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# Approximate Maximum Likelihood Estimation for the Three-Parameter Weibull Distribution

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## Abstract

We obtain the approximate maximum likelihood estimators (AMLEs) for the scale and location parameters $\theta$ and $\mu$ in the three-parameter Weibull distribution based on Type-II censored samples. We also compare the AMLEs with the modified maximum likelihood estimators (MMLEs) in the sense of the mean squared error (MSE) based on complete sample.

*Keywords* : Approximate maximum likelihood estimator, Mean squared error, Modified maximum likelihood estimator, Type-II censored sample, Weibull distribution

## 1. Introduction

The probability density function (pdf) of the random variable $X$ having a three-parameter Weibull distribution is given by

$$f(x;\mu,\beta,\theta)=\frac{\beta}{\theta}\left(\frac{x-\mu}{\theta}\right)^{\beta-1}\exp\left\{-\left(\frac{x-\mu}{\theta}\right)^{\beta}\right\},\qquad(1.1)$$
$$\mu < x < \infty,\ \beta > 0,\ \theta > 0.$$

where $\theta$ is the scale parameter, $\beta$ is the shape parameter, and $\mu$ is the location parameter.

The Weibull distribution is named after the Swedish physicist, Waloddi Weibull, who used it to represent the distribution of the breaking strength of materials. The Weibull distribution has various applications in the field of reliability and life testing where samples are either complete or censored.

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In most cases of censored samples, the explicit estimators may not be obtained by the maximum likelihood estimation method. So we need another method that generates the explicit estimator. The approximate maximum likelihood estimation method was first developed by Balakrishnan (1989a, b) for the purpose of providing the explicit estimators of the scale parameter in the Rayleigh distribution and the mean and standard deviation in the normal distribution with censoring. Some historical remarks and a good summary of the approximate maximum likelihood estimation may be found in Balakrishnan and Cohen (1991).

Kang (1996) obtained the AMLE for the scale parameter of the double exponential distribution based on Type-II censored samples and he showed that the proposed estimator is generally more efficient than the best linear unbiased estimator and the optimum unbiased absolute estimator. Kang and Cho (1998) proposed the AMLEs of the location and scale parameters of the doubly truncated normal distribution. Woo et. al. (1998) obtained the AMLE of the scale parameter of the p-dimensional Rayleigh distribution with singly right censored samples. Kang et al. (2001) proposed the AMLE for the shape parameter in the three-parameter Weibull distribution based on Type-II right censored samples.

In this paper, we propose the AMLEs of the scale and location parameters $\theta$ and $\mu$. We also compare the AMLEs with the MMLEs based on complete samples in the sense of the MSE.

## 2. AMLE of the Scale and Location Parameters

Consider the pdf (1.1) and the cumulative distribution function (cdf) of the random variable $X$ having a three-parameter Weibull distribution as follows;

$$F(x;\mu,\beta,\theta)=1-\exp\left\{-\left(\frac{x-\mu}{\theta}\right)^{\beta}\right\},\qquad x>\mu, \tag{2.1}$$

where $\beta$ is known.

Let

$$X_{r+1:n}\le X_{r+2:n}\le\cdots\le X_{n-s:n} \tag{2.2}$$

be a doubly Type-II censored sample from the Weibull distribution with pdf (1.1), where the first $r$ and the last $s$ observations are censored.

The likelihood function based on the censored sample in (2.2) is given by

$$L=\frac{n!}{r!s!}\{F(X_{r+1:n})\}^{r}\{1-F(X_{n-s:n})\}^{s}\prod_{i=r+1}^{n-s}f(X_{i:n}). \tag{2.3}$$

By putting $Z_{i:n}=\dfrac{X_{i:n}-\mu}{\theta}$, the equation (2.3) can be rewritten as

$$L=\frac{n!}{r!s!}\theta^{-A}\{F(Z_{r+1:n})\}^{r}\{1-F(Z_{n-s:n})\}^{s}\prod_{i=r+1}^{n-s}f(Z_{i:n}), \tag{2.4}$$

where $A=n-r-s$ is the size of the censored sample, $f(z)=\beta z^{\beta-1}\exp(-z^{\beta})$ and $F(z)=1-\exp(-z^{\beta})$ are the pdf and cdf of the standard Weibull distribution, respectively.

