퍼지 및 지능시스템학회 논문지 2001, Vol. 11, No. 3, pp. 286-291

Some Properties of BL-Algebras

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Abstract

We inverstigate the properties of BL-hommorphisms on BL-algebras. In particular, we find the BL-algebra in duced by lattice-isomorphism. From these facts, we obtain the generalized Lukasiewicz structure. More-over, we study the properties of quotient BL-algebras and deductive systems.

Key Words : BL-algebras, Quotient BL-algebras, Deductive systems

1. Introduction and preliminaries

Ward and Dilworth [7]introduced residuated lattices as the foundation of the algebraic structures of fuzzy logic. Hâjeck [1] introduced a BL-algebra which is a general tool of fuzzy logic. Recently, Höhle [2,3] extended the fuzzy set f : X→L where L is a BL-algebra in stead of an unit interval I.

In this paper, we investigate the properties of BL-homomorphisms on BL-algebras. In particular, we find the BL-algebra induced by lattice-isomorphism. From these facts, we can obtain the generalized Lukasiewicz structure. Moreover, we prove the first isomorphism theorem on BL-algebras. We study the properties of quotient BL-algebras. We give the examples of them. In general, the intersection of deductive systems is a deductive system. We construct the smallest deductive system containing the union of deductive systems.

Definition 1.1 ([1,6]). A lattice $(L,\leq,\wedge,\vee,\odot,\rightarrow,0,1)$ is called a residuated lattice if it satisfies the following conditions : for each $x,y,z\in L$,

(R1) $(L,\odot,1)$ is a commutative monoid,

(R2) if $x\leq y$, then $x\odot z\leq y\odot z$ ($\odot$ is an isotone operation),

(R3) (Galois correspondence) : $(x\odot y)\leq z$ iff $x\leq y\rightarrow z$.

In a residuated lattice L, $x^*=(x\rightarrow 0)$ is called *complement* of $x\in L$.

Lemma 1.2 ([6]). In a residuated lattice $(L,\leq,\wedge,\vee,\odot,\rightarrow,0,1)$ we have the following properties : for $x,y,z\in L$,

(1) $x=1\rightarrow x$,

(2) $1=x\rightarrow x$,

(3) $x\odot y\leq x,y$,

(4) $x\odot y\leq x\wedge y$,

(5) $y\leq x\rightarrow y$,

(6) $x\odot y\leq x\rightarrow y$,

(7) $x\leq y$ iff $1=x\rightarrow y$,

(8) $x=y$ iff $1=x\rightarrow y=y\rightarrow x$,

Definition 1.3 ([1.6]). A residuated lattice $(L,\leq,\wedge,\vee,\odot,\rightarrow,0,1)$ is called a *BL-algebra* if it satisfies the following conditions : for each $x,y,\in L$,

(B1) $x\wedge y=x\odot(x\rightarrow y)$,

(B2) $x\vee y=[(x\rightarrow y)\rightarrow y]\wedge[(y\rightarrow x)\rightarrow x]$,

(B3) $(x\rightarrow y)\vee(y\rightarrow x)=1$

Definition 1.4 ([6]). Let L be a BL-algebra. A subset D of L is a deductive system of L, *ds* for short, if it satisfies the following conditions :

(1) $1\in D$,

(2) if $x, x\rightarrow y\in D$, then $y\in D$.

Theorem 1.5. Let L be a BL-algebra. A nonempty subset D of L is *ds* iff it satisfies the following conditions :

(1) if $a,b\in D$, then $a\odot b\in D$,

(2) if $a\in D$ and $a\leq b$, then $b\in D$.

Proof. ($\Rightarrow$) Let $a,b\in D$, Since $(a\odot b)\leq(a\odot b)$, by Galois correspondence, $a\leq[b\rightarrow(a\odot b)]$. Since $a=1\odot a$, we have $a\rightarrow[b\rightarrow(a\odot b)]=1$. Since D is a *ds*, $b\rightarrow(a\odot b)\in D$. Thus, $(a\odot b)\in D$.

