퍼지 및 지능시스템학회 논문지 2001, Vol. 11, No. 2, pp. 173-177

Products of *TL*-Finite State Machines

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ABSTRACT

We introduce cascade products, wreath products, sums and joins of *TL*-finite state machines and investigate their algebraic structures. Also we study the relations with other products of *TL*-finite state machines.

Key Words : Fuzzy finite state machine; *TL*-finite state machine; covering; cascade product; wreath product; sums and joins.

1. Introduction

Since Wee [9] in 1967 introduced the concept of fuzzy automata following Zadeh [11], fuzzy automata theory has been developed by many researchers. Recently Malik et al. [4-7] introduced the concepts of fuzzy finite state machines and fuzzy transformation semigroups based on Wee's concept [9] of fuzzy automata and related concepts and applied algebraic technique. In this paper, we introduce cascade products, wreath products, sums and joins of *TL*-finite state machines that are generalizations of crisp concepts in algebraic automata theory and investigate their algebraic structures. Also we study the relations with other products of *TL*-finite state machines.

For the terminology in (crisp) algebraic automata theory, we refer to [2].

2. Preliminaries

We let L denote a complete lattice that contains at least two distinct elements. The meet, join, and partial ordering will be written as $\wedge, \vee$, and $\leq$, respectively. We also write 1 and 0 for the greast element and least element of L, respectively.

Definition 2.1 A triple $M=(Q,X,\tau)$ where Q and X are finite nonempty sets and τ is a L-subset of $Q\times X\times Q$, i.e., τ is a function from $Q\times X\times Q$ to L, is called an L-finite state machine.

Let $M=(Q,X,\tau)$ be an L-finite state machine. Then Q is called the set of states and X is called the set of input symbols. Let X^+ denote the set of all words of elements of X of finite length with the empty word λ.

접수일자 : 2000년 11월 27일
완료일자 : 2001년 2월 6일

Definition 2.2 [8]. A binary operation T on L is called a t-norm if

(1) $T(a,1)=a$
(2) $T(a,b)\leq T(a,c)$ whenever $b\leq c$,
(3) $T(a,b)=T(b,a)$,
(4) $T(a,T(b,c))=T(T(a,b),c)$

for all $a,b,c\in L$.

Definition 2.3 [10] (i) A t-norm T on L is said to be $\vee$-distributive if $T(a,b\vee c)=T(a,b)\vee T(a,c)$ for all $a,b,c\in L$.

(ii) T is said to be positive-definite if $T(a,b)>0$ for all $a,b\in L\setminus\{0\}$.

One gets immediately $T(0,a)=0$ and $T(a,b)\leq a\wedge b$ for all $a,b\in L$. Throughout this paper, T shall mean a positive-definite and $\vee$-distributive t-norm on L unless otherwise specified.

We will denote $T(a_1,T(a_2,\cdots,T(a_{n-2},T(a_{n-1},a_n))\cdots))$ by $T(a_1,\cdots,a_n)$ where $a_1,\cdots,a_n\in L$.

Example 2.4 Let $L=[0,1]\times\{1\}$. Define a partial order $\leq$ on L by for $a=(a_1,1), b=(b_1,1)\in L$, $a\leq b$ if $a_1\leq b_1$. Define $T(a,b)=(a_1,b_1,1)$ where $a=(a_1,1)$, $b=(b_1,1)\in L$. Then T is a positive-definite and $\vee$-distributive t-norm on L.

Definition 2.5 Let $M=(Q,X,\tau)$ be an L-finite state machine. Define $\tau^+:Q\times X^+\times Q\to L$ by

$$\tau^+(q,\lambda,p)=\begin{cases}1 & \text{if } q=p\\ 0 & \text{if } q\neq p\end{cases}$$

$$\begin{aligned}&\tau^+(p,a_1\cdots a_n,q)\\ &=\vee\{T(\tau(p,a_1,r_1),\tau(r_1,a_2,r_2),\cdots,\\ &\quad \tau(r_{n-2},a_{n-1},r_{n-1}),\tau(r_{n-1},a_n,q))\mid r_i\in Q\}\end{aligned}$$

where $p,q\in Q$ and $a_1,\cdots,a_n\in X$. When T is applied to M as above, M is called a TL-finite state machine(briefly, a TL-fsm).

