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On The Generalization of Approach Cauchy Spaces

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ABSTRACT

We construct several supercategories of ACHY (of approach Cauchy spaces) and AULim (of approach uniform limit spaces) and investigate the relation among them.

Key Words : approach uniform limit space, approach Cauchy space.

1. Introduction

Uniform limit space was introduced by C. H. Cook and H. R. Fischer [2] as a generalization of uniform space and a slight modification of the definition of uniform limit space was proposed by O. Wyler [15] so that the resulting category **ULim** satisfies 'cartesian closedness' [6]. Endowing the underlying set of a uniform limit space with the set of all Cauchy filters one obtains a Cauchy space which has been studied as a useful tool for constructing completions. In 1987, since it was shown that Cauchy spaces form a cartesian closed topological category **CHY** but quotient maps in **CHY** are neither countably productive nor hereditary [1], the question arose whether it was possible to embed **CHY** into better behaved supercategories. In [12], **CHY** was shown to be a finally dense, bireflective subcategory of topological universe **FIL** of filter-merotopic spaces which was introduced by M. Katětov [4] and could be implicitly embedded into **ULim** as a bicoreflective subcategory [5]. By omitting some of the axioms for uniform limit spaces, the notions of preuniform convergence space, semiuniform convergence space and semiuniform limit space were introduced and based on the relation between uniform limit space and Cauchy space, the notion of semi-Cauchy space was introduced as the object of the bicoreflective hull of **CHY** in **ULim** [13, 14]. The category **SCHY** of semi-Cauchy spaces is regarded not only to be better behaved than **CHY** but also to preserve more structure of **CHY** than any other categories.

In [10], the notion of approach structure was introduced by means of various axiom systems and by the notion of 'distance' between a point and a set, metric spaces and topological spaces can be viewed as entities of the same kind. In [8] and [9], the nations of approach Cauchy structure and approach uniform limit structure were introduced by measuring smallness of the members which quantify the notions of Cauchy structure and uniform limit structure, respectively. Now, we generalize the notions of approach Cauchy structure and approach uniform limit structure and show that these quantified structures yield supercategories of **ACHY** (of approach Cauchy spaces) and **AULim** (of approach uniform limit spaces) which preserve the classical relation among supercategories of **CHY** and **ULim**.

For any set X, we denote the set of all filters on X by $\mathcal{F}(X)$. The filter generated by a filter basis $\mathcal{B}$ is denoted by $[\mathcal{B}]$. In particular, the point filter generated by the set $\{x\}$ is denoted by $\dot{x}$. If $\mathcal{F}, \mathcal{G} \in \mathcal{F}(X)$, then $\mathcal{F} \times \mathcal{G} = [\{F \times G \mid F \in \mathcal{F}, G \in \mathcal{G}\}]$. If $f: X \to Y$ is a map and $\mathcal{F} \in \mathcal{F}(X)$, then $f(\mathcal{F}) = [f(F) \mid F \in \mathcal{F}\}]$, IF $\Phi, \Psi \in \mathcal{F}(X \times X)$, then $\Phi \circ \Psi = [\{U \circ V \mid U \in \Phi, V \in \Psi\}]$, provided that every $U \circ V = \{(x, y) \in X \times Y \mid \exists z \in X$ such that $(x, z) \in U, (z, y) \in V\}$ is empty.

A *filter (-merotopic) space* [4] is a pair (X, c), where X is a set and c is a subset of $\mathcal{F}(X)$ such that the following conditions are satisfied :

(FIL1) $\dot{x} \in c$ for all $x \in X$,

(FIL2) if $\mathcal{F} \in c$ and $\mathcal{F} < \mathcal{G}$, then $\mathcal{G} \in c$

A filter space (X, c) is called a *semi-Cauchy space* [13] if the following condition is fulfilled :

(SACHY) If $\mathcal{F} \in \mathcal{F}(X)$ is such that there exists finite family $(\mathcal{F}_j)_{j=1}^{n}$ in c with $\bigcap_{j=1}^{n} (\mathcal{F}_j \times \mathcal{F}_j) < \mathcal{F} \times \mathcal{F}$, then $\mathcal{F} \in c$.

A filter space (X, c) is called a *Cauchy space* if the following condition is satisfied : (CHY) if $\mathcal{F}, \mathcal{G} \in c$ and $\exists$ F∨G exists, then $\mathcal{F} \cap \mathcal{G} \in c$.

Given filter spaces (X, c) and (Y, c'), a map $f: X \to Y$ is called *continuous* if $\mathcal{F} \in c$ implies $f(\mathcal{F}) \in c'$.

Let **FIL** denote the category of filter spaces and continuous maps and denote the full subcategory of **FIL** consisting of all semi-Cauchy (Cauchy) spaces by **SCHY** (CHY), respectively.

A *preuniform convergence space* is a pair (X, L), where X is a set and L is a subset of $\mathcal{F}(X \times X)$ such that the following conditions are satisfied:

(ULim1) $\dot{x} \times \dot{x} \in L$ for all $x \in X$,

(ULim2) if $\Phi \in L$ and $\Phi < \Psi$, then $\Psi \in L$.

A preuniform convergence space (X, L) is called a *semiuniform convergence space* if the following condi-

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tion is satisfied:

(ULim3) $\Phi \in L$ implies $\Phi^{-1} \in L$.

A semiuniform convergence space (X, L) is called a *semiuniform limit space* if the following condition is satisfied:

(ULim4) if $\Phi, \Psi \in L$, then $\Phi \cap \Psi \in L$.

A semiuniform limit space (X, L) is called a *uniform limit space* if the following condition is satisfied:

(ULim5) if $\Phi, \Psi \in L$ and $\exists\, \Phi \circ \Psi$, then $\Phi \circ \Psi \in L$.

For any preuniform convergence spaces (X, L) and (Y, L'), a map $f: X \to Y$ is called *uniformly continuous* if $\Phi \in L$ implies $(f \times f)(\Phi) \in L'$.