Let $a\in D$ and $a\leq b$. Since $a\leq b$, by Lemma 1.2(7), $a\rightarrow b=1\in D$. Hence $b\in D$.

($\Leftarrow$)Since $D\neq\emptyset$, $a\leq 1$ for each $a\in D$. By (2), $1\in D$. Let $a, a\rightarrow b\in D$. By (1), $a\odot(a\rightarrow b)\in D$. Since $(a\rightarrow b)\leq(a\rightarrow b)$, we have $[(a\rightarrow b)\odot a]\leq b$. By (2), $b\in D$.

Definition 1.6([6]). Let $\sim$ be an equivalence relation on A. Let $f: A^m\rightarrow A$ be an m-ary operation on A. We say that $\sim$ is a *congruence* with respect to f if $a_i\sim b_i$ for each $i=1,\dots,m$, then $f(a_1,\dots,a_m)\sim f(b_1,\dots,b_m)$.

Theorem 1.7 ([6]). If $\sim$ is a congruence relation on a BL-algebra L. then $D=\{a\in L\mid a\sim 1\}$ is a *ds*.

접수일자 : 2001년 2월 1일
완료일자 : 2001년 3월 12일

Theorem 1.8 ([6]). Let L be a BL-algebra. Let D be a *ds* of L. Define $a\sim b$ iff $(a\to b)\odot(b\to a)\in D$.
Then $\sim$ is a congruence relation with respect to $\to,\odot,{}^*,\vee,\wedge$.

Theorem 1.9 ([6]). Let D be a *ds* of a BL-algebra L. Define on L/D which is the set of equivalence classes $\{|a| \mid a\in L\}$, for all $a,b\in L$,

$$|a|\le|b| \text{ iff } a\to b\in D.$$

then

$$(L/D,\ \le,\wedge,\vee,\odot,\to,|0|,|1|)$$

is a BL-algebra where $|a|\wedge|b| = |a\wedge b|$,
$|a|\vee|b| = |a\vee b|$, $|a|\odot|b| = |a\odot b|$
$|a|\to|b| = |a\to b|$.

Theorem 1.10 ([6]). Let L, K be two BL-algebras. Let $h: L\to K$ be a BL-homomorphism. Then, for all $x,y\in L$,

(1) $h(x^*) = h(x)^*$, $h(1)=1$
(2) if $x\le y$, then $h(x)\le h(y)$,
(3) $h(x\wedge y)=h(x)\wedge h(y)$, $h(x\vee y)=h(x)\vee h(y)$,
(4) if D is a *ds* of L, then $h(D)$ is a *ds* of K.

2. BL-homomorphism

Definition 2.1 ([6]). Let L, K be two BL-algebras. A map $h: L\to K$ is called a *BL-homomorphism* if for all $x,y\in L$, it satisfies the following conditions:

(1) $h(x\to y) = h(x)\to h(y)$,
(2) $h(x\odot y) = h(x)\odot h(y), h(0) = 0$.

A BL-homomorphism $h: L\to K$ is called a *BL-isomrphism* if $h^{-1}$is a BL-homomorphism and h is bijective.

Theorem 2.2. Let L, K be two BL-algebras. If $h: L\to K$ is a bijective BL-homomorphism, then h a BL- isomorphism.

Proof. We only show that $f^{-1}$is a BL-homomorphism. Put $f^{-1}(y_1)=x_1$ and $f^{-1}(y_2)=x_2$ for each $y_1,y_2\in K$. Since $f$is a BL-homomorphism,

$$f(x_1\odot x_2) = f(x_1)\odot f(x_2) = y_1\odot y_2,$$

$$f(x_1\to x_2) = f(x_1)\to f(x_2) = y_1\to y_2.$$

If implics $x_1\odot x_2 = f^{-1}(y_1\odot y_2)$ and $x_1\to x_2 = f^{-1}(y_1\to y_2)$. Thus,

