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OPTIMALITY CONDITIONS AND AN ALGORITHM FOR LINEAR-QUADRATIC BILEVEL PROGRAMMING

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ABSTRACT

The linear fractional - quadratic bilevel programming problem, in which the leader's objective function is a linear fractional function and the follower's objective function is a quadratic function, is studied in this paper. The leader's and the follower's variables are related by linear constraints. The derivations of the optimality conditions are based on Kuhn-Tucker conditions and the duality theory. It is also shown that the original linear fractional - quadratic bilevel programming problem can be solved by solving a standard linear fractional program and the optimal solution of the original problem can be achieved at one of the extreme point of a convex polyhedral formed by the new feasible region. The algorithm is illustrated with the help of an example.

1. INTRODUCTION

In this papers we consider a bilevel programming problem in which the leader's objective function is linear fractional and the follower's objective function is quadratic.

There are many applications of bilevel programming problem. The problem can be presented in terms of an economic policy. The government determine certain goals to be achieved during the planning period and in order to optimize their achievement, the government can use certain policy measures such as taxes and subsidies. Once the policy measures are announced the private sector reacts to government policy measures by optimally forming a plan of action. This private sector plan however may not be what the government anticipated. The reaction of the private sector together with the government's policy measures will jointly determine the degree of achievement of the government's economic goals. The government's objectives are at least in partial conflict with the private sector goals. The policy maker faces an optimization problem subject to the optimization problem for industries and consumers. Hence there are two types of objectives involved in the BLP. These objectives can be of any type depending upon the problem of the policy makers and the industries. In this paper we have chosen the leader's objective to be linear fractional function and the follower to be a quadratic function.

In a bilevel programming situation, the higher level decision maker is the central government or a central authority which sets the policies and the lower level decision maker is the state government, industrial managers and the like, who work within the frame work of these policies.

In a bilevel program, the leader optimizes his objective function independently and is affected by the reaction of the follower who makes his decision after the former. Bilevel programming has been developed and studied by many authors such as Bard [1, 2, 3]. Bialas and Karwan [4], Wang, Wang and Romano-Rodriques [9] and Candler and Townsley [6].

The purpose of this paper is to find the optimality conditions and a solution procedure to solve a linear fractional bilevel programming problem in which the leader's objective is linear fractional and the follower's objective is quadratic. It is proved that the given problem can be solved by a linear fractional program. The techniques used is to replace the follower's problem by the corresponding Kuhn-Tucker necessary and sufficient optimality conditions. Alternate representation of the original problem is made by appending these conditions to the leader's constraint set. It is shown that the optimality solution of the Bilevel Programming Problem is at an extreme point satisfying complementary condition.

2. MATHEMATICAL FORMULATION

The linear fractional quadratic bilevel programming problem (FQP) is

$$\text{(FQP)}: \qquad \max_{x} \quad F(x, y) = \frac{a^T x + b^T y + \alpha}{C^T x + d^T y + \beta} \qquad \text{where } y \text{ solves}$$

$$\max_{y} \; f(x, y) = p^T x + q^T y + (x, y)^T Q \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\text{subject to} \quad Ax + By \leq r$$

where $a, c, p \in R^{n_1}$, $b, d, q \in R^{n_2}$, $r \in R^m$, Q is an $(n_1 + n_2) \times (n_1 + n_2)$ real symmetric matrix with

$$Q = \begin{bmatrix} Q_3 & Q_2^T \\ Q_2 & Q_1 \end{bmatrix}$$

A, B are $m \times n_1$ and $m \times n_2$ matrices respectively. It is assumed that $c^T x + d^T y + \beta > 0$ for all $(x, y) \in S = \{(x, y) \mid Ax + By \leq r\}$. f can be written as

$$f(x, y) = y^T Q_1 y - (q + 2Q_2 x)^T y + (x^T Q_3 x + p^T x)$$

Because x is fixed prior to the maximization of f, the follower's problem is equivalent to

$$(P_x): \qquad \max_{y} \; f_1(x, y) = Y^T Q_1 y + (q + 2Q_2 x)^T y$$

$$\text{subject to} \quad By \leq r - Ax$$

Therefore, (FQP) is equivalent to

$$(\text{FQP}'): \qquad \max_{x} \; F(x, y) = \frac{a^T x + b^T y + \alpha}{c^T x + d^T y + \beta} \quad \text{where } y \text{ solves}$$

$$\max_{y} \; f_1(x, y) = y^T Q_1 y + (q + 2Q_2 x)^T y$$

$$\text{subject to} \quad By \leq r - Ax$$

Leader's solution space is given by

$$P = \{ x \in R^{n_1} \mid \text{there exists } y \text{ such that } (x, y) \in S\}$$

