduced to
\[
2(1 - m^2)B^{ij} = \alpha^2u^{ij}.
\]

(4.7)

If $m \neq 1$, that is, $F^n$ is not a Kropina space, then (4.7) gives $B^{ij}$ and (4.5) can be rewritten in the form
\[
\{(1 - m)\beta^2 + mb^2\alpha^2\}\left\{\frac{\beta^{-m}u^{ij}}{1 - m} + 2(m\beta^{-1-m} - \alpha^{-1-m})(s^i_0y^j - s^j_0y^i)\right\}
- m\{(1 + m)r_00\beta^{-m} + 2s_0(m\alpha^2\beta^{-1-m} - \alpha^{1-m})\{b^iy^j - b^jy^i\} = 0.
\]

(4.8)

Collecting the terms of (4.8) which seemingly do not contain $\beta$, we can put
\[
2m\alpha^{-1-m}\{b^2(s^i_0y^j - s^j_0y^i) - s_0(b^iy^j - b^jy^i)\} = \beta u^{ij}_{2-m},
\]

where $u^{ij}_{2-m}$ are $hp(2 - m)$. Consequently we have
\[
b^2(s^i_0y^j - s^j_0y^i) - s_0(b^iy^j - b^jy^i) = \beta u^{ij}
\]

and $u^{ij}_{2-m} = 2m\alpha^{-1-m}u^{ij}$ with $hp(1)u^{ij}$. Thus (4.8) is reduced to
\[
\{(1 - m)\beta^2 + mb^2\alpha^2\}\frac{\beta^{-m}u^{ij}}{1 - m} + 2m^2\alpha^2\beta^{-m}u^{ij}
+ 2[m(1 - m)\beta^{-1-m} - \alpha^{-1-m}\{(1 - m)\beta^2 + mb^2\alpha^2\}(s^i_0y^j - s^j_0y^i)
- m\{(1 + m)r_00\beta^{-m} - 2s_0\alpha^{1-m}\{b^iy^j - b^jy^i\} = 0.
\]

(4.9)
Consequently (4.9) is obtained as follows:

\[(4.11) \quad b^2 s_{ij} = b_i s_j + b_j s_i, \quad \text{provided that} \quad b^2 \neq 0.\]

That is, (4.1). From (4.11), (4.9) is reduced to \(v^{ij} = y^i s^j - y^j s^i\) and (4.10) is rewritten in the form

\[(4.12) \quad \{(1 - m)\beta^2 + mb^2 \alpha^2\} \left\{ \frac{\beta^{-m} u^{ij}}{1 - m} - \frac{2(m \beta^{-m} - \alpha^{-1-m} \beta)}{b^2} (s^i y^j - s^j y^i) \right\} \\
+ \left\{ \frac{2m(1 - m)\beta^{1-m} - 2\alpha^{-1-m} \{(1 - m)\beta^2 + mb^2 \alpha^2\}}{b^2} \right\} \frac{s_0}{b^2} \\
- m \{(1 + m)r_{00} \beta^{-m} - 2s_0 \alpha^{1-m}\} \right\} (b^i y^j - b^j y^i) = 0.\]

Multiplying (4.12) by \(\beta^m\), we obtain

\[(4.13) \quad \{(1 - m)\beta^2 + mb^2 \alpha^2\} \left\{ \frac{u^{ij}}{1 - m} - \frac{2(m - \alpha^{-1-m} \beta^{1+m})}{b^2} (s^i y^j - s^j y^i) \right\} \\
+ \left\{ \frac{2m(1 - m)\beta - 2\alpha^{-1-m} \beta^m \{(1 - m)\beta^2 + mb^2 \alpha^2\}}{b^2} \right\} \frac{s_0}{b^2} \\
- m \{(1 + m)r_{00} - 2s_0 \alpha^{1-m} \beta^m\} \right\} (b^i y^j - b^j y^i) = 0.\]

Transvecting (4.13) by \(b_is_j\), we have

\[(4.14) \quad \{(1 - m)\beta^2 + mb^2 \alpha^2\} \left\{ \frac{1}{1 - m} u^{ij} b_i s_j + \frac{2}{b^2} (m - \alpha^{-1-m} \beta^{1+m}) s^i s_j \beta \right\} \\
= \left\{ m \{(1 + m)r_{00} - 2s_0 \alpha^{1-m} \beta^m\} \\
- 2 \left[ m(1 - m)\beta - \alpha^{-1-m} \beta^m \{(1 - m)\beta^2 + mb^2 \alpha^2\} \right] \frac{s_0}{b^2} \right\} b^2 s_0.\]