$$f^{-1}(y_1)\odot f^{-1}(y_2) = x_1\odot x_2 = f^{-1}(y_1\odot y_2),$$

$$f^{-1}(y_1)\to f^{-1}(y_2) = x_1\to x_2 = f^{-1}(y_1\to y_2).$$

Theorem 2.3. Let $(L,\wedge,\vee,0,1)$ be a lattice and $(K,\le,\wedge,\vee,\odot_K,\to,0,1)$ be a BL-algebra. Let $h: L\to K$ be a lattice-isomorphism (h is bijective, $h(x\wedge y) = h(x)\wedge h(y)$ and $h(x\vee y)=h(x)\vee h(y)$). Define two operations as follows:

$$x\to y = h^{-1}(h(x)\to h(y)),$$
$$x\odot_L y = h^{-1}(h(x)\odot_K h(y)).$$

Then:

(1) $(L,\le,\wedge,\vee,\odot_L,\to,0,1)$ is a BL-algebra.
(2) A map $h: L\to K$ is a BL-isomorphism.

Proof. (1) (A) For each $x,y\in L$, Define $x\le y$ iff $x\vee y=y$. Since $h: L\to K$ is a lattice-isomorphism, $x\le y$ iff $x\vee y=y$ iff $h(x)\vee h(y)=h(y)$. Thus h and h^{-1} are order preserving maps.

(R1) $(L,\odot_L,1)$ is a commutative monoid from:
It is trivial that $\odot_L$ is commutative.

$$\begin{aligned} x\odot_L 1 &= h^{-1}(h(x)\odot_K h(1))\\ &= h^{-1}(h(x)\odot_K 1)\\ &= h^{-1}(h(x))=x.\end{aligned}$$

$$\begin{aligned}(x\odot_L y)\odot_L z &= [h^{-1}(h(x)\odot_K h(y))]\odot_L z\\ &= h^{-1}(h([h^{-1}(h(x)\odot_K h(y))])\odot_K h(z))\\ &= h^{-1}(h(x)\odot_K (h(y))\odot_K h(z))\\ &= h^{-1}(h(x)\odot_K (h(y)\odot_K h(z)))\\ &= h^{-1}(h(x)\odot_K h([h^{-1}(h(y)\odot_K h(z))]))\\ &= x\odot_L[h^{-1}(h(y)\odot_K h(z))]\\ &= x\odot_L(y\odot_L z).\end{aligned}$$

(R2) If $x\le y$, then $h(x)\le h(y)$.
Thus $h(x)\odot_K h(z)\le h(y)\odot_K h(z)$.
From (A), since h^{-1} is oder preserving map,

$$x\odot_L z = h^{-1}(h(x)\odot_K h(z))\le h^{-1}(h(y)\odot_K h(z)) = y\odot_L z$$

(R3) (Galois correspondence): $(x\odot_L y)\le z$ iff $x\le y\to z$.

$$\begin{aligned}(x\odot_L y)\le z &\text{ iff } h^{-1}(h(x)\odot_K h(y)\le z\\ &\text{ iff } (h(x)\odot_K h(y))\le h(z)\\ &\text{ iff } h(x)\le[h(y)\to h(z)]\\ &\text{ iff } x\le h^{-1}[h(y)\to h(z)]\\ &\text{ iff } x\le(y\to z).\end{aligned}$$

(B1) Since $h(x)\wedge h(y)=h(x)\odot_L(h(x)\to h(y))$,

$$\begin{aligned}x\wedge y &= h^{-1}(h(x))\wedge h^{-1}(h(y))\\ &= h^{-1}(h(x)\wedge h(y))\\ &= h^{-1}(h(x)\odot_K[h(x)\to h(y)])\\ &= h^{-1}(h(x)\odot_K[h(h^{-1}(h(x)\to h(y)))])\\ &= x\odot_L[h^{-1}[h(x)\to h(y)]]\\ &= x\odot_L(x\to y).\end{aligned}$$