Follower's solution space is given by

$S(x) = \{ y \in R^{n_2} \mid (x, y) \in S\}$ For each $x \in P$, let $Y(x)$ denote the set of optimal solutions to the follower's problem (P_x)

Let $\overline{S} = \{(x, y) \mid (x, y \in S,\ y \in Y(x)\}$, i.e., it denotes the set of feasible points to (FQP').

Definition 2: A point (x^*,y^*) is said to be optimal to (FQP') if

(i) $(x^*, y^*) \in \overline{S}$ and

(ii) $F(x^*, y^*) \geq F(x, y)$ $\quad \forall\ (x,y) \in \overline{S}$

We assume that Q_1 is negative definite and (FQP) has at least an optimal solution. The set $Y(x)$ is singleton for each $x \in P$.

3. OPTIMALITY CONDITIONS FOR BILEVEL FRACTIONAL-QUADRATIC PROGRAMMING PROBLEM

In this section necessary and sufficient optimality conditions for a pair (x, y) to be an optimal solution of (FQP') are derived.

Theorem 1: (x^*, y^*) is an optimal solution to (FQP') if and only if there exists a vector $w^* \in R^m$ such that (x^*, y^*, w^*) solves the following nonlinear programming problem.

(P1) :
$$\max_{(x,y,w)} F(x,y) = \frac{a^T x + b^T y + \alpha}{c^T x + d^T y + \beta}$$

subject to
$$Ax + By \leq r \tag{1}$$
$$w^T(Ax + By) = w^T r \tag{2}$$
$$2Q_2 x + 2Q_1 y + q = B^T w \tag{3}$$
$$w \geq 0 \tag{4}$$

Proof. It can be easily proved by applying the Kuhn-Tucker conditions to the follower's problem

For a given $w \geq 0$, define

$$S[w] = \{(x,y) \mid (x,y) \text{ satisfies (1) - (4)}\}$$
$$= \text{the set of points which satisfy Kuhn – Tucker conditions for a given } w \geq 0$$

If we define $\max\ \{F(x,y) \mid (x,y) \in S[w] = -\infty\}$, where $S[w] = \phi$, we can reformulate (P1) as

$$\max_{w} \max_{(x,y)\in S[w]} F(x, y) \tag{5}$$

$$(\mathrm{Pw}): \quad \max_{(x,y)} F(x,y) = \frac{a^T x + b^T x + \alpha}{c^T x + d^T x + \beta}$$

$$\text{subject to} \quad Ax + By \le r \tag{6}$$

$$w^T(Ax + By) = w^T r \tag{7}$$

$$2Q_2 x + 2Q_1 y = B^T w - q \tag{8}$$

which is a linear fractional programming problem. For a given $w \ge 0$, constraint (7) can be replaced by $w^T(Ax + By) \ge w^T r$, i.e. $-w^T(Ax + By) \le -w^T r$, as $w^T(Ax+By) \le w^T r$ holds for every $w \ge 0$. Dual of (P_w) is given by

$$(D_w): \quad \min_{(u_1,u_2,u_3)} h_w(u_1,u_2,u_3) = \frac{(u_1 - u_2 w)^T r + u_3^T (B^T w - q) + \alpha}{\beta}$$

$$\text{subject to} \quad (u_1 - u_2 w)^T A + 2u_3^T Q_2 +$$

$$\frac{(u_1 - u_2 w)^T r + u_3^T (B^T w - q) + \alpha}{\beta} c^T = a^T \tag{9}$$

$$(u_1 - u_2 w)^T B + 2u_3^T Q_1 +$$

$$\frac{(u_1 - u_2 w)^T r + u_3^T (B^T w - q) + \alpha}{\beta} d^T = b^T \tag{10}$$

$$u_1,\ u_2 \ge 0 \tag{11}$$

where $u_1 \in R^m$, $u_2 \in R^l$ and $u_3 \in R^{n_2}$.