Let **PUConv** denote the category of preuniform convergence spaces and uniformly continuous maps and **SUConv (SULim, ULim)** denote the full subcategory of **PUConv** consisting of all semiuniform convergence (semiuniform limit, uniform limit) spaces, respectively.

2. The category AULim

Definition 2.1 (1) An *approach preuniform convergence* (shortly, APUConv-) *space* is a pair (X, η), where X is a set and $\eta: \mathcal{F}(X \times X) \to [0, \infty]$ is a map such that the following conditions are satisfied:

(AULim1) $\eta(\dot{x} \times \dot{x}) = 0$ for all $x \in X$,

(AULim2) if $\Phi, \Psi \in \mathcal{F}(X \times X)$ and $\Phi < \Psi$, then $\eta(\Phi) \geq \eta(\Psi)$.

(2) An approach preuniform convergence space (X, η) is called an *approach semiuniform convergence* (shortly, ASUConv-) *space* if the following condition is satisfied :

(AULim3) $\eta(\Phi^{-1}) = \eta(\Phi)$ for all $\Phi \in \mathcal{F}(X \times X)$.

(3) An approach semiuniform convergence space (X, η) is called an *approach semiuniform limit* (shortly, ASULim-) *space* if the following condition is satisfied :

(AULim4) $\eta(\Phi \cap \Psi) \leq \eta(\Phi) \vee \eta(\Psi)$ for all $\Phi, \Psi \in \mathcal{F}(X \times X)$.

(4) An approach semiuniform limit space (X, η) is called an *approach uniform limit* (shortly, AULim- and in [9], *ultra approach uniform convergence*) *space* if the following condition is satisfied:

(AULim5) if $\Phi, \Psi \in \mathcal{F}(X \times X)$ are such that $\exists\, \Phi \circ \Psi$, then $\eta(\Phi \circ \Psi) \leq \eta(\Phi) \vee \eta(\Psi)$.

Definition 2.2 Given APUConv-spaces (X, η) and (Y, η'), a map $f: X \to Y$ is called a *uniform contraction* if $\eta'((f \times f)(\Phi)) \leq \eta(\Phi)$ for all $\Phi \in \mathcal{F}(X \times X)$.

Let **APUConv** denote the category of APUConv-spaces and uniform contractions and **ASUConv (ASULim, AULim)** denote the full subcategory of **APUConv** consisting of all ASUConv(ASULim, AULim)-spaces, respectively.

Recall from [9] that **AULim** contains **ULim** as a bireflectively and bicoreflectively embedded full subcategory and an approach uniform limit space (X, η) is a uniform limit space iff $\eta\mathcal{F}(X \times X)) \subseteq \{0, \infty\}$. Analogously, we have the following.

For any preuniform convergence (semiuniform convergence, semiuniform limit) space (X, L), the map $\eta_L: \mathcal{F}(X \times X) \to [0, \infty]$ defined by

$$\Phi \to \eta_L(\Phi) = \begin{cases} 0 & \text{for } \Phi \in L \\ \infty & \text{for } \Phi \notin L \end{cases}$$

is clearly an APUConv(ASUConv, ASULim)-structure on X, respectively. Furthermore, for any preuniform convergence spaces (X, L) and (Y, L'), a map $f: (X, L) \to (Y, L')$ is uniformly continuous iff $f: (X, \eta_L) \to (Y, \eta_{L'})$ is a uniform contraction. So **PUConv (SUConv, SULim)** is embedded as a full subcategory in **APUConv (ASUConv, ASULim)**, respectively.

Proposition 2.3. *An* APUConv(ASUConv, ASULim)*-space* (X, η) *is a preuniform convergence (semiuniform convergence, semiuniform limit) space, respectively iff* $\eta(\mathcal{F}(X \times X)) \subseteq \{0, \infty\}$.

Theorem 2.4. *The category* **PUConv (SUConv, SULim)** *is a bireflective and bicoreflective subcategory of* **APUConv (ASUConv, ASULim)**, *respectively.*

Theorem 2.5. *The category* **APUConv** *is a topological construct.*

Proof. To show that the category **APUConv** is initially complete, take any source $(X \xrightarrow{f_i} (X_i, \eta_i))_{i \in J}$ in **APUConv**. Then the map $\eta: \mathcal{F}(X \times X) \to [0, \infty]$\$ defined by

$$\Phi \mapsto \eta(\Phi) = \sup_{i \in J} \eta_i((f_i \times f_i)(\Phi))$$

is the initial approach preuniform convergence structure on X and the remainder is trivial.

Since final structure is the dual concept of initial structure, Theorem 2.5 guarantees the existence of final structures in **APUConv**. Here, we present the explicit form of the final APUConv-structure.

Proposition 2.6. *For any sink* $((X_j, \eta_j) \xrightarrow{f_j} X)_{j \in J}$ *in* **APUConv**, *the map* $\eta: \mathrm{F}(X \times X) \to [0, \infty]$ *defined by*

$$\eta(\Phi) = \begin{cases} 0 & \text{if } \Phi = x \times x \text{ for } \textit{some } x \in X \\ \inf\{\eta_j(\Phi_j) \mid (f_j \times f_j)(\Phi_j) < \Phi & \text{for } \textit{some } j \in J, \Phi_j \in F(X_j \times X_j)\} \\ & \textit{otherwise} \end{cases}$$

is the final approach preuniform convergence structure on X.

Propostion 2.7. *The category* **ASUConv** *is a bireflective and bicoreflective subcategory of* **APUConv**.