(B2) Since $h(x)\vee h(y) = [(h(x)\to h(y))\to h(y)] \wedge[h(y)\to h(x))\to h(x)]$,

$$\begin{aligned} x\vee y &= h^{-1}(h(x))\vee h^{-1}(h(y)) \\ &= h^{-1}(h(x)\vee h(y)) \\ &= h^{-1}([h(x)\to h(y)]\to h(y))\wedge h^{-1}([h(y)\to h(x)]\to h(x)) \\ &= h^{-1}(h(h^{-1}[h(x)\to h(y)])\to h(y)) \\ &\quad \wedge h^{-1}(h(h^{-1}[h(y)\to h(x)])\to h(x)) \\ &= [h^{-1}(h(x)\to h(y))\to y] \wedge [h^{-1}(h(y)\to h(x))\to x] \\ &= [(x\to y)\to y] \wedge [(y\to x)\to x]. \end{aligned}$$

(B3) Since $(h(x)\to h(y))\vee(h(y)\to h(x))=1$,

$$\begin{aligned} 1 &= h^{-1}(1) \\ &= h^{-1}([h(x)\to h(y)] \vee [h(y)\to h(x)]) \\ &= h^{-1}((h(x)\to h(y))) \vee h^{-1}(h(y)\to h(x)) \\ &= (x\to y) \vee (y\to x). \end{aligned}$$

Thus, $(L,\le,\vee,\wedge,\odot_L,\to,0,1)$ is a BL-algebra.

(2) From the definition of two operations $\odot_L$ and $\to$ and Theorem 2.2, h is a BL-isomorphism.

From the above theorem, we obtain the important results.

Example 2.4. Let $I=[0,\ 1]$ be an unit interval and $(I,\le,min,max,0,1)$ be a lattice. Define on I binary operations $\odot$ and $\to$ by

$$\begin{aligned} x\odot y &= max\{0,x+y-1\}, \\ x\to y &= min\{1,1-x+y\}. \end{aligned}$$

We have $(x\odot y)\odot z=x\odot(y\odot z)$ from

$$(x\odot y)\odot z=x\odot(y\odot z)=0,\ \text{if}\ x+y+z\le 2,$$

$$(x\odot y)\odot z=x\odot(y\odot z)=x+y+z-2,\ \text{if}\ x+y+z>2.$$

We easily show that (R1) $(L,\odot,1)$ is a commutative monoid and (R2) if $x\le y$, then $x\odot z\le y\odot z$ ($\odot$ is an isotone operation).

(R3) (Galois correspondence): $(x\odot y)\le z$ iff $x\le y\to z$ from

$$\begin{aligned} (x\odot y)\le z\ &\text{iff}\ x+y-1\le z \\ &\text{iff}\ x\le 1-y+z \\ &\text{iff}\ x\le min\{1,1-y+z\}. \end{aligned}$$

(B1) $x\wedge y=x\odot(x\to y)$ from:

If $x\le y$, $x\odot(x\to y)=x\odot 1=x$ and $x\wedge y=x$.

If $x>y$, $x\odot(x\to y)=x\odot(1-x+y)=y$ and $x\wedge y=y$.

(B2) $x\vee y=[(x\to y)\to y]\wedge[(y\to x)\to x]$ from

If $x\le y$, $[(x\to y)\to y]\wedge[(y\to x)\to x]=$
$y\wedge[(1-y+x)\to x]=y$.

If $x>y$, $[(x\to y)\to y]\wedge[(y\to x)\to x]=x$.

Similarly, (B3) $(x\to y)\vee(y\to x)=1$.

Then $(I,\le,min,max,\odot,\to,0,1)$ is a BL-algebra, called *Lukasiewicz structure*.

(1) Define $h: I\to(I,\le,min,max,\odot,\to,0,1)$ by $h(x)=x^p$ where $p>0$. Then h is a lattice-isomorphism. From Theorem 2.3, we can obtain the generalized Lukasiewicz structure as follows:

$$\begin{aligned} x\to y &= h^{-1}(h(x)\to h(y)) \\ &= h^{-1}(min\{1,1-h(x)+h(y)\}) \\ &= min\left\{1,(1-x^p+y^p)^{\frac{1}{p}}\right\} \end{aligned}$$

$$x\odot_I y=h^{-1}(h(x)\odot h(y))=(max\{0,x^p+y^p-1\})^{\frac{1}{p}}$$

Then $(I,\le,\wedge,\vee,\odot_I,\to,0,1)$ is a BL-algebra and $h: L\to K$ is a BL-isomorphism.