Suppose that $s[w] \ne \phi$ for a $w \ge 0$. Because (FQP) has at least an optimal solution, for this w, (P_w) has an optimal solution. By duality theory, (D_w) will also have an optimal solution [5] and the optimal objective values of (P_w) and (D_w) are equal

Define max – min problem

$$(\mathrm{P2}): \quad \max_{x} \min_{(u_1,u_2,u_3)} h_w(u_1,u_2,u_3)$$

$$\text{subject to} \quad (u_1 - u_2 w)^T A + 2u_3^T Q_2 +$$

$$\frac{(u_1 - u_2 w)^T r + u_3^T (B^T w - q) + \alpha}{\beta} c^T = a^T \tag{12}$$

$$(u_1 - u_2 w)^T B + 2u_3^T Q_1 +$$

$$\frac{(u_1 - u_2 w)^T r + u_3^T (B^T w - q) + \alpha}{\beta} d^T = b^T \tag{13}$$

$$u_1,\, u_2 \geq 0 \tag{14}$$

$$w \geq 0 \tag{15}$$

For a given $w \geq 0$, let

$$S\{w\} = \{(u_1, u_2, u_3) \mid (u_1, u_2, u_3) \text{ satisfying (12) - (15)}\}$$

and define

$$\min\ \{h_w(u_1, u_2, u_3) \mid (u_1, u_2, u_3) \in S\{w\}\} = -\infty \text{ when } S\{w\} = \phi.$$

(P2) can be rewritten as

$$\max_{w \geq 0} \min_{(u_1, u_2, u_3) \in S\{w\}} h_w(u_1, u_2, u_3) \tag{16}$$

Theorem 2 (Weak Duality Theorem): If (x,y,w) and (u_1,u_2,u_3,w) are feasible to the problem (P1) and (P2) respectively, then

$$h_w(u_1, u_2, u_3) \geq F(x, y)$$

Proof. It can easily be proved by multiplying (12) by x and (13) by y and adding and using the feasibility of (x, y, w) and (u_1, u_2, u_3, w) to (P1) and (P2) respectively.

Theorem 3 (Strong Duality Theorem): Suppose that (x^*, y^*, w^*) and $(u_1^*, u_2^*, u_3^*, w^*)$ are feasible solution to (P1) and (P2) respectively. Then (x^*, y^*, w^*) and $(u_1^*, u_2^*, u_3^*, w^*)$ are optimal to (P1) and (P2) respectively if an only if

$$h_w(u_1^*, u_2^*, u_3^*) = f(x^*, y^*) \tag{17}$$

$$(u_1^*, u_2^* w^*)^T (Ax + By - r) + u_3^{*T}(2Q_2 x + 2Q_1 y + q - B^T w^*) \leq 0 \tag{18}$$

$$\forall\ (x, y) \in \overline{S}$$

Proof. Suppose (x^*, y^*, w^*) and $(u_1^*, u_2^*, u_3^*, w^*)$ are optimal for (P1) and (P2) respectively.

If $S[w] \neq \phi$ for $w \geq 0$, By duality theory [5].

$$\min_{(u_1, u_2, u_3) \in S\{w\}} h_w(u_1, u_2, u_3) = \max_{(x,y) \in S[w]} F(x, y)$$

If $S[w] = \phi$ for some $w \geq 0$, according to convention

$$\min_{(u_1,u_2,u_3)\in S\{w\}} h_w(u_1,u_2,u_3) = \max_{(x,y)\in S[w]} F(x,y) = -\infty$$

Hence

$$\max_{w\geq 0} \min_{(u_1,u_2,u_3)\in S\{w\}} h_w(u_1,u_2,u_3) = \max_{w\geq 0} \max_{(x,y)\in S[w]} F(x,y) \tag{19}$$

therefore (17) holds.

Since (u_1^*,u_2^*,u_3^*,w^*) is feasible to (P2) we have from (12) and (13)

$$\begin{aligned}&(u_1^* - u_2^* w^*)^T(Ax+By-r) + u_3^{*T}(2Q_2x+2Q_1y+q-B^Tw^*)\\&+\frac{(u_1^*-u_2^*w^*)^T r + u_3^{*T}(B^Tw^*-q)+\alpha}{\beta}(c^Tx+d^Ty+\beta)\\&= a^Tx+b^Ty+\alpha\end{aligned} \tag{20}$$

As $(x,y)\in\overline{S}$ and (x^*,y^*,w^*) soles (P1), therefore (x^*,y^*) solves (FQP'). Using Theorem 1 and (17), we get (18)

Conversely let (17) and (18) hold. To prove that (x^*,y^*,w^*) and (u_1^*,u_2^*,u_3^*,w^*) are optimal for (P1) and (P2) respectively.