Suppose that there exists \(u = u_i(x) y^i\) such that \((1 - m)\beta^2 + mb^2 \alpha^2 = b^2 s_0 u\). Then this is written in the form

\[2 \{(1 - m)b_i b_j + mb^2 a_{ij}\} = b^2(s_i u_j + s_j u_i).\]

Transvection by \(b_i b^i\) leads to the contradiction \(b^2 = 0\). Therefore (4.14) shows that we have a function \(h_1(x)\) satisfying

\[\frac{1}{1 - m} u^{ij} b_i s_j + \frac{2}{b^2} (m - \alpha^{-1-m} \beta^{1+m}) s^i s_j \beta = h_1(x) b^2 s_0,\]
\[
\begin{align*}
\{m(1 + m)r_{00} - 2ms_0\alpha^{-m}\beta^m\} - 2[m(1 - m)\beta - \alpha^{-1-m}\beta^m]. \\
\{(1 - m)\beta^2 + mb^2\alpha^2\}\frac{s_0}{\beta^2}\} s_0 = \{(1 - m)\beta^2 + mb^2\alpha^2\} h_1(x)s_0.
\end{align*}
\]

If \(s_0 \neq 0\), then we get from the latter
\[(4.15)\]
\[r_{00} = \frac{h_1(x)}{m(1 + m)}\{(1 - m)\beta^2 + mb^2\alpha^2\} + \frac{2(1 - m)s_0\beta}{m(1 + m)\beta^2}(m - \alpha^{-1-m}\beta^{1+m}).\]

Thus (4.13) gives \(u_{ij}\) of the form
\[(4.16)\]
\[u_{ij} = \frac{2(1 - m)}{b^2}(m - \alpha^{-1-m}\beta^{1+m})(s^iy^j - s^jy^i) + h_1(x)(1 - m)(b^iy^j - b^jy^i).\]

Since \(r_{00}\) is \(hp\) (2) from (4.15), \(\alpha^{-1-m}\beta^{1+m}\) must be \(hp\) (0). The condition for \(\alpha^{-1-m}\beta^{1+m}\) to be \(hp\) (0) is \(m = -3\) alone. Thus substituting \(m = -3\) in (4.15), we have
\[(4.17)\]
\[r_{00} = \frac{h_1(x)}{6}(4\beta^2 - 3b^2\alpha^2) - \frac{4s_0}{3b^2\beta}(\alpha^2 + 3\beta^2).\]

(4.17) shows that there exists \(h_2(x)\) satisfying \(s_0 = h_2(x)\beta\). Then (4.17) is reduced to
\[(4.18)\]
\[r_{ij} = \left(\frac{2h_1(x)}{3} - \frac{4h_2(x)}{b^2}\right)b_ib_j - \left(\frac{b^2h_1(x)}{2} + \frac{4h_2(x)}{3b^2}\right)a_{ij}.\]

That is, (4.2). If \(s_0\) is assumed to vanish, then (4.11) gives \(s_{ij} = 0\) and (4.13) is reduced to
\[
\{(1 - m)\beta^2 + mb^2\alpha^2\} u_{ij} = m(1 - m^2)r_{00}(b^iy^j - b^jy^i).\]

Transvection by \(b_iy_j(y_j = a_{jr}y^r)\) leads to
\[
\{(1 - m)\beta^2 + mb^2\alpha^2\} u_{ij}b_iy_j = m(1 - m^2)r_{00}(b^2\alpha^2 - \beta^2).\]

It is easy to show that \((1 - m)\beta^2 + mb^2\alpha^2\) is not contained in \(b^2\alpha^2 - \beta^2\) (\(= \gamma^2\)). Consequently it is contained in \(r_{00}\); there exists a function \(h_3(x)\) such that \(r_{00} = h_3(x)(1 - m)\beta^2 + mb^2\alpha^2\). Therefore (4.18) holds in this case, too.
Next, we deal with \( m > 1 \). Multiplication of (4.3) by \( \alpha^{-1+m} \) leads to \( s_0 = 0 \) and \( s_{ij} = 0 \). Thus we obtain \( r_{00} = h_3(x)(1-m)\beta^2 + mb^2\alpha^2 \) in common with \( s_0 = 0 \).