Remark. In Definition 2.5 if we let $T=\wedge$ and $L=[0,1]$, then the concept of a TL-fsm is the concept of [7].

Hereafter a fuzzy state machine will always be written as a TL-finite state machine because a fuzzy state machine always induces a TL-finite state machine as in Definition 2.5.

Definition 2.6. Let $M_1=(Q_1, X_1, \tau_1)$ and $M_2=(Q_2, X_2, \tau_2)$ be TL-finite state machines. Let $\alpha: Q_1\rightarrow Q_2$ and $\beta: X_1\rightarrow X_2$ be mappings. Then the pair (α,β) is called a TL-finite state machine homomorphism (which is written by (α,β) if

$$\tau_1(p,a,q)\le\tau_2(\alpha(p),\beta(a),\alpha(q)), p,q\in Q_1, a\in X_1$$

The homomorphism (α,β) : $M_1\rightarrow M_2$ is called isomorphism if α and β are bijective respectively.

Definition 2.7 Let $M_1=(Q_1,X_1,\tau_1)$ and $M_2=(Q_2,X_2,\tau_2)$ be TL-finite state machines. If $\xi: X_1\rightarrow X_2$ is a function and $\eta: Q_2\rightarrow Q_{10}$ is a surjective partial function such that $\tau_1^+(\eta(p),a,\eta(q))\le\tau_2^+(p,\xi(a),q)$ for all p,q in the domain of η and $a\in X_1$, then we say that (η,ξ) is a covering of M_1 by M_2 and that M_2 covers M_1 and denote by $M_1\le M_2$. Moreover, if the inequality always turns out equality, then we say that (η,ξ) is a complete covering of M_1 by M_2 and that M_2 completely covers M_1 and denote by $M_1\le_c M_2$.

In Definition 2.7, we abused the function ξ. We will write the natural semigroup homomorphism from X_1^+ to $X_2^|$ induced by ξ by ξ also for convenience sake. We give an example that is elementary and important.

Example 2.8 Let $M=(Q,X,\tau)$ be a TL-finite state machine. Define an equivalence relation $\sim$ on X by $a\sim b$ if and only if $\tau(p,a,q)=\tau(p,b,q)$ for all $p,q\in Q$. Construct a fuzzy finite state machine $M_1=(Q,X\sim,\tau^{\sim})$ by defining $\tau^{\sim}(p,[a],q)=\tau(p,a,q)$. Now define $\xi: X\rightarrow X/\sim$ by $\xi(a)=[a]$ and $\eta=1_Q$. Then (η,ξ) is a complete covering of M by M_1 clearly.

Proposition 2.9 Let M_1, M_2 and M_3 be TL-finite state machines. If $M_1\le M_2$ [resp. $M_1\le_c M_2$] and $M_2\le M_3$ [$M_2\le_c M_3$], then $M_1\le M_3$ [$M_1\le_c M_3$].

Proof. It is straightforward.

3. Several products of TL-finite statemachines

Several products of finite state machines are in [2]. Some of these products have been fuzzified in [1], [3] and [5]. In this section we introduce cascade products, wreath products, sums and joins of TL-finite state machines.

Definition 3.1. Let $M_1=(Q_1, X_1, \tau_1)$ and $M_2=(Q_2, X_2, \tau_2)$ be TL-finite state machines. The cascade product $M_1\omega M_2$ of M_1 and M_2 with respect to $\omega: Q_2\times X_2\rightarrow X_1$ is the TL-finite state machine $(Q_1\times Q_2, X_2, \tau_1\omega\tau_2)$ with

$$\begin{aligned}&(\tau_1\omega\tau_2)((p_1,p_2),b,(q_1,q_2))\\ &= T(\tau_1(p_1,\omega(p_2,b),q_1),\tau_2(p_2,b,q_2))\end{aligned}$$

where $(p_1,p_2)\in Q_1\times Q_2$, $b\in X_2$ and $(q_1,q_2)\in Q_1\times Q_2$.