Proof. Note that $(\Phi^{-1})^{-1} = \Phi$ and $(f \times f)(\Phi) = ((f \times f)(\Phi^{-1}))^{-1}$ for any $\Phi \in \mathcal{F}(X \times X)$ and a map f on X. Then it can be easily proved that for any approach preuniform convergence space (X, η), the ASUConv-bireflector is $(X, \eta) \xrightarrow{1_X} (X, \eta_r)$, where $\eta_r: \mathcal{F}(X \times X) \to$

$[0, \infty]$ is defined by

$$\Phi \mapsto \eta_r(\Phi) = \eta(\Phi) \wedge \eta(\Phi^{-1})$$

and the ASUConv-bicoreflector is $(X, \eta_c) \xrightarrow{1_X} (X, \eta)$, where $\eta_c : \mathcal{F}(X \times X) \to [0, \infty]$ is given by

$$\Phi \mapsto \eta_c(\Phi) = \eta(\Phi) \vee \eta(\Phi^{-1}).$$

Proposition 2.8. *The category* **ASULim** *is a bireflective subcategory of the category* **ASUConv**.

Proof. For any approach semiuniform convergence space $(X; \eta)$, the ASULim-bireflector is $(X, \eta) \xrightarrow{1_X} (X, \eta_L)$, where $\eta_L: \mathcal{F}(X \times X) \to [0, \infty]$ is defined by

$$\eta_L(\Phi) = \inf\{ \sup_{j=1}^{n} \eta(\Phi_j) \mid (\Phi_j)_{j=1}^{n} \subseteq \mathcal{F}(X \times X) \text{ such that } \bigcap_{j=1}^{n} \Phi_j < \Phi\}.$$

Proposition 2.9. *The category* **AULim** *is a bireflective subcategory of* ASULim.

Proof. For any approach semiuniform limit space (X, η), the AULim-bireflector is $(X, \eta) \xrightarrow{1_X} (X, (\eta_S)_L)$, where $\eta_S: \mathcal{F}(X \times X) \to [0, \infty]$ is defined by

$\eta_S(\Phi) = \inf\{ \sup_{j=1}^{n} \eta(\Phi_j) \mid (\Phi_j)_{j=1}^{n} \subseteq \mathcal{F}(X \times X)$ such that $\Phi_1 \circ \cdots \circ \Phi_n < \Phi\}$.

It suffices to show that $(\eta_S)_L$ satisfies (AULim5). Let $\Phi, \Psi \in \mathcal{F}(X \times X)$ be such that $\Phi \circ \Psi$ exists and take any $(\Phi_i)_{i=1}^{n}, (\Psi_j)_{j=1}^{m} \subseteq \mathrm{F}(X \times X)$ such that $\bigcap_{i=1}^{n} \Phi_i < \Phi$, $\bigcap_{j=1}^{m} \Psi_j < \Psi$. Since $\Phi \circ \Psi$ exists and

$$\begin{aligned} \Phi \circ \Psi &> \bigcap_{i=1}^{n} \Phi_i \circ \bigcap_{j=1}^{m} \Psi_j \\ &= \bigcap_{\substack{i=1,\cdots,n \\ j=1,\cdots,m}} \Phi_i \circ \Psi_j, \end{aligned}$$

there exists at least one pair of indices (i, j) such that $\Phi_i \circ \Psi_j$ exists. Take all such pairs and rearrange by $k = 1, \cdots, l$. Then

$$\Phi \circ \Psi > \bigcap_{k=1}^{l} \Phi_k \circ \Psi_k$$

and since $\eta_S(\Phi_k \circ \Psi_k) \le \eta_S(\Phi_k) \vee \eta_S(\Psi_k)$ for all $k = 1, \cdots, l$, we have

$$\begin{aligned} \eta_S(\Phi_k \circ \Psi_k) &\le \sup_{k=1}^{l} (\eta_S(\Phi_k) \vee \eta_S(\Psi_k)) \\ &\le \left(\sup_{k=1}^{l} \eta_S(\Phi_k)\right) \vee \left(\sup_{k=1}^{l} \eta_S(\Psi_k)\right) \\ &\le \left(\sup_{i=1}^{n} \eta_S(\Phi_i)\right) \vee \left(\sup_{i=1}^{m} \eta_S(\Psi_j)\right) \end{aligned}$$

and consequently $(\eta_S)_L(\Phi \circ \Psi) \le (\eta_S)_L(\Phi) \vee (\eta_S)_L(\Psi)$.

Theorem 2.10. *The categories* **ASUConv, ASULim** *and* **AULim** *are topological constructs.*

Proof. This is an immediate consequence of Theorem 2.5 and theorem A.10 in [3].

For any APUConv-spaces (X, η) and (Y, η'), let $C(X, Y)$ be the set of all uniform contractions from X to Y. Then for any $\Phi \in \mathcal{F}(X \times X)$ and $\Theta \in \mathcal{F}(C(X, Y) \times C(X, Y))$, the set $\{H(A) : A \in \Phi, H \in \Theta\}$, where $H(A) = \{(h(a), k(b)) \mid (a, b) \in A, (h, k) \in H\}$ for each $A \in \Phi$ and $H \in \Theta$, forms a filter basis on $Y \times Y$. Let $\Theta(\Phi)$ be the filter on $Y \times Y$ generated by above and define a map $\eta^* : \mathrm{F}(C(X, Y) \times C(X, Y)) \to [0, \infty]$ by

$$\Theta \mapsto \eta^*(\Theta) = \inf L(\Theta),$$

where $L(\Theta) = \{a \in [0, \infty] \mid \eta'(\Theta(\Phi)) \le \eta(\Phi) \vee a$ *for all* $\Phi \in \mathcal{F}(X \times X)\}$.