The case (II): Since \( \alpha^{1-m} \beta^{m+1} \) is irrational in \( (y^i) \), (4.4) is divided into two equations as follows:

\[
2[(1-m)\beta^2 + mb^2\alpha^2]((1+m)\beta B^{ij} + m\alpha^2(s^i_0y^j - s^j_0y^i)) - m\alpha^2(1 + m)r_{00}\beta + 2ms_0\alpha^2(b^iy^j - b^jy^i) = 0,
\]

\[(4.19)\]

\[
((1-m)\beta^2 + mb^2\alpha^2)(s^i_0y^j - s^j_0y^i) - ms_0\alpha^2(b^iy^j - b^jy^i) = 0.
\]

\[(4.20)\]

Transvecting (4.20) by \( b_iy_j \), we get

\[s_0\alpha^2((1-m)\beta^2 + mb^2\alpha^2) - ms_0\alpha^2(b^2\alpha^2 - \beta^2) = 0,
\]

which implies \( s_0\alpha^2\beta = 0 \). Hence we get \( s_0 = 0 \), that is, \( s_i = 0 \). (4.20) is reduced to \( s^i_0y^j - s^j_0y^i = 0 \). Transvection of this by \( y_i \) leads to \( s^i_0 = 0 \). Therefore \( s_{ij} = 0 \). Substituting \( s_{ij} = 0 \) in (4.19), we obtain

\[
2[(1-m)\beta^2 + mb^2\alpha^2]B^{ij} - m\alpha^2r_{00}(b^iy^j - b^jy^i) = 0.
\]

\[(4.21)\]

The term in (4.21) which seemingly does not contain \( \alpha^2 \) is \( 2(1-m)\beta^2B^{ij} \) only, and hence we must have \( hp(3) \) \( u^{ij}_3 \) satisfying

\[
2(1-m)\beta^2B^{ij} = \alpha^2u^{ij}_3.
\]

\[(4.22)\]

Suppose \( \alpha^2 \not\equiv 0 \) (mod. \( \beta \)). Then (4.22) is reduced to \( B^{ij} = \alpha^2u^{ij} \), where \( u^{ij} \) are \( hp(1) \). Hence (4.21) leads to

\[
2[(1-m)\beta^2 + mb^2\alpha^2]u^{ij} - r_{00}(b^iy^j - b^jy^i) = 0.
\]

\[(4.23)\]

Transvecting (4.23) by \( b_iy_j \), we obtain

\[
2[(1-m)\beta^2 + mb^2\alpha^2]u^{ij}b_iy_j - r_{00}(b^2\alpha^2 - \beta^2) = 0.
\]

Thus there exists a function \( h_4(x) \) such that

\[
2(m-1)u^{ij}b_iy_j - r_{00} = h_4(x)\alpha^2, \quad 2mb^2u^{ij}b_iy_j - b^2r_{00} = h_4(x)\beta^2.
\]
Eliminating $u^{ij}b_{ij}y_j$ from the above equations, we have

$$b^2 r_{00} = h_4(x)\{(m - 1)\beta^2 - mb^2\alpha^2\},$$

which implies

$$r_{ij} = \frac{h_4(x)}{b^2}\{(m - 1)b_i b_j - mb^2a_{ij}\}. \quad (4.24)$$

From $s_{ij} = 0$ and (4.24) we obtain

$$b_{i;j} = h_5(x)\{(m - 1)b_i b_j - mb^2a_{ij}\}, \quad (4.25)$$

where $h_5(x) = h_4(x)/b^2$.

Consequently, if (4.25) is satisfied, then $s_{ij} = 0$ and

$$r_{00} = h_5(x)\{(m - 1)\beta^2 - mb^2\alpha^2\},$$

from which $\overline{B}^{ij}$ of (4.4) are $hp(3)$. Hence (4.18) holds in this case, too.

In any case we obtain $b_{i;j}$ by (4.11) and (4.18), then $\overline{B}^{ij}$ are given by (4.7) together with (4.16). Consequently a Finsler space $\overline{F}^n = (M^n, L + \beta)$ ($n > 2$) with non zero $b^2$ which is obtained by Randers change of a generalized Kropina space $F^n = (M^n, L = \alpha^{1+m}\beta^{-m}, m \neq \pm1, 0)$ is a Douglas space, if and only if $b_{i;j}$ are given (4.11) and (4.18). That is, (4.1) and (4.2) hold.