Now

$$\begin{aligned}&\frac{a^Tx^*+b^Ty^*+\alpha}{c^Tx^*+d^{Ty^*}+\beta} = \frac{(u_1^*-u_2^*w^*)^T + u_3^{*T}(B^Tw^*-q)+\alpha}{\beta}\\&= \frac{a^Tx^*+b^Ty^*+\alpha}{c^Tx^*+d^Tx^*+\beta} - \left\{\frac{(u_1^*-u_2^*w^*)^T r + u_3^{*T}(2Q_2x+2Q_1y+q-B^Tw^*)}{c^Tx+d^Ty+\beta}\right\}\end{aligned}$$

[(Using feasibility of (u_1^*,u_2^*,u_3^*,w^*) and (20)]

$$\geq \frac{a^Tx+b^Ty+\alpha}{c^Tx+d^Ty+\beta} \qquad \text{[by (18)]}$$

Therefore $F(x^*,y^*) \geq F(x,y)$ for all $(x,y)\in\overline{S}$

i.e. (x^*,y^*) solves (FQP') [by definition 2]

i.e. (x^*,y^*,w^*) solves (P1) [by Theorem 1]

From (19), optimum objective values of (P1) and (P2) are equal. Since (x^*, y^*, w^*) solves (P1) and objective function values are equal, therefore,

$(u_1^*, u_2^*, u_3^*, w^*)$ is the optimal solution to (P2).

Combining Theorem 1 and Theorem 3 we obtain the following necessary and sufficient optimality conditions.

Theorem 4: (x^*, y^*) is an optimal solution to (FQP') if and only if there exists $w^* \in R^m$, $u_1^* \in R^m$, $u_2^* \in R^1$ and $u_3^* \in R^{n_2}$ satisfying $w^* \geq 0$, $u_1^* \geq 0$ and $u_2^* \geq 0$ such that

$$2Q_2x^* + 2Q_1y^* + q = B^T w^* \tag{21}$$

$$w^{*T}(Ax^* + By^* - r) = 0 \tag{22}$$

$$(u_1^* - u_2^* w^*)^T A + 2u_3^{*T} Q_2 + \frac{(u_1^* - u_2^* w^*)^T r + u_3^T (B^T w^* - q) + \alpha}{\beta} c^T = a^T \tag{23}$$

$$(u_1^* - u_2^* w^*)^T B + 2u_3^{*T} Q_1 + \frac{(u_1^* - u_2^* w^*)^T r + u_3^T (B^T w^* - q) + \alpha}{\beta} d^T = b^T \tag{24}$$

$$u_1^{*T}(Ax^* + By^* - r) = 0 \tag{25}$$

$$(u_1^* - u_2^* w^*)^T (Ax + By - r) + u_3^{*T}(2Q_2 x + 2Q_1 y + d - B^T w^*) \leq 0 \quad \forall (x, y) \in \overline{S} \tag{26}$$

Proof. Let (x^*, y^*) be optimal to (FQP').

By Theorem 1, there exists a $w^* \in R^m$ such that (x^*, y^*, w^*) satisfies (2)-(5). Hence, (x^*, y^*, w^*) satisfies (21) and (22).

Clearly $S[w^*] \neq \phi$ and (x^*, y^*) is an optimal solution to (P_{w^*}). This implies that there exists $u_1^* \in R^m$, $u_2^* \in R^1$ and $u_3^* \in R^{n_2}$ such that (u_1^*, u_2^*, u_3^*) solves (D_{w^*}) [8].

Hence (23) and (24) hold. Further, since (x^*, y^*) solves (P_{w^*}) and (u_1^*, u_2^*, u_3^*) solves (D_{w^*}) therefore

$$\frac{a^T x^* + b^T y^* + \alpha}{c^T x^* + d^T y^* + \beta} = \frac{(u_1^* - u_2^* w^*)^T \gamma + u_3^{*T}(B^T w^* - q) + \alpha}{\beta} \tag{27}$$

Multiplying (23) by x^*, (24) by y^* and using (8), (9) and (27), we get $u_1^{*T}(Ax^* + By^* - r) = 0$.