Propostion 2.11. η^* *is the coarsest* APUConv-structure on $C(X, Y)$ with respect to which the evaluation map $ev: X \times C(X, Y) \to Y$ defined by $(x, f) \mapsto f(x)$ is a uniform contraction.

proof. Clearly, η^* is well-defined. (AULim1) follows from the inequality

$$\eta'((\dot{f} \times \dot{f})(\Phi)) = \eta'((f \times f)(\Phi)) \le \eta(\Phi)$$

for all $f \in C(X, Y)$ and $\Phi \in \mathcal{F}(X \times X)$. Since it holds that $\Theta < \Theta'$ in $\mathrm{F}(C(X, Y) \times C(X, Y))$ implies $\Theta(\Phi) < \Theta'(\Phi)$ in $\mathcal{F}(Y \times Y)$ for all $\Phi \in \mathrm{F}(X \times X)$, (AULim2) is immediate and consequently η^* is an APUConv-structure on $C(X, Y)$. Since for any $\Psi \in \mathcal{F}((X \times C(X, Y)) \times (X \times C(X, Y)))$

$$\begin{aligned} \eta'((ev \times ev)(\Psi)) &\le \eta'((ev \times ev)((\pi_1 \times \pi_1)(\Psi) \times (\pi_2 \times \pi_2)(\Psi))) \\ &= \eta'((\pi_2 \times \pi_2)(\Psi)((\pi_1 \times \pi_1)(\Psi))) \\ &\le \eta((\pi_1 \times \pi_1)(\Psi)) \vee \eta^*((\pi_2 \times \pi_2)(\Psi)) \\ &= (\eta \times \eta^*)(\Psi) \end{aligned},$$

where π_1 and π_2 are the canonical projection maps from $X \times C(X, Y)$ to X and $C(X, Y)$ respectively, the map $ev: X \times C(X, Y) \to Y$ is a uniform contraction with respect to η^*. Let η_* be another APUConv-structure on $C(X, Y)$ with respect to which ev is a uniform contraction. Then for all $\Phi \in \mathcal{F}(X \times X)$ and $\Theta \in \mathcal{F}(C(X, Y) \times C(X, Y))$, we have

$$\eta'((ev \times ev)(\Phi \times \Theta)) = \eta'(\Theta(\Phi)) \le \eta(\Phi) \vee \eta_*(\Theta)$$

and consequently $\eta_*(\Theta) \in L(\Theta)$ for all $\Theta \in \mathcal{F}(C(X, Y) \times C(X, Y))$. So $\eta^*(\Theta) \le \eta_*(\Theta)$ for all $\Theta \in \mathcal{F}(C(X, Y) \times C(X, Y))$ and hence we have the result.

Proposition 2.12. *Let* $(X, \eta), (Y, \eta')$ *and* (Z, η'') *be* APUConv-*spaces and let* $f: X \times Z \to Y$ *be a uniform contraction.* Then there exists a unique uniform contraction $\hat{f}: Z \to C(X, Y)$ such that $ev \circ (1_X \times \hat{f}) = f$.

proof. Define a map $\hat{f}: Z \to C(X, Y)$ by

$$z \mapsto \hat{f}(z): X \to Y$$
$$x \mapsto \hat{f}(z)(x) = f(x, z).$$

Then for each $z \in Z$, $\hat{f}(z) = f \circ (1_X \times [z])$, where $[z]: X \to Z$ is a map defined by $x \mapsto z$ for all $x \in X$. Since the identity map, the constant map and the composition of uniform contractions are uniform contractions, $\hat{f}(z) \in C(X, Y)$ and the map $\hat{f}$ is well-defined. Furthermore, since for any $\Phi \in \mathcal{F}(X \times X)$ and $\Psi \in \mathcal{F}(Z \times Z)$, we have

$$\begin{aligned} \eta'(((\hat{f} \times \hat{f})(\Psi))(\Phi)) &= \eta'((f \times f)(\Phi \times \Psi)) \\ &\le (\eta \times \eta'')(\Phi \times \Psi) \\ &= \eta(\Phi) \vee \eta''(\Psi), \end{aligned}$$

$\eta^*((\hat{f} \times \hat{f})(\Psi)) \le \eta''(\Psi)$ for all $\Psi \in \mathcal{F}(Z \times Z)$ and hence $\hat{f}$ is a uniform contraction. Clearly, $ev \circ (1_X \times \hat{f}) = f$ and such an $\hat{f}$ is unique.

Combining Proposition 2.11 and 2.12, we have

Theorem 2.13. *The category* **APUConv** *is cartesian closed.*

Proposition 2.14. **ASULim** *is closed under the formation of quotients in* **ASUConv**.

proof. For any ASULim-space (X, η) and an onto map $f: X \to Y$, the quotient ASUConv-structure

$$\begin{aligned} \eta' : \mathcal{F}(Y \times Y) &\to [0, \infty] \\ \Phi &\mapsto \eta'(\Phi) = \inf\{\eta(\Psi) \mid (f \times f)(\Psi) < \Phi\} \end{aligned}$$

on Y satisfies (AULim4).

Proposition 2.15. **AULim** *is closed under the formation of coproducts in* **ASULim**.

proof. For any family $((X_j, \eta_j))_{j \in J}$ of AULim-spaces and a sink of canonical injections $(\iota_j: X_j \to \coprod_{j \in J} X_j)_{j \in J}$, the coproduct ASULim-structure

$$\eta: \mathcal{F}(\coprod_{j \in J} X_j \times \coprod_{j \in J} X_j) \to [0, \infty]$$

defined by

$$\eta(\Phi) = \inf\Big\{ \sup_{j=1}^{n} \eta_{i_j}(\Phi_j) \;\Big|\; \text{, for } each\ j = 1, \cdots, n, \Phi_j \in \mathcal{F}(X_{i_j} \times X_{i_j}) \text{ for } some\ i_j \in J\ such\ that\ \bigcap_{j=1}^{n} (\iota_{i_j} \times \iota_{i_j})(\Phi_j) < \Phi \Big\}$$