Definition 3.2 Let $M_1=(Q_1, X_1, \tau_1)$ and $M_2=(Q_2,X_2,\tau_2)$ be TL-finite state machines. The wreath product $M_1\circ M_2$ of M_1 and M_2 is the TL-finite state machine $(Q_1\times Q_2, X_1^{Q_2}\times X_2, \tau_1\circ\tau_2)$ with

$$\begin{aligned}&(\tau_1\circ\tau_2)((p_1,p_2),(f,b),(q_1,q_2))\\ &= T(\tau_1(p_1,f(p_2),q_1),\tau_2(p_2,b,q_2))\end{aligned}$$

where $(p_1,p_2)\in Q_1\times Q_2$, $b\in X_2$, $(q_1,q_2)\in Q_1\times Q_2$ and $f\in X_1^{Q_2}$.

Definition 3.3 Let $M_1=(Q_1,X_1,\tau_1)$ and $M_2=(Q_2,X_2,\tau_2)$ be TL-finite state machines, where $Q_1\cap Q_2=\emptyset$ and $X_1\cap X_2=\emptyset$. The join $M_1\vee M_2$ of M_1 and M_2 is the TL-finite state machine $(Q_1\cup Q_2, X_1\cup X_2, \tau_1\vee\tau_2)$ with

$$(\tau_1\vee t_2)(p,a,q)=\begin{cases}\tau_1(p,a,q) & \text{if } (p,a,q)\in Q_1\times X_1\times Q_1\\ \tau_2(p,a,q) & \text{if } (p,a,q)\in Q_2\times X_2\times Q_2\\ 0, & \text{otherwise}\end{cases}$$

Definition 3.4 Let $M_1=(Q_1,X_1,\tau_1)$ and $M_2=(Q_2,X_2,\tau_2)$ be TL-finite state machines, where $Q_1\cap Q_2=\emptyset$ and $X_1\cap X_2=\emptyset$. The join* $M_1\vee^* M_2$ of M_1 and M_2 is the TL-finite state machine $(Q_1\cup Q_2, X_1\cup X_2, \tau_1\vee^*\tau_2)$ with

$$(\tau_1\vee^*\tau_2)(p,a,q)$$
$$=\begin{cases}\tau_1(p,a,q) & \text{if } (p,a,q)\in Q_1\times X_1\times Q_1\\ \tau_2(p,a,q) & \text{if } (p,a,q)\in Q_2\times X_2\times Q_2\\ 1 & \text{if } (p,a,q)\in(Q_1\times X_2\times Q_2)\cup(Q_2\times X_1\times Q_1)\\ 0, & \text{otherwise}\end{cases}$$

Definition 3.5 Let $M_1=(Q_1,X_1,\tau_1)$ and $M_2=(Q_2,X_2,\tau_2)$ be TL-finite state machines, where

$Q_1 \cap Q_2 = \emptyset$. The sum $M_1 + M_2$ of M_1 and M_2 is the TL-finite state machine $(Q_1 \cup Q_2, X_1 \times X_2, \tau_1 + \tau_2)$ with

$$(\tau_1 + \tau_2)(p, (a, b), q) = \begin{cases} \tau_1(p, a, q) & \text{if } p, q \in Q_1 \\ \tau_2(p, b, q) & \text{if } p, q \in Q_2 \\ 0, & \text{otherwise} \end{cases}$$

Definition 3.6. Let $M_1 = (Q_1, X_1, \tau_1)$ and $M_2 = (Q_2, X_2, \tau_2)$ be TL-finite state machines, where $Q_1 \cap Q_2 = \emptyset$. The sum* $M_1 +^* M_2$ of M_1 and M_2 is the TL finite state machine $(Q_1 \cup Q_2, X_1 \times X_2, \tau_1 +^* \tau_2)$ with

$$(\tau_1 +^* \tau_2)(p, (a, b), q) = \begin{cases} \tau_1(p, a, q) & \text{if } p, q \in Q_1 \\ \tau_2(p, b, q) & \text{if } p, q \in Q_2 \\ 1 & \text{if } (p, q) \in (Q_1 \times Q_2) \cup (Q_2 \times Q_1) \\ 0, & \text{otherwise} \end{cases}$$

4. Associative properties

Proposition 4.1 Let M_1, M_2 and M_3 be TL-finite state machines. Then the following are hold :

(i) $(M_1 \wedge M_2) \wedge M_3 = M_1 \wedge (M_2 \wedge M_3)$.