On the other hand, it has been known ([8]) that a generalized Kropina space $F^n$ ($n > 2$) with non zero $b^2$ is a Douglas space, if and only if $b_{i;j}$ are given by (4.1) and (4.2). That is to say, the case $s_0 \neq 0$ for $F^n$ to be a Douglas space corresponds to the case $m = -3$ for $\overline{F}^n$ to be a Douglas space and the case $s_0 = 0$ for $F^n$ to be of Douglas type corresponds to the case $m \neq -3$, $m \in \mathbb{R}$ for $\overline{F}^n$ to be of Douglas type. Thus we obtain the following

**Theorem 4.1.** Let $F^n$ ($n > 2$) be a generalized Kropina space with $L = \alpha^{1+m}\beta^{-m}$, $m$ being a constant $\neq \pm1, 0$. A Finsler space $\overline{F}^n$ which is obtained by a special Randers change of $F^n$ with non zero $b^2$ of Douglas type is also of Douglas type, and vice versa.
5. Kropina space

Let $F^n$ be a Kropina space with $L = \alpha^2/\beta$ and $\overline{F}^n = (M^n, \overline{L})$ a Finsler space which is obtained by Randers change of $F^n = (M^n, L)$. From (2.4), $\overline{B}^{ij} = \overline{B}^i y_j - \overline{B}^j y_i$ in $\overline{F}^n$ are written as

$$\overline{B}^{ij} = B^{ij} + \frac{\alpha}{L_\alpha} (s^i_0 y^j - s^j_0 y^i) - \frac{\alpha^4 s_0 L_{\alpha\alpha}}{L_\alpha (\beta^2 L_\alpha + \alpha \gamma^2 L_{\alpha\alpha})} (b^i y^j - b^j y^i).$$

Suppose $F^n$ is a Douglas space. Since $B^{ij}$ are $hp(3)$, the necessary and sufficient condition for $\overline{F}^n$ to be also a Douglas space is that

$$\frac{\alpha}{L_\alpha} (s^i_0 y^j - s^j_0 y^i) - \frac{\alpha^4 s_0 L_{\alpha\alpha}}{L_\alpha (\beta^2 L_\alpha + \alpha \gamma^2 L_{\alpha\alpha})} (b^i y^j - b^j y^i)$$

are $hp(3)$. Thus we have the following

**PROPOSITION 5.1.** Let $F^n = (M^n, L)$ be a Finsler space with an $(\alpha, \beta)$-metric of Douglas type. Then $\overline{F}^n = (M^n, L + \beta)$ which is obtained by a special Randers change of $F^n$ is also a Douglas space, if and only if

$$W^{ij} = \frac{\alpha}{L_\alpha} (s^i_0 y^j - s^j_0 y^i) - \frac{\alpha^4 s_0 L_{\alpha\alpha}}{L_\alpha (\beta^2 L_\alpha + \alpha \gamma^2 L_{\alpha\alpha})} (b^i y^j - b^j y^i)$$

are $hp(3)$.

We suppose $F^n$ is a Douglas space. The condition for $\overline{F}^n = (M^n, L + \beta)$ to be a Douglas space is that (5.2) is $hp(3)$. From (5.2) we have

$$W^{ij} = \frac{\beta}{2} (s^i_0 y^j - s^j_0 y^i) - \frac{s_0 \beta}{b^2} (b^i y^j - b^j y^i).$$

Since $B^{ij}$ and $W^{ij}$ are $hp(3)$, $\overline{B}^{ij}$ are $hp(3)$, that is, $\overline{F}^n$ is a Douglas space. Thus a Kropina space $F^n$ is of Douglas type, then a Finsler space $\overline{F}^n$ which is obtained by a special Randers change of $F^n$ is of Douglas type also. We consider the condition for a Finsler space which is obtained by a special Randers change of a Kropina space to be of Douglas type. For $\overline{F}^n = (M^n, \overline{L} = \alpha^2/\beta + \beta)$, (2.4) gives

$$\overline{B}^{ij} = \frac{1}{2 \beta} (\beta^2 - \alpha^2) (s^i_0 y^j - s^j_0 y^i) + \frac{1}{2 b^2 \beta} (r_{00} \beta - s_0 (\beta^2 - \alpha^2)) (b^i y^j - b^j y^i).$$
Since the terms \((\beta/2)(s^0_jy^j - s^j_0y^i) + (1/2\beta^2)(r_{00} - s_0\beta)(b^i_jy^j - b^j_iy^i)\) are \(hp(3)\), these terms may be neglected in our discussion and we treat only of

\[
W^{ij} = \frac{\alpha^2}{2\beta} \left\{ \frac{s_0}{b^2} (b^i_jy^j - b^j_iy^i) - (s^i_0y^j - s^j_0y^i) \right\}.
\]