satisfics (AULim5). For this, let $\Phi, \Psi \in \mathcal{F}(\coprod_{j \in J} X_j \times \coprod_{j \in J} X_j)$ be such that $\exists, \Phi \circ \Psi$ and take any $(\Phi_j)_{j=1}^{n}, (\Psi_l)_{l=1}^{m}$, such that $\Phi_j \in \mathcal{F}(X_{i_j} \times X_{i_j}), \Psi_l \in \mathcal{F}(X_{k_l} \times X_{k_l})$ for some $i_j, k_l \in J$ and $\bigcap_{j=1}^{n} (\iota_{i_j} \times \iota_{i_j})(\Phi_j) < \Phi$, $\bigcap_{l=1}^{m} (\iota_{k_l} \times \iota_{k_l})(\Psi_l) < \Psi$, respectively. Since $\Phi \circ \Psi$ exists and

$$\begin{aligned} \Phi \circ \Psi &> \bigcap_{j=1}^{n} (\iota_{i_j} \times \iota_{i_j})(\Phi_j) \circ \bigcap_{l=1}^{m} (\iota_{k_l} \times \iota_{k_l})(\Psi_l) \\ &= \bigcap_{\substack{j=1,\cdots,n \\ l=1,\cdots,m}} (\iota_{i_j} \times \iota_{i_j})(\Phi_j) \circ (\iota_{k_l} \times \iota_{k_l})(\Psi_l), \end{aligned}$$

there exists at least one pair of indices (i_j, k_l) such that $(\iota_{i_j} \times \iota_{i_j})(\Phi_j) \circ (\iota_{k_l} \times \iota_{k_l})(\Psi_l)$ exists and for such a pair (i_j, k_l), we have $i_j = k_l$. Take all such pairs (i_j, k_l) and rearrange by p_q for $q = 1, \cdots, r$. We note that

$$(\iota_{i_j} \times \iota_{i_j})(\Phi_j) \circ (\iota_{k_l} \times \iota_{k_l})(\Psi_l) = \begin{cases} 0 & \text{if } i_j \ne k_l \\ (\iota_{i_j} \times \iota_{k_l})(\Phi_j \circ \Psi_l) & \text{if } i_j = k_l \end{cases}$$

then

$$\Phi \circ \Psi > \bigcap_{q=1}^{r} (\iota_{p_q} \times \iota_{p_q})(\Phi_q \circ \Psi_q)$$

and since $\eta_{p_q}(\Phi_q \circ \Psi_q) \le \eta_{p_q}(\Phi_q) \vee \eta_{p_q}(\Psi_q)$ for all $q = 1, \cdots, r$, we have

$$\sup_{q=1}^{r} \eta_{p_q}(\Phi_q \circ \Psi_q) \le \sup_{j=1}^{n} \eta_{i_j}(\Phi_j) \vee \sup_{l=1}^{m} \eta_{k_l}(\Psi_l)$$

and consequently $\eta(\Phi \circ \Psi) \le \eta(\Phi) \vee \eta(\Psi)$.

Theorem 2.16. *The categories* **ASUConv**, **ASULim** *and* **AULim** *are cartesian closed.*

Proof. The fact that the categories **ASUConv** and **ASULim** are cartesian closed is immediate from Theorem 2.13 and Proposition 2.5 in [11] and for the category **AULim**, refer to [9].

3. The category ACHY

Recall from [7] that an *approach filter* (shortly, AFIL-) *space* is a pair (X, γ), where X is a set and $\gamma: \mathcal{F}(X) \to [0, \infty]$ is a map such that the following conditions are satisfied :

(AFIL1) $\gamma(\dot{x}) = 0$ for all $x \in X$,

(AFIL2) if $\mathcal{F}, \mathcal{G} \in \mathcal{F}(X)$ and $\mathcal{F} < \mathcal{G}$, then $\gamma(\mathcal{F}) \ge \gamma(\mathcal{G})$

and an approach filter space (X, γ) is called an *approach Cauchy* (shortly, ACHY- and in [8], *ultra approach Cauchy*) *space* if the following condition is satisfied :

(ACHY) if $\mathcal{F}, \mathcal{G} \in \mathcal{F}(X)$ are such that $\mathcal{F} \vee \mathcal{G}$ exists, then $\gamma(\mathcal{F} \cap \mathcal{G}) \le \gamma(\mathcal{F}) \vee \gamma(\mathcal{G})$.

Definition 3.1. An approach filter space (X, γ) is called an *approach semi-Cauchy* (shortly, ASCHY-) *space* if the following condition is satisfied :

(SACHY) $\gamma(\mathcal{F}) = \inf\{\sup_{j=1}^{n} \gamma(\mathcal{F}_j) \mid (\mathcal{F}_j)_{j=1}^{n} \in S(\mathcal{F})\}$,

where $S(\mathcal{F})$ is the collection of finite families $(\mathcal{F}_j)_{j=1}^{n}$ of filters on X such that $\bigcap_{j=1}^{n} (\mathcal{F}_j \times \mathcal{F}_j) < \mathcal{F} \times \mathcal{F}$.

Given AFIL-spaces (X, γ) and (Y, γ'), a map $f: X \to Y$ is called a *contraction* if $\gamma'(f(\mathcal{F})) \le \gamma(\mathcal{F})$ for all $\mathcal{F} \in \mathcal{F}(X)$.

Let **AFIL** denote the category of **AFIL**-spaces and contractions and denote the full subcategory of **AFIL** consisting of all ASCHY(ACHY)-spaces by **ASCHY** **(ACHY)**, respectively.

Recall from [7, 8] that **FIL** **(CHY)** is embedded as a bireflective and bicoreflective full subcategory in **AFIL** **(ACHY)**, respectively and an approach filter (Cauchy) space (X,γ) is a filter (Cauchy) space, respectively iff $\gamma(\mathcal{F}(X)) \subseteq \{0,\infty\}$. Analogously, we have the following.

For any semi-Cauchy space (X,c), the map $\gamma_c: \mathcal{F}(X) \to [0,\infty]$ defined by

$$\mathcal{F} \mapsto \gamma_c(\mathcal{F}) = \begin{cases} 0 & \text{for } \mathcal{F} \in c \\ \infty & \text{for } \mathcal{F} \notin c \end{cases}$$

is clearly an approach semi-Cauchy structure on X. Furthermore, for any semi-Cauchy spaces (X,c) and (Y,c'), a map $f:(X,c)\to(Y,c')$ is continuous iff $f:(X,\gamma_c)\to(Y,\gamma_{c'})$ is a contraction. So **SCHY** is embedded as a full subcategory in **ASCHY**.