(ii) $(M_1 \times M_2) \times M_3 = M_1 \times (M_2 \times M_3)$.

Proof. It is straightforward.

Now we prove that wreath product, join and sum of TL-finite state machines are associative.

Theorem 4.2 Let $M_1 = (Q_1, X_1, \tau_1)$, $M_2 = (Q_2, X_2, \tau_2)$ and $M_3 = (Q_3, X_3, \tau_3)$ be TL-finite state machines. Then the following are hold :

(i) $(M_1 \circ M_2) \circ M_3 \cong M_1 \circ (M_2 \circ M_3)$

(ii) $(M_1 \vee M_2) \vee M_3 \cong M_1 \vee (M_2 \vee M_3)$, where $Q_1 \cap Q_2 \cap Q_3 = \emptyset$ and $X_1 \cap X_2 \cap X_3 = \emptyset$

(iii) $(M_1 + M_2) + M_3 \cong M_1 + (M_2 + M_3)$, where $Q_1 \cap Q_2 \cap Q_3 = \emptyset$

Proof. (i) Let $\alpha: (Q_1 \times Q_2) \times Q_3 \to Q_1 \times (Q_2 \times Q_3)$ be the natural mapping. Then α is a bijective mapping. Let $g_1: X_1^{Q_2} \times X_2 \to X_1^{Q_2}$ and $g_2: X_1^{Q_2} \times X_2 \to X_2$ be the natural projection mappings. Given a mapping $f: Q_3 \to X_1^{Q_2} \times X_2$ let $f_1 = g_1 \circ f$ and $f_2 = g_2 \circ f$. Define $\beta: (X_1^{Q_2} \times X_2)^{Q_3} \times X_3 \to X_1^{Q_2 \times Q_3} \times (X_2^{Q_3} \times X_3)$ by $\beta((f, b_3)) = (h, (f_2, b_3))$, where $h: Q_2 \times Q_3 \to X_1$ by $h((p_2, p_3)) = f_1(p_3)(p_2)$. We can easily show that β is injective. Let $(w, (v, b_3)) \in X_1^{Q_2 \times Q_3} \times (X_2^{Q_3} \times X_3)$ and define $u: Q_3 \to X_1^{Q_2 \times Q_3}$ by $u(p_3) = (v^{p_3}, w(p_3))$ where $v^{p_3}(p_2) = v(p_2, p_3)$. Then $b((u, b_3)) = (w, (v, b_3))$ and thus β is surjective. Now

$$\begin{aligned}
& (\tau_1 \circ (\tau_2 \circ \tau_3))(\alpha((p_1, p_2), p_3), \beta((f, b_3)), \alpha((q_1, q_2), q_3))) \\
= & (\tau_1 \circ (\tau_2 \circ \tau_3))((p_1, (p_2, p_3)), (h, (f_2, b_3)), (q_1, (q_2, q_3))) \\
= & T(\tau_1(p_1, h(p_2, p_3), q_1), (\tau_2 \circ t_3)((p_2, p_3), (f_2, b_3), (q_2, q_3))) \\
= & T(\tau_1(p_1, h(p_2, p_3), q_1), T(\tau_2(p_2, f_2(p_3), q_2), \tau_3(p_3, b_3, q_3))) \\
= & T(T(\tau_1(p_1, h(p_2, p_3), q_1), \tau_2(p_2, f_2(p_3), q_2)), \tau_{3(p_3}, b_3, q_3))) \\
= & T(T(\tau_1(p_1, f_1(p_3)(p_2), q_1), \tau_2(p_2, f_2(p_3), q_2)), \tau_3(p_3, b_3, q_3))) \\
= & T((\tau_1 \circ \tau_2)((p_1, p_2), (f_1(p_3), f_2(p_3)), (q_1, q_2)), \tau_3(p_3, b_3, q_3))) \\
= & T((\tau_1 \circ \tau_2)((p_1, p_2), f(p_3), (q_1, q_2)), \tau_3(p_3, b_3, q_3))) \\
& \text{since } (f_1(p_3), f_2(p_3)) = f(p_3) \\
= & ((\tau_1 \circ \tau_2) \circ \tau_3)(((p_1, p_2), p_3), (f, b_3), ((q_1, q_2), q_3)))
\end{aligned}$$

(ii) Let α be an identity mapping on $Q_1 \cup Q_2 \cup Q_3$ and β be an identity mapping on $X_1 \cup X_2 \cup X_3$. Then (α, β) be a required isomorphism.