(2) $g: ([1,2],min,max,1,2)\to(I,\le,min,max,\odot,\to,0,1)$ by $g(x)=\log_2 x$. The g is a lattice-isomorphism. From Theorem 2.3, we can obtain the generalized Lukasiewicz structure as follows:

$$\begin{aligned} x\to y &= g^{-1}(g(x)\to g(y)) \\ &= g^{-1}(min\{1,1-\log_2 x+\log_2 y\}) \\ &= min\{2,2^{1-\log_2 x+\log_2 y}\}, \\ &= min\left\{2,\frac{2y}{x}\right\}, \end{aligned}$$

$$x\odot_{[1,2]}y=g^{-1}(g(x)\odot g(y))=max\left\{1,\frac{xy}{2}\right\}.$$

Then $([1,2],\le,\wedge,\vee,\odot_{[1,2]},\to,1,2)$ is a BL-algebra.

(3) Define $k: I\to(I,\le,min,max,\odot,\to,0,1)$ by $k(x)=\log_2(x+1)$.

We can obtain the generalized Lukasiewicz structure as follows:

$$\begin{aligned} x\to y &= min\left\{1,\frac{2y+1}{x+1}\right\}, \\ x\odot_I y &= max\left\{0,\frac{xy+x+y-1}{2}\right\}. \end{aligned}$$

Then $(I,\le,\wedge,\vee,\odot_I,\to,0,1)$ is a BL-algebra.

We prove the first isomorphism theorem on BL-algebras from the following theorem.

Theorem 2.5. Let L,K be two BL-algebras. Let $h: L\to K$ be a BL-homomorphism. Then

(1) If H is a *ds* of K, then $D=\{a\in L\mid h(a)\sim_H 1\}$ is a *ds* of L.

(2) (The first isomorphism theorem) If h is surjective and $H=\{1\}$, then $D=\{a\in L\mid h(a)=1\}$ is a *ds* of L and the map $\overline{h}: L/D\to K$ defined by $\overline{h}(\mid a\mid)=h(a)$ is a BL-isomorphism.

Proof. (1) Let $a,b\in D$. Then $h(a),h(b)\sim_H 1$. Since $\sim_H$ is a congruence relation with respect to the operation $\odot$, we have $h(a)\odot h(b)\sim_H 1\odot 1$. Since $h(a\odot b)=h(a)\odot h(b)$, $h(a\odot b)\sim_H 1$ that is $a\odot b\in D$. Let $a\le b$ and $a\in D$. From Theorem 1.10(2) and Lemma 1.2(7), $h(a)\le h(b)$ implies $h(a)\to h(b)=1$. Since $h(a)\sim_H 1$ and $h(b)\sim_H h(b)$ and $\sim_H$ is a congruence relation with respect to the operation $\to$, we have $1=[h(a)\to h(b)]\sim_H[1\to h(b)]$. By Lemma 1.2(1),

$h(b)=[1\to h(b)]$. Thus $b\in D$.

(2) Let $h(a)\sim_H 1$. By Theorem 1.8,

$$([h(a)\to 1]\odot[1\to h(a)])\in H.$$

Since $H=\{1\}$,

$$([h(a)\to 1]\odot[1\to h(a)])=1.$$

By Lemma 1.2(3,8), $h(a)=1$. Thus $D=\{a\in L\mid h(a)=1\}$.

Let $a\sim_D b$. Then $(a\to b)\odot(b\to a)\in D$. It implies

$$([h(a)\to h(b)]\odot[h(b)\to h(a)])=1.$$

By Lemma 1.2(3,8), $h(a)=h(b)$. Thus, $\overline{h}$ is well defined.