Proposition 3.2. *An approach semi-Cauchy space* (X,γ) *is a semi-Cauchy space iff* $\gamma(\mathcal{F}(X)) \subseteq \{0,\infty\}$.

Theorem 3.3. *The category* **SCHY** *is a bireflective and bicoreflective subcategory of* **ASCHY**.

Theorem 3.4. *The category* **AFIL** *is a topological construct.*

For any source $(X \xrightarrow{f_j} (X_j,\gamma_j))_{j\in J}$ in **AFIL**, the map $\gamma: \mathcal{F}(X) \to [0,\infty]$ defined by

$$\mathcal{F} \mapsto \gamma(\mathcal{F}) = \sup_{j\in J} \gamma_j(f_j(\mathcal{F}))$$

is the initial approach filter structure on X and for any sink $((X_j,\gamma_j) \xrightarrow{f_j} X)_{j\in J}$ in **AFIL**, the map $\gamma: \mathcal{F}(X) \to [0,\infty]$ defined by

$$\gamma(\mathcal{F}) = \begin{cases} 0 & \text{if } \mathcal{F} = \dot{x} \text{ for } some\ x \in X \\ \inf\{\gamma_j(\mathcal{F}_j) \mid f_j(\mathcal{F}_j) < \mathcal{F} \text{ for } some\ j \in J,\ \mathcal{F}_j \in \mathcal{F}(X_j)\} & otherwise \end{cases}$$

is the final approach filter structure on X.

Proposition 3.5. *The category* **ASCHY** *is a bireflective subcategory of* **AFIL**.

Proof. For any AFIL-space (X,γ), the ASCHY-bireflector is $(X,\gamma) \xrightarrow{1_X} (X,\gamma_S)$, where $\gamma_S: \mathcal{F}(X) \to [0,\infty]$ is defined by

$$\gamma_S(\mathcal{F}) = \inf\Big\{\sup_{j=1}^{n} \gamma(\mathcal{F}_j) \mid (\mathcal{F}_j)_{j=1}^{n} \in S(\mathcal{F})\Big\}.$$

Proposition 3.6. *The category* **ACHY** *is a bireflective subcategory of the category* **ASCHY**.

Proof. First, to show that the condition (ACHY) implies (SACHY), take any $\mathcal{F} \in \mathcal{F}(X)$ and $(\mathcal{F}_j)_{j=1}^{n} \in S(\mathcal{F})$. We may assume that $\mathcal{F}_i \vee \mathcal{F}_j = 0$ for $i \neq j$, otherwise $\mathcal{F}_i$ and $\mathcal{F}_j$ can be replaced by $\mathcal{F}_i \cap \mathcal{F}_j$ and in that case $\sup_{j=1}^{n} \gamma(\mathcal{F}_j)$ is not changed by the condition (ACHY). So we can take $(A_j)_{j=1}^{n}$ such that $A_j \in \mathcal{F}_j$ for each $j = 1,\cdots,n$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. Then there is $A \in \mathcal{F}$ such that $A \times A \subseteq \bigcup_{j=1}^{n} (A_j \times A_j)$ and hence we have $A \subseteq A_k$ for some $k \in \{1,\cdots,n\}$. Thus we obtain $\mathcal{F}_k < \mathcal{F}$ and consequently it follows that $\gamma(\mathcal{F}) \le \gamma(\mathcal{F}_k) \le \sup_{j=1}^{n} \gamma(\mathcal{F}_j)$.

For any approach semi-Cauchy space (X,γ), the ACHY-bireflector is $(X,\gamma) \xrightarrow{1_X} (X,\gamma_C)$, where $\gamma_C: F(X) \to [0,\infty]$ is defined by

$$\mathcal{F} \mapsto \gamma_C(\mathcal{F}) = \inf\Big\{\sup_{j=1}^{n} \gamma(\mathcal{F}_j) \mid (\mathcal{F}_j)_{j=1}^{n} \in C(\mathcal{F})\Big\},$$

where $C(\mathcal{F})$ is the collection of finite families $(\mathcal{F}_j)_{j=1}^{n}$ of filters on X such that $\exists\, \mathcal{F}_j \vee \mathcal{F}_{j+1}$ for $j = 1, \cdots, n-1$ and $\bigcap_{j=1}^{n} \mathcal{F}_j < \mathcal{F}$.

Theorem 3.7. *The categories* **ASCHY** *and* **ACHY** *are topological constructs.*

Proof. This is an immediate consequence of Theorem 3.4 and theorem [3].

Proposition 3.8. The category **ACHY** is closed under the formation of coproducts in **AFIL**.

Proof. For any family $((X_j,\gamma_j))_{j\in J}$ of ACHY-spaces and a sink of canonical injections $(\iota_j : X_j \to \coprod_{j\in J} X_j)_{j\in J}$, the ACHY-coproduct structure

$$\gamma: \mathcal{F}(\coprod_{j\in J} X_j) \to [0,\infty]$$

arises from the AFIL-coproduct structure applying the ACHY-bireflector. So

$$\gamma:(\mathcal{F}) = \inf\{\sup_{j=1}^{n} \gamma_{i_j}(\Phi_j) \mid \text{for each } j = 1,\cdots,n,\ \mathcal{F}_j \in \mathcal{F}(X_{i_j}) \text{ for some } i_j \in J \text{ such that } (\iota_{i_j}(\mathcal{F}_j))_{j=1}^{n} \in C(\mathcal{F})\}$$

Taks any $(\iota_{i_j}(\mathcal{F}_j))_{j=1}^{n} \in C(\mathcal{F})$, then it must be $i_1 = \cdots = i_n = i_0$ for some $i_0 \in J$ and $\exists\, \mathcal{F}_j \vee \mathcal{F}_{j+1}$ for $j = 1,\cdots, n-1$. Furthermore,

$$\bigcap_{j=1}^{n} \iota_{i_j}(\mathcal{F}_j) = \bigcap_{j=1}^{n} \iota_{i_0}(\mathcal{F}_j) = (\iota_{i_0} \bigcap_{j=1}^{n} \mathcal{F}_j) < \mathcal{F}$$

and

$$\sup_{j=1}^{n} \gamma_{i_j}(\mathcal{F}_j) = \sup_{j=1}^{n} \gamma_{i_0}(\mathcal{F}_j) = \gamma_{i_0}(\bigcap_{j=1}^{n} \mathcal{F}_j)$$

Consequently, γ is in fact the AFIL-coproduct structure.