(iii) Let α be an identity mapping on $Q_1 \cup Q_2 \cup Q_3$ and $\beta: (X_1 \times X_2) \times X_3 \to X_1 \times (X_2 \times X_3)$ be the natural mapping. Then (α, β) be a required isomorphism.

Remark. $\vee^*$ and $+^*$ are not associative operations.

5. Coverings

Proposition 5.1 Let $M_1 = (Q_1, X_1, \tau_1)$ and $M_2 = (Q_2, X_2, \tau_2)$ be TL-finite state machines. Then

(i) $M_1 \wedge M_2 \leq_c M_1 \times M_2$ where $X_1 = X_2$.

(ii) $M_1 \omega M_2 \leq_c M_1 \circ M_2$.

Proof. We only prove (ii).

(ii) Let $\xi: X_2 \to X_1^{Q_2} \times X_2$ be a function such that $\xi(x_2) = (\xi_1(x_2), \xi_2(x_2))$ where $x_2 \in X_2$, $\xi_1(x_2): Q_2 \to X_1$ is a function defined by $\xi_1(x_2)(p_2) = \omega(p_2, \xi_2(x_2))$ and $\xi_2 = I_{X_2}$. And let $\eta = 1_{Q_1 \times Q_2}$. Then for each $(p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$ and $x_2 \in X_2$, we have

$$\begin{aligned}
& (\tau_1 \omega \tau_2)(\eta((p_1, p_2)), x_2, \eta((q_1, q_2))) \\
= & (\tau_1 \omega \tau_2)((p_1, p_2), \xi_2(x_2), (q_1, q_2)) \\
= & T(\tau_1(p_1, \omega(p_2, \xi_2(x_2)), q_1), \tau_2(p_2, \xi_2(x_2), q_2)) \\
= & T(\tau_1(p_1, \xi_1(x_2)(p_2), q_1), \tau_2(p_2, \xi_2(x_2), q_2)) \\
= & (\tau_1 \circ \tau_2)((p_1, p_2), (\xi_1(x_2), \xi_2(x_2)), (q_1, q_2)) \\
= & (\tau_1 \circ t_2)((p_1, p_2), \xi_2(x_2), (q_1, q_2)).
\end{aligned}$$

Hence

$$M_1 \omega M_2 \leq M_1 \circ M_2.$$

Proposition 5.2 Let $M_1=(Q_1,X_1,\tau_1)$ and $M_2=(Q_2,X_2,\tau_2)$ be TL-finite state machines such that $Q_1\cap Q_2=\varnothing$ and $X_1\cap X_2=\varnothing$. Then

(i) $M_1\le M_1\vee M_2$

(ii) $M_1\le M_1\vee^* M_2$

Proof. We only prove (i).
Let $\eta: Q_1\cup Q_2\to Q_1$ be a partial surjective function defined by $\eta(p_1)=p_1$, where $p_1\in Q_1$. And $\xi: X_1\to X_1\cup X_2$ be the natural projection. Then (η,ξ) is a required covering of M_1 by $M_1\vee M_2$.

Theorem 5.3 Let $M_1=(Q_1,X,\tau_1)$ and $M_2=(Q_2,X,\tau_2)$ be TL-finite state machines. Then

(i) $M_1\vee M_2\le M_1\vee^* M_2$.

(ii) $M_1+M_2\le M_1+^* M_2$.