The logarithm of the likelihood function (2.4) is given by

$$\ln L=\ln\left(\frac{n!}{r!s!}\right)-A\ln\theta+r\ln\{F(Z_{r+1:n})\}+s\ln\{1-F(Z_{n-s:n})\}+\sum_{i=r+1}^{n-s}\ln f(Z_{i:n}). \quad (2.5)$$

On differentiating with respect to $\theta$ and $\mu$ in turn and equating to zero, we obtain the estimating equations as

$$\begin{aligned}\frac{\partial\ln L}{\partial\theta} = & -\frac{1}{\theta}\Big[A+r\frac{f(Z_{r+1:n})}{F(Z_{r+1:n})}Z_{r+1:n}-s\frac{f(Z_{n-s:n})}{1-F(Z_{n-s:n})}Z_{n-s:n} \\ & +\sum_{i=r+1}^{n-s}\frac{f'(Z_{i:n})}{f(Z_{i:n})}Z_{i:n}\Big]=0,\end{aligned} \quad (2.6)$$

$$\frac{\partial\ln L}{\partial\mu}=-\frac{1}{\theta}\left[r\frac{f(Z_{r+1:n})}{F(Z_{r+1:n})}-s\frac{f(Z_{n-s:n})}{1-F(Z_{n-s:n})}+\sum_{i=r+1}^{n-s}\frac{f'(Z_{i:n})}{f(Z_{i:n})}\right]=0. \quad (2.7)$$

Since the likelihood equations (2.6) and (2.7) are very complicated, the equations do not admit explicit solutions for $\theta$ and $\mu$. But we may expand the functions $\frac{f(Z_{r+1:n})}{F(Z_{r+1:n})}$, $\frac{f(Z_{n-s:n})}{1-F(Z_{n-s:n})}$ and $\frac{f'(Z_{i:n})}{f(Z_{i:n})}$ in Taylor series around the points $F^{-1}(p_{r+1})=(-\ln q_{r+1})^{1/\beta}$, $F^{-1}(p_{n-s})=(-\ln q_{n-s})^{1/\beta}$ and $F^{-1}(p_i)=(-\ln q_i)^{1/\beta}$, respectively, where $p_i=i/(n+1)$ and $q_i=1-p_i$. So we may approximate these functions as

$$\frac{f(z_{r+1:n})}{F(z_{r+1:n})}\simeq\alpha-\delta Z_{r+1:n}, \quad (2.8)$$

$$\frac{f(z_{n-s:n})}{1-F(z_{n-s:n})}\simeq k+\eta Z_{n-s:n}, \quad (2.9)$$

$$\frac{f'(z_{i:n})}{f(z_{i:n})}\simeq v_i+\gamma_i Z_{i:n}, \quad (2.10)$$

where

$$\begin{aligned}\alpha= & \frac{\beta\left\{(-\ln q_{r+1})^{\frac{1}{\beta}}\right\}^{\beta-1}q_{r+1}}{p_{r+1}}-(-\ln q_{r+1})^{\frac{1}{\beta}}\Bigg[\frac{\beta(\beta-1)\left\{(-\ln q_{r+1})^{\frac{1}{\beta}}\right\}^{\beta-2}q_{r+1}}{p_{r+1}} \\ & -\frac{\beta^2\left\{(-\ln q_{r+1})^{\frac{1}{\beta}}\right\}^{2\beta-2}q_{r+1}}{p_{r+1}}-\frac{\beta^2\left\{(-\ln q_{r+1})^{\frac{1}{\beta}}\right\}^{2\beta-2}q_{r+1}^2}{p_{r+1}^2}\Bigg],\end{aligned}$$

$$\delta = \frac{\beta^2\left\{(-\ln q_{r+1})^{\frac{1}{\beta}}\right\}^{2\beta-2} q_{r+1} - \beta(\beta-1)\left\{(-\ln q_{r+1})^{\frac{1}{\beta}}\right\}^{\beta-2} q_{r+1}}{p_{r+1}}$$

$$+\frac{\beta^2\left\{(-\ln q_{r+1})^{\frac{1}{\beta}}\right\}^{2\beta-2} q_{r+1}^2}{p_{r+1}^2},$$

$$k = \beta\left\{(-\ln q_{n-s})^{\frac{1}{\beta}}\right\}^{\beta-1} - (-\ln q_{n-s})^{\frac{1}{\beta}}\beta(\beta-1)\left\{(-\ln q_{n-s})^{\frac{1}{\beta}}\right\}^{\beta-2},$$

$$\eta = \beta(\beta-1)\left\{(-\ln q_{n-s})^{\frac{1}{\beta}}\right\}^{\beta-2},$$

$$v_i = (\beta-1)(-\ln q_i)^{-\frac{1}{\beta}} - \beta\left\{(-\ln q_i)^{\frac{1}{\beta}}\right\}^{\beta-1} - (-\ln q_i)^{\frac{1}{\beta}}\Big[(\beta-1)(\beta-2)(-\ln q_i)^{-\frac{2}{\beta}}$$

$$-3\beta(\beta-1)\left\{(-\ln q_i)^{\frac{1}{\beta}}\right\}^{\beta-2} + \beta^2\left\{(-\ln q_i)^{\frac{1}{\beta}}\right\}^{2\beta-2}$$

$$-\left\{(\beta-1)(-\ln q_i)^{-\frac{1}{\beta}} - \beta\left\{(-\ln q_i)^{\frac{1}{\beta}}\right\}^{\beta-1}\right\}^2\Big],$$

and

$$\gamma_i = (\beta-1)(\beta-2)(-\ln q_i)^{-\frac{2}{\beta}} - 3\beta(\beta-1)\left\{(-\ln q_i)^{\frac{1}{\beta}}\right\}^{\beta-2}$$

$$+\beta^2\left\{(-\ln q_i)^{\frac{1}{\beta}}\right\}^{2\beta-2} - \left\{(\beta-1)(-\ln q_i)^{-\frac{1}{\beta}} - \beta\left\{(-\ln q_i)^{\frac{1}{\beta}}\right\}^{\beta-1}\right\}^2.$$