Since (x^*, y^*, w^*) solves (P1) and optimal objective function values of (P1) and (P2) are equal $(u_1^*, u_2^*, u_3^*, w^*)$ solves (P2) which in turn by Theorem 3 proves that

$$(u_1^* - u_2^* w^*)^T (Ax + by - r) + u_3^{*T}(2Q_2 x + 2Q_1 y + q - B^T w^*) \leq 0$$

$$\forall\ (x, y) \in \overline{S} \qquad \text{[by (18)]}$$

Therefore (26) holds.

Hence if (x^*, y^*) solves (FQP'), then there exists $w^* \geq 0$, $u_1^* \geq 0$, $u_2^* \geq 0$ and u_3^* such that (21)-(26) are true.

Conversely, let $(x^*, y^*) \in S$ and there exists $w^* \geq 0$, $u_1^* \geq 0$, $u_2^* \geq 0$ and u_3^* such that (21)-(26) hold, then to prove that (x^*, y^*) solves (FQP').

(21)-(24) imply that (x^*, y^*, w^*) is feasible to (P1) and $(u_1^*, u_2^*, u_3^*, w^*)$ is feasible to (P2). Multiplying (23) by x^*, (24) by y^* and adding we get

$$\frac{(u_1^* - u_2^* w^*)^T \gamma + u_3^{*T}(B^T w^* - q) + \alpha}{\beta} = \frac{a^T x^* + b^T y^* + \alpha}{c^T x^* + d^T y^* + \beta}$$

[Using (21), (22), (25)]

This with (26) and Theorem 3 implies that (x^*, y^*, w^*) is optimal to (P1) and hence (x^*, y^*) is optimal to (FQP') by Theorem 1.

4. ALGORITHM

Denote $S'(w) = S \cap \{(x, y) \mid 2Q_2 x + 2Q_1 y + q = B^T w\}$

$$T = \left\{ \begin{array}{c} (x, y, w) \mid Ax + By \leq r \\ 2Q_2 x + 2Q_1 y + q = B^T w \\ w \geq 0 \end{array} \right\}$$

and $F_{(M,w)}(x,y) = \dfrac{a^T x + b^T y + \alpha + Mw^T(Ax + By - r)}{c^T x + d^T y + \beta}$

Theorem 5: For any optimal solution (x^*, y^*) of problem (FQP') , there exists a $u_2^* \geq 0$ and a $w^* \geq 0$ such that (x^*, y^*) is also optimal to the following linear fractional programming problem $(FP_{(u_2^*, w^*)})$:

$$\max \quad F_{(u_2^*,w^*)}(x,y) = \frac{a^T x + b^T y + \alpha + u_2^* w^{*T}(Ax + By - r)}{c^T x + d^T y + \beta} \tag{28}$$

$$\text{subject to} \quad (x,y) \in S'(w^*) \tag{29}$$

Proof. Suppose (x^*, y^*) is an optimal solution to (FQP'). By Theorem 1, there exists a $w^* \geq 0$ such that (x^*, y^*, w^*) is an optimal solution to problem (P1).Obviously $(x^*, y^*) \in S'(w^*)$ and is optimal to problem (P_{w^*}) . Therefore, there exists (u_1^*, u_2^*, u_3^*) such that (u_1^*, u_2^*, u_3^*) is optimal to (D_{w^*}) [5] and

$$F(x^*, y^*) = h_{w^*}(u_1^*, u_2^*, u_3^*)$$

From (19) the optimal objective values of problem (P1) and (P2) are equal as (x^*, y^*, w^*) and (u_1^*, u_2^*, u_3^*) are optimal to (P1) and (P2) respectively.