Proof. (i) Let η and ξ identity mappings on $Q_1\cup Q_2$ and $X_1\cup X_2$ respectively.
Case (a) : If $(p,a,q)\in Q_1\times X_1\times Q_1$, then
$(\tau_1\vee\tau_2)(\eta(p),a,\eta(q))=\tau_1(p,a,q)=(\tau_1\vee^*\tau_2)(p,\xi(a),q)$.
Case (b) : If $(p,a,q)\in Q_2\times X_2\times Q_2$, then
$(\tau_1\vee\tau_2)(\eta(p),a,\eta(q))=t_2(p,a,q)=(\tau_1\vee^*\tau_2)(p,\xi(a),q)$.
Case (c) : If $(p,a,q)\in(Q_1\times X_1\times Q_2)\cup(Q_2\times X_2\times Q_1)$, then $(\tau_1\vee\tau_2)(\eta(p),a,\eta(q))=(\tau_1\vee\tau_2)(p,a,q)=0\le 1$ $=(\tau_1\vee^*\tau_2)(p,\xi(a),q)$.

(ii) The proof is similar to the proof of (i).

Theorem 5.4 Let $M_1=(Q_1,X_1,\tau_1)$, $M_2=(Q_2,X_2,\tau_2)$ and $M_3=(Q_3,X_3,\tau_3)$ be TL-finite state machines such that $M_1\le M_2$. Then

(i) $M_1\vee M_3\le M_2\vee M_3$.

(ii) $M_1\vee^* M_3\le M_2\vee^* M_3$.

(iii) $M_1+M_3\le M_2+M_3$.

(iv) $M_1+^* M_3\le M_2+^* M_3$.

Proof. We only show that (i) holds. Since $M_1\le M_2$, there exist a partial surjective mapping $\eta: Q_2\to Q_1$ and a mapping $\xi: X_1\to X_2$ such that
$\tau_1(\eta(p),a,\eta(q))\le\tau_2(p,\xi(a),q)$.

Define $\eta': Q_2\cup Q_3\to Q_1\cup Q_3$ by $\eta'(p)=\begin{cases} p & \text{if } p\in Q_3\\ \eta(p) & \text{if } p\in Q_2\end{cases}$

and $\xi': X_1\cup X_3\to X_2\cup X_3$ by $\xi'(a)=\begin{cases} a & \text{if } a\in X_3\\ \xi(a) & \text{if } p\in X_1\end{cases}$.

Then η' is a partial surjective mapping and ξ' is a mapping. Show that $(\tau_1\vee\tau_3)(\eta'(p),a,\eta'(q))\le(\tau_2\vee\tau_3)(p,\xi'(a),q)$, where $p,q\in Q_2\cup Q_3$ and $a\in X_1\cup X_3$.

(i) If $p,q\in Q_2$ and $a\in X_1$, then

$$\begin{aligned}(\tau_1\vee\tau_3)(\eta'(p),a,\eta'(q)) &= \tau_1(\eta(p),a,\eta(q))\\ &\le \tau_2(p,\xi(a),q)\\ &=(\tau_2\vee t_3)(p,\xi'(a),q)\end{aligned}$$

(ii) If $p,q\in Q_3$ and $a\in X_3$, then

$$\begin{aligned}(\tau_1\vee\tau_3)(\eta'(p),a,\eta'(q)) &= \tau_3(p,a,q)\\ &=\tau_3(p,\xi'(a),q)\\ &=(\tau_2\vee\tau_3)(p,\xi'(a),q)\end{aligned}$$

(iii) In all other cases

$$\begin{aligned}(\tau_1\vee\tau_3)(\eta'(p),a,\eta'(q)) &=0\\ &\le(\tau_2\vee\tau_3)(p,\xi'(a),q)\end{aligned}$$

This completes the proof.

Theorem 5.5 Let $M_1=(Q_1,X_1,\tau_1)$, $M_2=(Q_2,X_2,\tau_2)$ and $M_3=(Q_3,X_3,\tau_3)$ be TL-finite state machines such that $M_1\le M_2$. Then

(i) $M_1\circ M_3\le M_2\circ M_3$.

(ii) $M_3\circ M_1\le M_3\circ M_2$.

Proof. Since $M_1\le M_2$, there exist $\eta: Q_2\to Q_1$ and $\xi: X_1\to X_2$ such that $\tau_1(\eta(p_2),a_1,\eta(q_2))\le\tau_2(p_2,\xi(a_1),q_2)$.