Since $h: L\to K$ is a BL-homomorphism, $\overline{h}: L/D\to K$ is a BL-homomorphism from the following statements:

$$\overline{h}(|x|\to|y|)=\overline{h}(|x\to y|)=h(x\to y)$$
$$=h(x)\to h(y)=\overline{h}(|x|)\to\overline{h}(|y|),$$
$$\overline{h(|x|\odot|y|)}=\overline{h}(|x\odot y|)=h(x\odot y)$$
$$=h(x)\odot h(y)=\overline{h}(|x|)\odot\overline{h}(|y|),$$
$$\overline{h}(|0|)=h(0)=0.$$

By Theorem 2.2, we only show that $\overline{h}$ is bijective. Let $h(a)=h(b)$. From Lemma 1.2(8),

$$[h(a)\to h(b)]=[h(b)\to h(a)]=1.$$

It implies

$$([h(a)\to h(b)]\odot[h(b)\to h(a)])=1.$$

Then $(a\to b)\odot(b\to a)\in D$. Thus, $a\sim_D b$, that is, $|a|=|b|$. Hence $\overline{h}$ is injective. Since h is surjective, $\overline{h}$ is surjective.

Example 2.6. Let X be a nonempty set and $P(X)$ be a family of all subsets of X. Then $(P(X),\subset,\cap,\cup,\emptyset,X)$ is a lattice. For each $A,B\in P(X)$, we define the operations $\odot$ and $\to$ by

$$A\odot B=A\cap B,\ A\to B=A^c\cup B.$$

It satisfies (R1) and (R2) of Definition 1.1.

We show that $A\cap B\subset C$ iff $A\subset B^c\cup C$ (Galois correspondence) from the following statements:

($\Rightarrow$) Since $A\subset(A\cup B^c)\cap(B\cup B^c)=(A\cap B)\cup B^c$ and $A\cap B\subset C$, we have $A\subset B^c\cup C$.

($\Leftarrow$) Since $A\subset B^c\cup C$, we have $A\cap B\subset(B^c\cup C)\cap B=C\cap B\subset C$.

It satisfies (B1),(B2) and (B3) of Definition 1.3.

(B1)

$$A\odot(A\to B)=A\cap(A^c\cup B)=A\cap B.$$

(B2)

$$[(A\to B)\to B]\cap[(B\to A)\to A]$$
$$=[(A^c\cup B)^c\cup B]\cap[(B^c\cup A)^c\cup A]$$
$$=[(A\cap B^c)\cup B]\cap[(B\cap A^c)\cup A]$$
$$=A\cup B.$$

(B3)

$$(A\to B)\cup(B\to A)=[(A^c\cup B)\cup(B^c\cup A)]=(A^c\cup B^c)\cup(A\cup B)=X.$$

Thus, $(P(X),\subset,\cap,\cup,\odot,\to,\emptyset,X)$ is a BL-algebra.

Example 2.7. Let $X=\{x_1,x_2,x_3\}$ and $Y=\{y_1,y_2\}$ be two sets. Define $h: P(X)\to P(Y)$ as follows:

$$h(\emptyset)=\emptyset,\quad h(X)=Y,$$
$$h(\{x_1\})=\{y_1\},\ h(\{x_2\})=\{y_2\},\ h(\{x_3\})=\emptyset,$$
$$h(\{x_1,x_2\})=\{y_1,y_2\},\ h(\{x_1,x_3\})=\{y_1\},\ h(\{x_2,x_3\})=\{y_2\}.$$

It satisfies the following conditions: for each $A,\ B\in P(X)$,

$$h(A\cap B)=h(A)\cap h(B),\ h(A\cup B)=h(A)\cup h(B),$$
$$h(A^c)=h(A)^c.$$

Since $A\to B=A^c\cup B$,

$$h(A\to B)=h(A^c\cup B)=h(A)^c\cup h(B)=h(A)\to h(B).$$

Hence $h: P(X)\to P(Y)$ is a BL-homomorphism. From Theorem 2.5(2), $D=\{A\in P(X)\mid h(A)=X\}=\{\{x_1,x_2\},X\}$ is *ds*. From Theorem 1.8, since

$$A\sim B \text{ iff } [(A\to B)\odot(B\to A)]\in D$$
$$\text{iff } [(A^c\cup B)\cap(B^c\cup A)]\in D$$
$$\text{iff } [(A\cap B)\cup(A\cup B)^c]\in D$$