Hence , from (12) and (13) for any $(x, y) \in S$, we get

$$\begin{aligned}
&(u_1^* - u_2^* w^*)^T (Ax + By - r) u_3^{*T} (2Q_2 x + 2Q_1 y + q - B^T w^*) \\
&+ \frac{(u_1^* - u_2^* w^*)^T + u_3^{*T}(B^T w^* - q) + \alpha}{\beta}(c^T x + d^T y + \beta) \\
&= a^T x + b^T y + \alpha
\end{aligned} \tag{30}$$

Since $u_1^*(Ax + By - r) \leq 0$ and

$$\frac{a^T x^* + b^T y^* + \alpha}{c^T x^* + d^T y^* + \beta} = \frac{(u_1^* - u_2^* w^*)^T r + u_3^{*T}(B^T w^* - q) + \alpha}{\beta}$$

(30) gives

$$\frac{a^T x^* + b^T y^* + \alpha}{c^T d^* + d^T y^* + \beta} \geq \max_{(x,y)\in S} \left\{ \frac{a^T x + b^T y + \alpha + u_2^* w^T (Ax + By - r)}{c^T x + d^T y + \beta} \right.$$

$$
\left. - \frac{u_3^{*T}\{2Q_2x+2Q_1y+q-B^Tw^*\}}{c^Tx+d^Ty+\beta} \right\}
$$

$$
\geq \max_{(x,y)\in S'(w^*)} \left\{ \frac{a^Tx+b^Ty+\alpha+u_2^*w^T(Ax+By-r)}{c^Tx+d^Ty+\beta} \right.
$$

$$
\left. - \frac{u_3^{*T}\{2Q_2x+2Q_1y+q-B^Tw^*\}}{c^Tx+d^Ty+\beta} \right\}
$$

$$
\geq \max_{(x,y)\in S[w^*]} \left\{ \frac{a^Tx+b^Ty+\alpha+u_2^*w^{*T}(Ax+By-r)}{c^Tx+d^Ty+\beta} \right\}
$$

$$
= \max_{(x,y)\in S[w^*]} \left\{ \frac{a^Tx+b^Ty+\alpha}{c^Tx+d^Ty+\beta} \right\} = \frac{a^Tx^*+b^Ty^*+\alpha}{c^Tx^*+d^Ty^*+\beta} \tag{31}
$$

which implies that they should be equal. Therefore

$$
\frac{a^Tx^*+b^Ty^*+\alpha}{c^Tx^*d^Ty^*+\beta} = \frac{a^Tx^*+b^Ty^*+u_2^*w^{*T}(Ax+By^*-r)}{c^Tx^*+d^Ty^*+\beta} \tag{32}
$$

This shows that (x^*, y^*) is optimal to problem $FP_{(u_2^*,w^*)}$.

Theorem 6: There exists a $w^* \geq 0$ and $M_0 > 0$ such that for any $M > M_0$ any optimal solution of problem $FP_{(M,w^*)}$.

$$
F_{(M,w^*)} : \max F_{(M,w^*)}(x,y)
$$

$$
\text{subject to} \quad (x,y) \in S'(w^*)
$$

is also an optimal solution to (FQP').

Proof. Let (x^*, y^*) be any optimal solution to (FQP'). By Theorem 5, there exists a $w^* \geq 0$ and $u_2^* \geq 0$ such that

$$
\frac{a^Tx^*+b^Ty^*+\alpha}{c^Tx^*+d^Ty^*+\beta} = \max_{(x,y)\in S'(w^*)} \left\{ \frac{a^Tx+b^Ty+\alpha}{c^Tx+d^Ty+\beta} + \frac{u_2^*w^{*T}(Ax+By-r)}{c^Tx+d^Ty+\beta} \right\}
$$

Since $w^*(Ax+By-r) \leq 0$ for any $(x, y) \in$ S.

Therefore, (30), (31), (32) hold if u_2^* is replaced by a fixed positive number M greater than or equal to u_2^*. Let

$$E = \{(x,y) \mid (x,y) \text{ is an extreme point of } S'(w^*)\}$$

let
$$\upsilon = \min_{(x,y)\in E}\left\{\frac{-w^{*T}(Ax+By-r)}{c^Tx+d^Ty+\beta}\right\}$$
$$\upsilon_1 = \max_{(x,y)\in S'(w^*)}\left\{\frac{a^Tx+b^Ty+\alpha}{c^Tx+d^Ty+\beta}\right\}$$
$$\upsilon_2 = \frac{a^Tx^*+b^Ty^*+\alpha}{c^Tx^*+d^Ty^*+\beta}$$

Because $(x, y) \in S$, $V \geq 0$. If $\upsilon = 0$, then $w^*(Ax + By\text{-}r) = 0$ for all $(x,y)\in E$ and hence for all $(x,y)\in S'(w^*)$. Hence $F_{(M,w^*)}(x,y) = F(x,y)$ for all $(x,y)\in S'(w^*)$ and the theorem follows. If v > 0, let M_0 be the smallest integer greater than u_2^* and $\frac{\upsilon_1-\upsilon_2}{\upsilon}$.