(i) Define $\eta': Q_2\times Q_3\to Q_1\times Q_3$ by $\eta'((p_2,p_3))=(\eta(p_2),p_3)$ and define $\xi': X_1^{Q_1}\times X_3\to X_2^{Q_1}\times X_3$ by $\xi'((f,a_3))=(\xi\circ f,a_3)$. Then

$$\begin{aligned}&(\tau_1\circ\tau_3)(\eta'(p_2,p_3),(f,a_3),\eta'(q_2,q_3))\\ =\;&(\tau_1\circ\tau_3)((\eta(p_2),p_3),(f,a_3),(\eta(q_2),q_3))\\ =\;&T(\tau_1(\eta(p_2),f(p_3),\eta(q_2)),\tau_3(p_3,a_3,q_3))\\ \le\;&T(\tau_2(p_2,(\xi\circ f)(p_3),q_2),\tau_3(p_3,a_3,q_3))\\ =\;&(\tau_2\circ\tau_3)((p_2,p_3),\xi'((f,a_3)),(q_2,q_3))\end{aligned}$$

(ii) Define $\eta': Q_3\times Q_2\to Q_3\times Q_1$ by $\eta'((p_3,p_2))=(p_3,\eta(p_2))$ and define $\xi': X_3^{Q_1}\times X_1\to X_3^{Q_1}\times X_2$ by $\xi'((f,a_1))=(f\circ\eta,\xi(a_1))$. Then

$$\begin{aligned}&(\tau_3\circ\tau_1)(\eta'(p_3,p_2),(f,a_1),\eta'(q_3,q_2))\\ =\;&(\tau_3\circ\tau_1)((p_3,\eta(p_2)),(f,a_1),(q_3,\eta(q_2)))\\ =\;&T(\tau_3(p_3,f(\eta(p_2)),q_3),\tau_1(\eta(p_2),a_1,\eta(q_2)))\\ \le\;&T(\tau_3(p_3,(f\circ\eta)(p_2),q_3),\tau_2(p_2,\xi(a_1),q_2))\\ =\;&(\tau_3\circ\tau_2)((p_3,p_2),\xi'((f,a_1)),(q_3,q_2))\end{aligned}$$

This completes the proof.

Theorem 5.6 Let $M_1=(Q_1,X_1,\tau_1)$, $M_2=(Q_2,X_2,\tau_2)$ and $M_3=(Q_3,X_3,\tau_3)$ be TL-finite state machines such that $Q_2\cap Q_3=\varnothing$. Then

(i) $M_1\circ(M_2\vee M_3)\le_c(M_1\circ M_2)\vee(M_1\circ M_3)$ where $X_2\cap X_3=\varnothing$

(ii) $M_1\circ(M_2\vee^* M_3)\le(M_1\circ M_2)\vee^*(M_1\circ M_3)$ where $X_2\cap X_3=\varnothing$

(iii) $M_1 \circ (M_2 + M_3) \leq_c (M_1 \circ M_2) + (M_1 \circ M_3)$

(iv) $M_1 \circ (M_2 +^* M_3) \leq (M_1 \circ M_2) +^* (M_1 \circ M_3)$

Proof. We only prove (i) and (iii).

(i) Recall $M_1 \circ (M_2 \vee M_3) = (Q_1 \times (Q_2 \cup Q_3) X_1^{Q_2 \cup Q_3}, \times (X_2 \cup X_3), \tau_1 \circ (\tau_2 \vee \tau_3))$ and $(M_1 \circ M_2) \vee (M_1 \circ M_3) = ((Q_1 \times Q_2) \cup (Q_1 \times Q_3), (X_1^{Q_2} \times X_2) \cup (X_1^{Q_3} \times X_3), (\tau_1 \circ \tau_2) \vee (\tau_1 \circ \tau_3))$. Define $\eta: (Q_1 \times Q_2) \cup (Q_1 \times Q_3) \to Q_1 \times (Q_2 \cup Q_3)$ by $\eta((p, q)) = (p, q)$. And define $\xi: X_1^{Q_2 \cup Q_3} \times (X_2 \cup X_3) \to (X_1^{Q_2} \times X_2) \cup (X_1^{Q_3} \times X_3)$ by

$$\xi((f, b)) = \begin{cases} (f|_{Q_2}, b) & \text{if } b \in X_2 \\ (f|_{Q_3}, b) & \text{if } b \in X_3 \end{cases}$$