Remark 3.9. The category **ACHY** is not closed under the formation of quotients **AFIL**. We use the same example in Remark 3.4 [12] which proves that the category **CHY** is not closed under the formation of quotients in **FIL**. Let $X = \{a, b, c, d\}$ be a set with distinct four elements and

$$c=\{\mathcal{F}\in \mathcal{F}(X) \mid \{a,\ b\}\in\mathcal{F} \text{ or } \{c,\ d\}\in \mathcal{F}\}.$$

Then c is a Cauchy structure on X and the map γ_c: $\mathcal{F}(X)\rightarrow[0,\infty]$ defined by

$$\gamma_c: (\mathcal{F})=\begin{cases} 0 & \text{if}\{a,\ b\}\in\mathcal{F} \text{ or}\{c,\ d\}\in\mathcal{F} \\ \infty & \text{otherwis} \end{cases}$$

is an approach Cauchy structure on X.

Let R be the equivalence relation on X given by the partition $\{\{a\},\ \{b,\ c\},\ \{d\}\}$, $\omega:X\rightarrow X/R$ be the natural map and c_R be the quotient filter structure on X/R with respect to ω. Then the induced AFIL-structure γ_{c_R} on X/R turns out to be the quotient AFIL-structure $(\gamma_c)_R$ with respect to ω and it fails to be an approach Cauchy structure on X/R.

Proposition 3.10. *The category* **ACHY** *is a finally dense subcategory of* **AFIL**.

proof. It suffices to show that every AFIL-space is the quotient object of some ACHY-space. Take any AFIL-space $(X,\ \gamma)$ and for each $\mathcal{F}\in\mathcal{F}(X)$, let

$$\gamma_{\mathcal{F}}:\mathcal{F}(X)\rightarrow[0,\ \infty]$$

be a map defined by

$$\mathcal{G}\mapsto\gamma_{\mathcal{F}}(\mathcal{G})=\begin{cases} 0 & \text{if}\mathcal{G}=\dot{x} \text{ for some } x\in X \\ \gamma(\mathcal{F}) & \text{if}\mathcal{F}<\mathcal{G} \\ \infty & \text{otherwis}. \end{cases}$$

Then $\gamma_{\mathcal{F}}$ is an approach Cauchy structure on X and $(1_X)_{\mathcal{F}}$: $(X,\ \gamma_{\mathcal{F}})\rightarrow(X,\ \gamma)$ is a contraction. Furthermore, $((X,\ \gamma_{\mathcal{F}})\xrightarrow{(1_X)_{\mathcal{F}}}(X,\ \gamma))_{\mathcal{F}\in\mathcal{F}(X)}$ constitute a final epi-sink in **AFIL** and hence the unique map

$$\coprod_{\mathcal{F}\in\mathcal{F}(X)}(1_X)_{\mathcal{F}}:\coprod_{\mathcal{F}\in\mathcal{F}(X)}(X,\ \gamma_{\mathcal{F}})\rightarrow(X,\ \gamma)$$

such that $(\coprod_{\mathcal{F}\in\mathcal{F}(X)},(1_X)_{\mathcal{F}}),\circ,\iota_{\mathcal{F}}=(1_X)_{\mathcal{F}}$ is the quotient map in **AFIL**, where $\iota_{\mathcal{F}}:(X,\ \gamma)\rightarrow\coprod_{\mathcal{F}\in\mathcal{F}(X)}$, $(X,\ \gamma_{\mathcal{F}})$ is the canonical injection map for each $\mathcal{F}\in\mathcal{F}(X)$. Moreover, since $\coprod_{\mathcal{F}\in\mathcal{F}(X)}$, $(X,\ \gamma_{\mathcal{F}})$ is an approach Cauchy space by Proposition 3.8, we have the result.

4 The relation of the categories AULim and ACHY

For any AFIL-space $(X,\ \gamma)$, define a map $\eta_\gamma:\mathcal{F}\ X\times X\rightarrow[0,\infty]$ by

$$\Phi\mapsto\eta_\gamma(\Phi)=\inf\{\gamma(\mathcal{F})\mid\mathcal{F}\times\mathcal{F}<\Phi\}.$$

Proclaim 4.1. *For any* AFIL-*space* $(X,\ \gamma)$, *the map* η_γ *is an* ASUConv-*structure on* X.

Proclaim 4.2. *For any* AFIL-*spaces* $(X,\ \gamma)$ *and* $(Y,\ \gamma')$, *if a map* $f:(X,\ \gamma)\rightarrow(Y,\ \gamma')$ *is a contraction, then* $f:(X,\ \eta_\gamma)\rightarrow(Y,\ \eta_{\gamma'})$ *is a uniform contraction.*

Proclaim 4.3. *For any* ASCHY(ACHY)-*space* $(X,\ \gamma)$, *the map* $(\eta_\gamma)_L$ *is an* ASULim(AULim)-*structure on* X, *respectively.*

proof. For any ASCHY-space (X,γ), the map $(\eta_\gamma)_L:\mathcal{F}(X\times X)\rightarrow[0,\infty]$ is actually defined by

$$(\eta_\gamma)_{L(\Phi)}=\inf\{\sup_{j=1}^{n}\gamma(\mathcal{F}),\mid(\mathcal{F})_{j=1}^{n}\subseteq\mathcal{F}(X)\text{ such that }\bigcap_{j=1}^{n}(\mathcal{F}_j\times\mathcal{F}_j)<\Phi\}$$

and for the remainder, refer to Proposition 6.1 [9].