As $M > M_0 > \frac{\upsilon_1-\upsilon_2}{\upsilon}$.

Therefore, $M\upsilon > \upsilon_1 - \upsilon_2$

$$M\left[\min \frac{-w^{*T}(Ax+By-r)}{c^Tx+d^Ty+\beta}\right] > \max_{(x,y)\in S'(w^*)}\left[\frac{a^Tx+b^Ty+\alpha}{c^Tx+d^Ty+\beta}\right] - \frac{a^Tx^*+b^Ty^*+\alpha}{c^Tx^*+d^Ty^*+\beta}$$

i.e.
$$\frac{a^Tx^*+b^Ty^*+\alpha}{c^Tx^*+d^Ty^*+\beta} > \max_{(x,y)\in S'(w^*)}\left[\frac{a^Tx+b^Ty+\alpha+Mw^*(Ax+By-r)}{c^Tx+d^Ty+\beta}\right]$$
$$> \max_{(x,y)\in S'(w^*)}\left[\frac{a^Tx+b^Ty+\alpha+u_2^*w^{*T}(Ax+By-r)}{c^Tx+d^Ty+\beta}\right]$$
$$= \frac{a^Tx^*+b^Ty^*+\alpha}{c^tx^*+d^Ty^*+\beta}$$

Therefore, the optimal solution $(\overline{x},\overline{y})$ of $FP_{(M,w^*)}$ must satisfy

$$\frac{a^T\overline{x}+b^T\overline{y}+\alpha}{c^T\overline{x}+d^T\overline{y}+\beta} + \frac{Mw^{*T}(A\overline{x}+B\overline{y}-r)}{c^T\overline{x}+d^T\overline{y}+\beta} = \frac{a^Tx^*+b^Ty^*+\alpha}{c^Tx^*+d^Ty+\beta}$$

We will now show that $w^{*T}(A\overline{x}+B\overline{y}-r)=0$.

If $w^{*T}(A\overline{x}+B\overline{y}-r)<0$

$$\frac{a^T\overline{x}+b^T\overline{y}+\alpha}{c^T\overline{x}+d^T\overline{y}+\beta}+\frac{Mw^{*T}(A\overline{x}+B\overline{y}-r)}{c^T\overline{x}+d^T\overline{y}+\beta}$$

$$<\frac{a^T\overline{x}+b^T\overline{y}+\alpha}{a^T\overline{x}+d^{T\overline{y}}+\beta}+\frac{v_1-v_2}{v}w^{*T}\frac{(A\overline{x}+B\overline{y}-r)}{c^T\overline{x}+d^T\overline{y}+\beta} \quad \text{[because } M>\frac{v_1-v_2}{v}\text{]}$$

$$\leq\frac{a^T\overline{x}+b^T\overline{y}+\alpha}{c^T\overline{x}+d^T\overline{y}+\beta}-(v_1-v_2)\leq v_2=\frac{a^T\overline{x}+b^T\overline{y}+\alpha}{c^T\overline{x}+d^T\overline{y}+\beta}$$

which contradicts (33). Thus for any optimal solution $(\overline{x},\overline{y})$ to $FP_{(M,w^*)}$ which satisfy

$$w^{*T}(Ax+By-r)=0 \text{ is also an optimal solution to (FQP').}$$

Theorem 7 [9]: Suppose (x^*, y^*) is an optimal solution to (FQP'). There exists an M and an extreme point $(\overline{x},\overline{y},\overline{w})$ of the polyhedral convex set T such that

$$F_{(M,\overline{w})}(\overline{x},\overline{y}) = F_{(M,w^*)}(x^*,y^*)$$

Proof. For proof Ref. [9].

Algorithm:

Step-1: Solve the problem

$$\max_{(x,y,w)\in T} F(x,y)$$

Let (x^*,y^*,w^*) be its optimal solution.

Step-2: If the solution so obtained is such that $w_i^*(Ax+By-r)_i=0$ for all i i.e., $(x^*,y^*,w^*)\in T(l_1,l_2)$ go to step 4, otherwise go to step-3.