Then

$$\begin{aligned} & \tau_1 \circ (\tau_2 \vee \tau_3)(\eta(p, p'), (f, b), \eta(q, q')) \\ = \; & \tau_1 \circ (\tau_2 \vee \tau_3)((p, p'), (f, b), (q, q')) \\ = \; & T(\tau_1(p, f(p'), q), (\tau_2 \vee \tau_3)(p', b, q')) \\ = \; & \begin{cases} T(\tau_1(p, f(p'), q), \tau_2(p', b, q')), (p', b, q') \in Q_2 \times X_2 \times Q_2 \\ T(\tau_1(p, f(p'), q), \tau_3(p', b, q')), (p', b, q') \in Q_3 \times X_3 \times Q_3 \\ 0, \text{otherwise} \end{cases} \\ = \; & \begin{cases} (\tau_1 \circ t_2)((p, p'), (f|_{Q_2}, b), (q, q')), (p', b, q') \in Q_2 \times X_2 \times Q_2 \\ (\tau_1 \circ \tau_3)((p, p'), (f|_{Q_3}, b), (q, q')), (p', b, q') \in Q_3 \times X_3 \times Q_3 \\ 0, \text{otherwise} \end{cases} \\ = \; & ((\tau_1 \circ \tau_2) \vee (\tau_1 \circ \tau_3))((p, p'), \xi(f, b), (q, q')) \end{aligned}$$

(iii) Recall $M_1 \circ (M_2 + M_3) = (Q_1 \times (Q_2 \cup Q_3), X_1^{Q_2 \cup Q_3} \times (X_2 \times X_3), \tau_1 \circ (\tau_2 + \tau_3))$ and $(M_1 \circ M_2) + (M_1 \circ M_3) = ((Q_1 \times Q_2) \cup (Q_1 \times Q_3), (X_1^{Q_2} \times X_2) \times (X_1^{Q_3} \times X_3), (\tau_1 \circ \tau_2) + (\tau_1 \circ \tau_3))$. Define $\eta: (Q_1 \times Q_2) \cup (Q_1 \times Q_3) \to Q_1 \times (Q_2 \cup Q_3)$ by $\eta((p, q)) = (p, q)$. And define $\xi: X_1^{Q_2 \cup Q_3} \times (X_2 \times X_3) \to (X_1^{Q_2} \times X_2) \times (X_1^{Q_3} \times X_3)$ by $\xi((f, (b_{2, b_3}))) = ((f|_{Q_2}, b_2), (f|_{Q_3}, b_3))$ Then

$$\begin{aligned} & \tau_1 \circ (\tau_2 + \tau_3)(\eta(p, p'), (f, (b_2, b_3)), \eta(q, q')) \\ = \; & \tau_1 \circ (\tau_2 + \tau_3)((p, p'), (f, (b_2, b_3)), (q, q')) \\ = \; & T(\tau_1(p, f(p'), q), (\tau_2 + \tau_3)(p', (b_2, b_3), q')) \\ = \; & \begin{cases} T(\tau_1(p, f(p'), q), \tau_2(p', b_2, q')), p', q' \in Q_2 \\ T(\tau_1(p, f(p'), q), \tau_3(p', b_3, q')), p', q' \in Q_3 \\ 0, \text{otherwise} \end{cases} \\ = \; & \begin{cases} (\tau_1 \circ \tau_2)((p, p'), (f|_{Q_2}, b_2), (q, q')), p', q' \in Q_2 \\ (\tau_1 \circ \tau_3)((p, p'), (f|_{Q_3}, b_3), (q, q')), p', q' \in Q_3 \\ 0, \text{otherwise} \end{cases} \\ = \; & ((\tau_1 \circ \tau_2) + (\tau_1 \circ \tau_3))((p, p'), ((f|_{Q_2}, b_2), (f|_{Q_3}, b_3)), (q, q')) \\ = \; & ((\tau_1 \circ \tau_2) + (\tau_1 \circ \tau_3))((p, p'), \xi(f, (b_2, b_3)), (q, q')) \end{aligned}$$

This completes the proof.

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