For \(n > 2\), \(\alpha^2 \equiv 0 \mod(\beta)\) ([3]). Therefore there exist \(hp(1)\) \(v^{ij} = v^{ij}_k(x)y^k\) such that

\[
\frac{s_0}{b^2} (b^i_jy^j - b^j_iy^i) - (s^i_0y^j - s^j_0y^i) = \beta v^{ij}.
\]

This equation is written in the form

\[
\frac{1}{b^2} \left\{ b^i_j \left( s^k_0 \delta^i_k + s^i_k \delta^j_k \right) - b^j_i \left( s^k_0 \delta^i_k + s^i_k \delta^j_k \right) \right\} \nonumber
\]

\[-(s^i_0 \delta^j_k + s^i_k \delta^j_k) + (s^j_0 \delta^i_k + s^j_k \delta^i_k) = b^i_k v^{ij}_k + b^j_k v^{ij}_k.
\]

Transvection of (5.7) by \(a^{hk}\) leads to

\[
\frac{1}{b^2} (b^i_j s^j - b^j_i s^i) - 2s^{ij} = b^r v^{ij}_r.
\]

Next, transvecting (5.7) by \(b^h\), we have

\[
(s^i_0 \delta^j_k + b^i_j s^j_k) - (s^j_0 \delta^i_k + b^j_i s^i_k) = b^2 v^{ij}_k + b^i_k b^r v^{ij}_r.
\]

Contraction of (5.7) with \(j\) and \(h\) leads to

\[
n \left( \frac{1}{b^2} b^i s^i_k - s^i_k \right) = b^i_k v^{ir}_r - b^r v^{ir}_r.
\]

Substituting \(b^r v^{ij}_r\) of (4.8) in (4.9), we have

\[
b^2 v^{ij}_k = 2s^{ij}b^i_k + \left\{ b^i s^j_k - b^j s^i_k + s^i_0 \delta^j_k - s^j_0 \delta^i_k + \frac{1}{b^2} (s^i b^j b^k_k - s^j b^i b^k_k) \right\},
\]

which imply

\[
b^2 v^{ir}_r = (n - 1)s^i, \quad b^2 b^r v^{ir}_r = b^i s^i_k - b^2 s^i_k.
\]
Consequently (5.10) leads to

\[(5.11) \quad s_{ij} = \frac{1}{b^2}(b_is_j - b_js_i).\]

Then (5.5) gives

\[\bar{W}^{ij} = \frac{\alpha^2}{2b^2}(s^iy^j - s^jy^i),\]

which are \(hp(3)\). Therefore (5.11) is the necessary and sufficient condition for \(F^n\) to be of Douglas type.

On the other hand, it is known ([8]) that a Kropina space \(F^n(n > 2)\) with \(b^2 \neq 0\) is of Douglas type, if and only if (5.11) is satisfied. Thus we have the

**Theorem 5.2.** A Finsler space \(\bar{F}^n(n > 2)\) which is obtained by a special Randers change of a Kropina space \(F^n\) with \(b^2 \neq 0\) is of Douglas type, if and only if the Kropina space \(F^n\) is of Douglas type.

6. Matsumoto space

We consider the condition for a Finsler space \(\bar{F}^n = (M^n, L + \beta)\) which is obtained by a special Randers change of Matsumoto space \(F^n = (M^n, L = \alpha^2/(\alpha - \beta))\) to be of Douglas type. It is known ([6]) that a Matsumoto space \(F^n(n > 2)\) is of Douglas type, if and only if \(b_{i;j} = 0\). Hence, for a Matsumoto space \(F^n\) of Douglas type, (2.4) leads to \(\bar{W}^{ij} = 0\), that is, \(\bar{B}^{ij} = B^{ij}\). Thus if a Matsumoto space \(F^n\) is of Douglas type, then a Finsler space which is obtained by a special Randers change of \(F^n\) is also of Douglas type. It is known ([8]) that a Matsumoto space \(F^n(n > 2)\) is of Douglas type, if and only if \(b_{i;j} = 0\). Hence, for a Matsumoto space \(F^n\) of Douglas type, (5.2) leads to \(\bar{W}^{ij} = 0\), that is, \(\bar{B}^{ij} = B^{ij}\). Thus if a Matsumoto space \(F^n\) is of Douglas type, then a Finsler space which is obtained by a special Randers change of \(F^n\) is also of Douglas type. For \(\bar{F}^n\), (2.3) gives

\[(6.1) \quad \{\alpha(1 + 2b^2) - 3\beta\}\{(\alpha - 2\beta)\bar{B}^{ij} - (2\alpha^2 - 2\alpha\beta + \beta^2)(s^i_0y^j - s^j_0y^i)\}
+ \alpha\{2s_0(2\alpha^2 - 2\alpha\beta + \beta^2) - r_{00}(\alpha - 2\beta)\}(b^i_0y^j - b^j_0y^i) = 0.\]