Step-3: Find the next best solution to the above problem using Murti's method and go to Step-2.

Step-4: Stop (x^*,y^*) is the optimal solution of the given problem.

Example: Consider the linear fractional quadratic bilevel programming problem

$$\textbf{(FQP)}: \quad \max_x \quad F(x,y) = \frac{8x+3y}{5x+2y+1} \qquad \textbf{where } y \textbf{ solves}$$

$$\max_y \quad f(x, yF) = 2x + 3y - x^2 - y^2$$

$$\text{subject to} \qquad x + 4y \le 4$$
$$x + y \le 2$$
$$x, y \ge 0$$

For a given x, the follower's problem is equivalent to

$$\max_y \quad f_1(x,y) = 3y - y^2$$

$$\text{subject to} \qquad 4y \le 4 - x$$
$$y \le 2 - x$$
$$y \ge 0$$

(FQP') equivalent to (FQP) is Z

$$\max_x \quad F(x,y) = \frac{8x+3y}{5x+2y+1}$$

where y solves

$$\text{(FQP')} \qquad \max_y \quad f_1(x,y) = 3y - y^2$$

$$\text{subject to} \qquad 4y \le 4 - x$$
$$y \le 2 - x$$
$$x, y \ge 0$$

clearly $f_1(x,y)$ is a concave function. Problem (P1) equivalent to (GFP') is

$$\text{(P1)} \qquad \max \quad F(x,y) = \frac{8x+3y}{5x+4y+1}$$

$$\text{subject to} \qquad x + 4y \le 4$$
$$x + y \le 2$$
$$w_1(x + 4y - 4) + w_2(x + y - 2) = 0$$
$$2(-1) + 2.0.x + 3 = [4,1] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

i.e., $\qquad -2y + 3 = 4w_1 + w_2$

$$x, y, \ w_1, \ w_2 \ge 0$$

$$\text{Let } T = \left\{ (x,y,w) \left| \begin{array}{c} x+4y \le 4 \\ x+y \le 2 \\ -2y+3 \le 4w_1 + w_2 \\ x,y,w_1,w_2 \ge 0 \end{array} \right. \right\}$$

Solve the problem

$$\max_{(x,y,w)\in T} \quad F(x,y) = \frac{8x+3y}{5x+2y+1}$$

It's optimal solution is $x=$ 2, $y=0$, $w_1=2$ and $w_2=0$, with $F(x,y)=\dfrac{16}{11}$.

Clearly it does not satisfy the condition.

$$w_1(x+4y-u)+w_2(x+y-2)=0 .$$

Find the second best solution for this problem. It is $x=\dfrac{4}{3}$, $y=\dfrac{2}{3}$, $w_1=\dfrac{5}{3}$ and $w_2 = 0$. It is satisfies the above condition. Therefore, optimal solution for the fractional – quadratic bilevel programming problem is

$$x=\frac{4}{3}, \quad y=\frac{2}{3}, \quad F(x,y)=\frac{38}{27} \text{ and } f(x,y)=\frac{22}{9}.$$

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REFERENCES

[1] Bard, J. F. and Falk, J. E., "An explicit solution to the multilevel programming problem," Computers and Operations Research 9 (1982), 77-100.

[2] Bard, J. F., "An algorithm for solving the general bilevel programming problem," Mathematics of Operations Research, 8 (1983). No. 2., 260-272.

[3] Bard, J. F., "Optimality conditions for the bilevel programming problem," Naval Research Logistics Quarterly, 31 (1984), 13-26.

[4] Bialas, W. F. and Karwan, M. H., "Two level linear programming," Management Science, 30 (1984), No. 8, 1004-1020.

[5] Bector, C. R., "A note on Dual Fractional Program," Cahier, 16, No. 2 (1974), 107-115.

[6] Candler, W. and Townsley, R. J., "A linear two level programming problem," Computers and Operations Research, 9 (1982), 59-76.

[7] Charnes, A. and Cooper, W. W., "Programming with Linear Fractional Functionals," Naval Reserach Logistics Quarterly, Vol. 9 (1962), 181-186.

[8] Mangasarian, O. L., "Non Linear programming," McGraw Hill, New York, 1969.

[9] Wang, S., Wang, Q. and Romano - Rodriquez, "Optimality conditions and an algorithm for linear - quadratic bilevel programs," Optimization, 31 (1994), 127-139.