We can obtain :

$$\emptyset\sim\{x_3\},\ \{x_1\}\sim\{x_1,x_3\}$$
$$\{x_2\}\sim\{x_2,x_3\},\ X\sim\{x_1,x_2\}$$

We obtain $P(X)/D=\{|\emptyset|,|\{x_1\}|,|\{x_2\}|,|X|\}$. Define $\overline{h}: P(X)/D\to P(Y)$ by $\overline{h}(|A|)=h(A)$. Hence $\overline{h}$ is a BL-isomorphism.

Theorem 2.8. Let L,K be two BL-algebras. Let $h: L\to K$ be a BL-homomorphism. Let N,H be *ds*'s of L,K, respectively.

(1) If $N\subset D$ where $D=\{a\in L\mid h(a)\sim_H 1\}$ is a *ds*, then $a\sim_N b$ implies $h(a)\sim_H h(b)$.

(2) If $N\subset D$, a map $\overline{h}: L/N\to K/H$ defined by $\overline{h}(|a|)=|h(a)|$ is a BL-homomorphism.

(3) If $N=D$ and h is surjective, a map $\overline{h}: L/N\to K/H$ is a BL-isomorphism.

Proof. (1) Let $a\sim_N b$. From Theorem 1.8, $(a\to b)\odot$

$(b\to a)\in N$. Since $(a\to b)\odot(b\to a)\le(a\to b)\in N$ and $N\subset D$, we have $(h(a)\to h(b))\sim_H 1$. Since $\sim_H$ is a congruence relation with respect to the operation $\odot$, $h(a)\sim_H h(a)$ and $(h(a)\to h(b))\sim_H 1$

$$[h(a)\odot(h(a)\to h(b))]\sim_H[(h(a)\odot 1)=h(a)].$$

Since $h(a)\odot(h(a)\to h(b))=h(a)\wedge h(b)$ from (B1) of Definition 1.3, $(h(a)\wedge h(b))\sim_H h(a)$. By a similar method, $(h(b)\wedge h(a))\sim_H h(b)$. Since $(h(a)\wedge h(b))\sim_H h(b)\wedge h(a))$, then $h(a)\sim_H h(b)$.

(2) The map $\overline{h}$ is well defined from (1). Since $|x\to y| = |x|\to|y|$ and $|h(x)\to h(y)| = |h(x)|\to|h(y)|$ from Theorem 1.9, we have

$$\begin{aligned}\overline{h}(|x|\to|y|) &= \overline{h}(|x\to y|)\\ &= |h(x)\to h(y)|\\ &= |h(x)|\to|h(y)|\\ &= \overline{h}(|x|)\to\overline{h}(|y|).\end{aligned}$$

Similarly, $\overline{h}(|x|\odot|y|)=\overline{h}(|x|)\odot\overline{h}(|y|), \overline{h}(|0|) = |0|$. Thus, $\overline{h}$ is BL-homomorphism.

(3) Since h is surjective, $\overline{h}$ is surjective. We only show that $\overline{h}$ is injective.

Let $h(x)\sim_H h(y)$. Since $\sim_H$ is a congruence relation with respect to the operation $\to$,

$$(h(x)\to h(y))\sim_H(h(y)\to h(y)).$$

$$(h(y)\to h(x))\sim_H(h(x)\to h(x))$$

Since $(h(y)\to h(y))=1$, $(h(x)\to h(x))=1$ from Lemma 1.2(2) and $\sim_H$ is a congruence relation with respect to the operation $\odot$,

$$[(h(x)\to h(y))\odot(h(y)\to h(x))]\sim_H 1.$$

Thus

$$(x\to y)\odot(y\to x)\in D.$$

Hence $x\sim_D y$.