Suppose that \(\bar{F}^n\) be a Douglas space, that is, \(\bar{B}^{ij}\) be \(hp(3)\). Since \(\alpha\)
is irrational in \((y^i)\), (6.1) is divided as follows:

\begin{equation}
(1 + 2b^2)\alpha^2 + 6\beta^2)\overline{B}^{ij} + \{(2\alpha^2\beta(1 + 2b^2) + 3\beta(2\alpha^2 + \beta^2))(s^i_0 y^j - s^j_0 y^i) \nonumber \\
- (4s_0 \alpha^2 \beta + r_{00} \alpha^2)(b^i y^j - b^j y^i) = 0,
\end{equation}

\begin{equation}
(5 + 4b^2)\beta \overline{B}^{ij} + \{(1 + 2b^2)(2\alpha^2 + \beta^2) + 6\beta^2\}(s^i_0 y^j - s^j_0 y^i) \nonumber \\
- 2\{(s_0 (2\alpha^2 + \beta^2) + r_{00} \beta)\}(b^i y^j - b^j y^i) = 0.
\end{equation}

Eliminating \(\overline{B}^{ij}\) from these equations, we have

\begin{equation}
A(s^i_0 y^j - s^j_0 y^i) + B(b^i y^j - b^j y^i) = 0,
\end{equation}

where we put

\begin{align*}
A &= \alpha^2(21\beta^2 + 12\beta^2 b^2 + 12\beta^2 b^4 - 2\alpha^2 - 8\alpha^2 b^2 - 8\alpha^2 b^4) - 27\beta^4, \\
B &= \alpha^2\{s_0 (6\beta^2 - 12\beta^2 b^2 + 4\alpha^2 + 8\alpha^2 b^2) - 3r_{00}\beta\} + 12\beta^3(s_0 \beta + r_{00}).
\end{align*}

Transvection of (6.4) by \(b_i y_j\) leads to

\begin{equation}
A s_0 \alpha^2 + B(b^2 \alpha^2 - \beta^2) = 0.
\end{equation}

Since the terms \(12(s_0 \beta + r_{00})\beta^5\) of (6.5) seemingly do not contain \(\alpha^2\), we must have \(kp(5) v_5\) such that

\begin{equation}
12(s_0 \beta + r_{00})\beta^5 = \alpha^2 v_5.
\end{equation}

In the first case of \(v_5 = 0\), we have \(r_{00} = -s_0 \beta\) from (6.6), and (6.5) is reduced to

\begin{equation}
\alpha^2(17\beta^2 + 13\beta^2 b^2 - 2\alpha^2 - 4\alpha^2 b^2) + 12\beta^4(b^2 - 3)s_0 = 0.
\end{equation}

If the coefficient of \(s_0\) does not vanish, then

\begin{equation}
\alpha^2(17\beta^2 + 13\beta^2 b^2 - 2\alpha^2 - 4\alpha^2 b^2) = 12\beta^4(3 - b^2).
\end{equation}