Proclaim 4.4. *For any* ASCHY-*spaces* $(X,\ \gamma)$ *and* $(Y,\ \gamma')$, *if a map* $f:(X,\ \gamma)\rightarrow(Y,\ \gamma')$ *is a contraction, then* $f:(X,(\eta_\gamma)_L)\rightarrow(Y,(\eta_{\gamma'})_L)$ *is a uniform contraction.*

Proof Take any $\Phi\in\mathcal{F}(X\times X)$ and $(\mathcal{F}_j)_{j=1}^{n}\subseteq\mathcal{F}(X)$ such that $\bigcap_{j=1}^{n}(\mathcal{F}_j\times\mathcal{F}_j)<\Phi$, then $(f(\mathcal{F}_j))_{j=1}^{n}\subseteq\mathcal{F}(Y)$ are such that $\bigcap_{j=1}^{n}(f(\mathcal{F}_j)\times f(\mathcal{F}_j))<(f\times f)(\Phi)$ and $\sup_{j=1}^{n}\gamma'(f(\mathcal{F}_j))\leq\sup_{j=1}^{n}\gamma(\mathcal{F}_j)$ and hence we are done.

For any ASUConv-space $(X,\ \eta)$, let $\gamma_\eta:\mathcal{F}(X)\rightarrow[0,\infty]$ be the map defined by

$$\mathcal{F}\mapsto\gamma_\eta(\mathcal{F})=\eta(\mathcal{F}\times\mathcal{F}).$$

Proclaim 4.5. *For any* ASUConv(ASULim, AULim)*-space* $(X,\ \eta)$, *the pair* $(X,\ \gamma_\eta)$ *is an* AFIL(ASCHY, ACHY)}-*space.*

Proof Refer to Proposition 6.3 [9].

Proclaim 4.6. *For any* ASUConv-*spaces* $(X,\ \eta)$ *and* $(Y,\ \eta')$, if $f:(X,\ \eta)\rightarrow(Y,\ \eta')$ *is a uniform contraction, then* $f:(X,\ \gamma_\eta)\rightarrow(Y,\ \gamma_{\eta'})$ *is a contraction.*

Proclaim 4.7. (1) *For any* AFIL-*structure* γ *on a set* X, $\gamma=\gamma_{\eta_\gamma}$.(2) *For any* ASCHY-*structure* γ *on a set* X, $\gamma=\gamma_{(\eta_\gamma)_L}$.

Proof (1) For any $\mathcal{F}\in\mathcal{F}(X)$,

$$\gamma_{\eta_\gamma}(\mathcal{F})=\eta_\gamma(\mathcal{F}\times\mathcal{F})=\inf\{\gamma(\mathcal{G})\mid\mathcal{G}\times\mathcal{G}<\mathcal{F}\times\mathcal{F}$$

and since $\mathcal{G} \times \mathcal{G} \langle \mathcal{F} \times \mathcal{F}$ implies $\mathcal{G} \langle \mathcal{F}$, it is proved..

(2) For any $\mathcal{F} \in \mathcal{F}(X)$,

$$\begin{aligned} \gamma_{(\eta_\gamma)_L}(\mathcal{F}) &= (\eta_\gamma)_L(\mathcal{F} \times \mathcal{F}) \\ &= \inf\{\sup \gamma(\mathcal{F}_j) \mid (\mathcal{F}_j)_{j=1}^n \subseteq (X) \text{ such that } \\ &\quad \bigcap_{j=1}^n (\mathcal{F}_j \times \mathcal{F}_j) \langle \mathcal{F} \times \mathcal{F}\} \\ &= \gamma(\mathcal{F}) \end{aligned}$$

by the condition (ASCHY).

Proclaim 4.8. (1) *For any* ASUConv-*structure* eta on X, $\eta \leq \eta_{\gamma_\eta}$..

(2) *For any* ASULim-*structure* η on X, $\eta \leq (\eta_{\gamma_\eta})_L$.

Proof (1) For any $\Phi \in \mathcal{F}(X \times X)$,

$$\begin{aligned} \eta_{\gamma_\eta} &= \inf\{\gamma_\eta(\mathcal{F}) \mid \mathcal{F} \times \mathcal{F} \langle \Phi\} \\ &= \inf\{\eta(\mathcal{F} \times \mathcal{F}) \mid \mathcal{F} \times \mathcal{F} \langle \Phi\} \\ &\geq \eta(\Phi). \end{aligned}$$

(2) For any $\Phi \in \mathcal{F}(X \times X)$,

$$\begin{aligned} (\eta_{\gamma_\eta})_L &= \inf\{\sup_{j=1}^{n} \gamma_\eta(\mathcal{F}_j) \mid (\mathcal{F}_j)_{j=1}^n \subseteq \mathcal{F}(X) \text{ such that } \bigcap_{j=1}^n (\mathcal{F}_j \times \mathcal{F}_j) \langle \Phi\} \\ &= \inf\{\sup_{j=1}^{n} \eta(\mathcal{F}_j \times \mathcal{F}_j) \mid (\mathcal{F}_j)_{j=1}^n \subseteq \mathcal{F}(X) \text{ such that } \bigcap_{j=1}^n (\mathcal{F}_j \times \mathcal{F}_j) \langle \Phi\} \\ &\geq \eta(\Phi). \end{aligned}$$

Theorem 4.9. **AFIL (ASCHY, ACHY)** *is a bicoreflective subcategory of* **ASUConv (ASULim, AULim)**, *respectively.*

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