Example 2.9. We define X, Y and h as same in Example 2.7. Let $H=\{\{y_1\}, Y\}$ be a *ds*. Then

$$D=\{A\in P(X)\mid h(A)\sim_H Y\}=\{\{x_1\},\{x_1,x_2\},\{x_1,x_3\},X\}$$

from the following :

$$\{y_1\}\sim_H Y,\ \{y_2\}\sim_H\varnothing.$$

Also, we have

$$\{x_1\}\sim_D\{x_1,x_2\},\sim_D\{x_1,x_3\}\sim_D X,$$

$$\{x_2\}\sim_D\{x_3\}\sim_D\{x_2,x_3\}\sim_D\varnothing.$$

It implies $P(X)/D=\{|\{x_1\}|, |\{x_2\}|\}$ and $P(Y)/H=\{|\{y_1\}|, |\{y_2\}|\}$. Then $\overline{h}: P(X)/D\to P(Y)/H$ defined by $\overline{h}(|A|) = |h(A)|$ is a BL-isomorphism.

In general, the intersection of deductive systems is a deductive system. But the union of deductive systems need not be a deductive system. We construct the smallest deductive system containing the union of deductive systems from the following theorem.

Theorem 2.10. Let $\{D_i\mid i\in\Gamma\}$ be a family of *ds*'s on a BL-algebra L.

(1) $\bigcap_{i\in\Gamma} D_i$ is a *ds*.

(2) Define a set

$$D=\{a\in L\mid x_1\odot\cdots\odot x_m\le a,\ \exists x_1,\cdots,x_m\in\bigcup_{i\in\Gamma}D_i\}.$$

Then D is the smallest *ds* containing each D_i.

Proof. (1) It is easily proved.

(2) Since $1\in D_i$ and $1\le 1$, then $1\in D$. Let $a,(a\to b)\in D$. We will show that $b\in D$. Since $a\in D$, there exist $x_1,\cdots,x_m\in\bigcup_{i\in\Gamma}D_i$ such that

$$x_1\odot\cdots\odot x_m\le a.$$

Since $(a\to b)\in D$, there exist $y_1,\cdots y_p\in\bigcup_{i\in\Gamma}D_i$ such that

$$y_1\odot\cdots\odot y_p\le(a\to b).$$

By Galois correspondence, it implies

$$y_1\odot\cdots\odot y_p\odot a\le b.$$

Since $\odot$ is isotone,

$$y_1\odot\cdots\odot y_p\odot x_1\odot\cdots\odot x_m\le b.$$

Since D_i is closed by the operation $\odot$, we have $b\in D$. Let $x_i\in D_i$. Since $x_i\le x_i$, we have $x_i\in D$. Hence $D_i\subset D$. Finally, if $\bigcup_{i\in\Gamma}D_i\subset H$ and H is a *ds*, we show that $D\subset H$. Let $a\in D$. There exist $x_1,\cdots,x_m\in\bigcup_{i\in\Gamma}D_i$ such that

$$x_1\odot\cdots\odot x_m\le a.$$

Since $x_1,\cdots,x_m\in H$, by Theorem 1.5 (1), we have $x_1\odot\cdots\odot x_m\in H$. From Theorem 1.5(2), we have $a\in H$.

Example 2.11. Let $X=\{x_1,x_2,x_3\}. D_1=\{\{x_1,x_2\},X\}$ and $D_2=\{\{x_2,x_3\},X\}$ *ds*'s. Then $D_1\cap D_2=\{X\}$ is a *ds*. But, $D_1\cup D_2=\{\{x_1,x_2\},\{x_2,x_3\},X\}$ is not a *ds* bacause

$$\{x_1,x_2\}\odot\{x_2,x_3\}=\{x_1,x_2\}\cap\{x_2,x_3\}=\{x_2\},\notin D_1\cup D_2.$$

From Theorem 2.10(2), $D=\{\{x_2\},\{x_1,x_2\},\{x_2,x_3\},X\}$ is the smallest *ds* containing D_1 and D_2.

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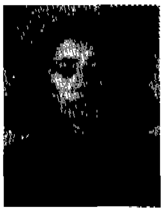

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