Since we suppose \(\alpha^2 \not\equiv 0 \pmod{\beta}\), the above assumption is a contradiction. Therefore we obtain \(s_0 = 0\) and \(r_{00} = 0\) from (6.6). Next, in the second case of \(v_5 \not\equiv 0\), (6.6) shows the existence of a function
\[
k_1(x) \text{ satisfying } v_5 = k_1(x)\beta^5, \text{ and hence } r_{00} = k_2(x)\alpha^2 - s_0\beta, \text{ where } k_2(x) = k_1(x)/12. \text{ Then (6.5) is reduced to (6.7)}
\]
\[A s_0 + \{s_0 (9\beta^2 - 12\beta^2 b^2 + 4\alpha^2 + 8\alpha^2 b^2) - 3k_2(x)\beta(\alpha^2 - 4\beta^2)\} (b^2 \alpha^2 - \beta^2) = 0.
\]
Only the terms \(-36s_0\beta^4 + 12\beta^4 b^2 s_0 - 12k_2(x)\beta^5\) of (6.7) seemingly do not contain \(\alpha^2\), and hence we must have \(hp(3) v_3\) such that
\[
12\{s_0 (b^2 - 3) - k_2(x)\beta\} \beta^4 = \alpha^2 v_3.
\]
From \(\alpha^2 \not\equiv 0 \text{ (mod.}\beta)\) it follows that \(v_3\) must vanish, and hence \(s_0 (b^2 - 3) - k_2(x)\beta\), that is, \((b^2 - 3) s_i = k_2(x) b_i\). Then transvection by \(b^4\) gives \(k_2(\beta) b^2 = 0\). In case of \(k_2(\beta) = 0\), we get \(b^2 = 3\) or \(s_i = 0\). If \(b^2 = 3\), then (6.7) is reduced to \(14s_0 (4\beta^2 - \alpha^2) \alpha^2 = 0\). Thus we obtain \(s_0 = 0\) and \(r_{00} = 0\). Next, if \(s_i = 0\), then we have \(s_0 = 0\) and \(r_{00} = 0\), too. On the other hand, in the case of \(b^2 = 0\), (6.7) is reduced to \(s_0 (17\alpha^2 \beta^2 - 2\alpha^4 - 36\beta^4) + 3k_2(x)\beta^3 (\alpha^2 - 4\beta^2) = 0\), which implies \(s_0 = 0\) and \(k_2(\beta) = 0\). Therefore, for \(n > 2\), both the cases of \(v_5 = 0\) and \(v_5 \neq 0\) lead to \(r_{00} = 0\) and \(s_0 = 0\). Hence (6.4) is reduced to \(s^i_0 y^j - s^j_0 y^i = 0\), and transvection by \(y_i\) gives \(s^i_0 = 0\). Finally \(r_{ij} = s_{ij} = 0\), that is, \(b_{i;j} = 0\).

Thus a Finsler space \(\overline{F}^n = (M^n, L + \beta)\) \((n > 2)\) which is obtained by a special Randers change of a Matsumoto space \(F^n = (M^n, L = \alpha^2/(\alpha - \beta))\) is Douglas space, if and only if \(b_{i;j} = 0\). On the other hand, M. Matsumoto proved ([8]) that a Matsumoto space \(F^n (n > 2)\) is of Douglas type, if and only if \(b_{i;j} = 0\). Thus we have the following

**Theorem 6.1.** A Finsler space \(\overline{F}^n (n > 2)\) which is obtained by a special Randers change of a Matsumoto space \(F^n\) of Douglas type is also of Douglas type, and vice versa.

On the other hand, it has been shown ([1]) that Matsumoto space is a Berwald space, if and only if \(b_{i;j} = 0\). Then according to Theorem 6.1 we have the following

**Corollary 6.2.** Let \(\overline{F}^n (n > 2)\) be a Finsler space which is obtained by a special Randers change of a Matsumoto space \(F^n\). If \(F^n\) is a Douglas space, then \(\overline{F}^n\) is a Berwald space.

**7. Finsler space with \(L = \alpha + \beta^2/\alpha\)**

We consider a Finsler space \(F^n = (M^n, L)\) with an \((\alpha, \beta)\)-metric \(L = \alpha + \beta^2/\alpha\). This metric may be regarded as constructed from \(\alpha\) and
one more Riemannian metric $\sqrt{\alpha^2 + \beta^2}$, and it is thought of as desirable
in the viewpoint of geometry and applications ([8]). For $\bar{F}^n = (M^n, \bar{L})$
which is obtained by a special Randers change of $F^n = (M^n, L = \alpha + \beta^2/\alpha)$, (2.3) gives

\[
\bar{B}^{ij} = \frac{\alpha^2(\alpha + 2\beta)}{(\alpha^2 - \beta^2)}(s^i_0y^j - s^j_0y^i) + \frac{\alpha^2\{r_{i0}(\alpha^2 - \beta^2) - 2s_0\alpha^2(\alpha + 2\beta)\}}{(\alpha^2 - \beta^2)\{\alpha^2(1 + 2b^2) - 3\beta^2\}}(b^iy^j - b^jy^i).
\]

Suppose that $\bar{F}^n$ be a Douglas space, that is, $\bar{B}^{ij}$ be $hp$ (3). Separating
(7.1) into the rational and irrational terms of $y^i$, we have

\[
\{\alpha^2(1 + 2b^2) - 3\beta^2\}\{(\alpha^2 - \beta^2)\bar{B}^{ij} - 2\alpha^2\beta(s^i_0y^j - s^j_0y^i)\}
- \alpha^2\{r_{i0}(\alpha^2 - \beta^2) - 4s_0\alpha^2\beta\}(b^iy^j - b^jy^i)
+ \alpha[2s_0\alpha^4(b^iy^j - b^jy^i) - \alpha^2\{\alpha^2(1 + 2b^2) - 3\beta^2\}(s^i_0y^j - s^j_0y^i)] = 0,
\]

which yield two equations as follows:

\[
\{\alpha^2(1 + 2b^2) - 3\beta^2\}\{(\alpha^2 - \beta^2)\bar{B}^{ij} - 2\alpha^2\beta(s^i_0y^j - s^j_0y^i)\}
- \alpha^2\{r_{i0}(\alpha^2 - \beta^2) - 4s_0\alpha^2\beta\}(b^iy^j - b^jy^i) = 0,
\]

(7.2)

\[
2s_0\alpha^2(b^iy^j - b^jy^i) - \{\alpha^2(1 + 2b^2) - 3\beta^2\}(s^i_0y^j - s^j_0y^i) = 0.
\]

(7.3)

Transvecting (7.3) by $b_iy_j$, we obtain

\[
2s_0\alpha^2(b^i\alpha^2 - \beta^2) - \{\alpha^2(1 + 2b^2) - 3\beta^2\}s_0\alpha^2 = 0,
\]

which implies $s_0\alpha^2(\beta^2 - \alpha^2) = 0$. Therefore we get $s_i = 0$. Hence (7.3)
is reduced to $s^i_0y^j - s^j_0y^i = 0$, and transvection by $y_i$ gives $s^i_0 = 0$. Consequently $s_{ij} = 0$. On the other hand, substituting (7.3) in (7.2), we have

\[
\{\alpha^2(1 + 2b^2) - 3\beta^2\}\bar{B}^{ij} - \alpha^2\{r_{i0}(b^iy^j - b^jy^i)\} = 0.
\]

(7.4)

Only the terms $3\beta^2\bar{B}^{ij}$ of (7.4) seemingly do not contain $\alpha^2$. Hence we must have $hp(3)$ $\nu_3^{ij}$ satisfying

\[
3\beta^2\bar{B}^{ij} = \alpha^2\nu_3^{ij}.
\]

(7.5)
For the sake of brevity we suppose $\alpha^2 \neq 0 \pmod{\beta}$. Then (7.5) is reduced to $B^{ij} = \alpha^2 v^{ij}$, where $v^{ij}$ are $h^p(1)$. Thus (7.4) leads to

$$\{\alpha^2 (1 + 2b^2) - 3\beta^2\} v^{ij} - r_{00}(b^i y^j - b^j y^i) = 0. \tag{7.6}$$

Transvecting (6.6) by $b_i y_j$, we get

$$\{\alpha^2 (1 + 2b^2) - 3\beta^2\} b_i v^{ij} y_j - r_{00}(b^2 \alpha^2 - \beta^2) = 0,$$

which imply

$$\alpha^2 \{(1 + 2b^2)b_i v^{ij} y_j - b^2 r_{00}\} = \beta^2 (3b_i v^{ij} y_j - r_{00}).$$

Therefore there exists a function $f_1(x)$ satisfying

$$(1 + 2b^2)b_i v^{ij} y_j - b^2 r_{00} = f_1(x) \beta^2, \quad 3b_i v^{ij} y_j - r_{00} = f_1(x) \alpha^2.$$

Eliminating $b_i v^{ij} y_j$ from above the equations, we obtain

$$r_{00} = f_1(x) \frac{(1 + 2b^2)\alpha^2 - 3\beta^2}{b^2 - 1}. \tag{7.7}$$

From (7.7) and $s_{ij} = 0$,

$$b_{ij} = f_2(x)\{(1 + 2b^2)a_{ij} - 3b_i b_j\}, \tag{7.8}$$

where $f_2(x) = f_1(x)/(b^2 - 1)$.

Conversely, if (7.8) is satisfied, then $s_{ij} = 0$ and

$$r_{00} = f_2(x)\{(1 + 2b^2)\alpha^2 - 3\beta^2\},$$

from which $B^{ij}$ of (7.1) are $h^p(3)$. Thus we have the following

**Theorem 7.1.** A Finsler space $F^n \ (n > 2)$ which is obtained by a special Randers change of a Finsler space $F^m$ with an $(\alpha, \beta)$-metric $L = \alpha + \beta^2/\alpha \ (b^2 \neq 1)$ of Douglas type, is also a Douglas space, and vice versa.
The Randers changes of Finsler spaces

References


Hong-Suh Park
Department of Mathematics
Yeungnam University
Gyongsan 712-749, Korea
E-mail: phs1230@unitel.co.kr

Il-Yong Lee
Department of Mathematics
Kyungsung University
Pusan 608-736, Korea
E-mail: iylee@star. kyungsung.ac.kr