By substituting (2.8), (2.9), and (2.10) into (2.6) and (2.7), we obtain the approximate likelihood equations for $\theta$ and $\mu$ as

$$\frac{\partial \ln L}{\partial\theta} \simeq \frac{\partial \ln L^*}{\partial\theta} = -\frac{1}{\theta}\Big[A + r(\alpha - \delta Z_{r+1:n})Z_{r+1:n} - s(k + \eta Z_{n-s:n})Z_{n-s:n}$$
$$+\sum_{i=r+1}^{n-s}(v_i + \gamma_i Z_{i:n})Z_{i:n}\Big] = 0 \qquad (2.11)$$

and

$$\frac{\partial \ln L}{\partial\mu} \simeq \frac{\partial \ln L^*}{\partial\mu} = -\frac{1}{\theta}\left[r(\alpha - \delta Z_{r+1:n}) - s(k + \eta Z_{n-s:n}) + \sum_{i=r+1}^{n-s}(v_i + \gamma_i Z_{i:n})\right] = 0. \qquad (2.12)$$

From the equation (2.12), we can derive the AMLE of $\mu$ as

$$\hat{\mu}_{AMLE} = \frac{W + a_1\hat{\theta}_{AMLE}}{a_2}, \qquad (2.13)$$

where

$$W = r\delta X_{r+1:n} + s\eta X_{n-s:n} - \sum_{i=r+1}^{n-s}\gamma_i X_{i:n},$$

$$a_1 = sk - r\alpha - \sum_{i=r+1}^{n-s} v_i,$$

and

$$a_2 = r\delta + s\eta - \sum_{i=r+1}^{n-s} \gamma_i.$$

By substituting the resulting value into (2.11), and simplifying, we obtain the quadratic equation for $\theta$ as

$$A\theta^2 + B_1\theta + C_1 = 0, \tag{2.14}$$

where

$$B_1 = r\alpha X_{r+1:n} - sk X_{n-s:n} + \sum_{i=r+1}^{n-s} v_i X_{i:n} + \frac{a_1}{a_2} W$$

and

$$C_1 = -r\delta X^2_{r+1:n} - s\eta X^2_{n-s:n} + \sum_{i=r+1}^{n-s} \gamma_i X^2_{i:n} + \frac{W^2}{a_2}.$$

From the equation (2.14), we can derive the AMLE of $\theta$ as

$$\hat{\theta}_{AMLE} = \frac{-B_1 + \sqrt{B_1^2 - 4AC_1}}{2A}. \tag{2.15}$$

We simulate the numerical values of the MSEs of $\hat{\theta}_{AMLE}$ and $\hat{\mu}_{AMLE}$ by a Monte Carlo simulation method. The simulation procedure is repeated 2000 times for $n = 10(10)30$, $r = 0(1)4$, $s = 0(1)4$, $\theta = 2$ and $\beta = 2$. The values are presented in Table 1. Since the MSEs do not depend on the location parameter $\mu$, we take $\mu = 1$. For fixed $n$, as $r$ and $s$ increase, the MSEs increase although there are some fluctuations.

## 3. MMLEs Based on Complete Samples

In this section, in order to observe the efficiency of the proposed AMLEs we compute the MSEs of the MMLEs and compare them with those of the AMLEs of the scale and location parameters when the samples are complete, i.e. $r = 0$ and $s = 0$. Since the equations are so complicate and do not admit explicit solutions for the MMLEs of the parameters when the samples are censored, we consider the complete sample.

Cohen and Whitten (1980, 1982a, 1985) proposed the MMLEs for parameters of the lognormal, the gamma and the inverse Gaussian. Also, Cohen and Whitten (1982b) proposed the modified moment and maximum likelihood estimators of the three-parameter Weibull distribution. These estimators also possess some advantages with respect to ease of computation, bias, and variance.

In the Weibull distribution with the pdf (1.1) and the cdf (2.1), the likelihood function of $X_1, \cdots, X_n$ is

$$L=\left(\frac{\beta}{\theta^{\beta}}\right)^{n}\prod_{i=1}^{n}(x_i-\mu)^{\beta-1}\exp\left\{-\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\theta}\right)^{\beta}\right\}. \tag{3.1}$$

When $\beta$ is known, on taking logarithms of (3.1), differentiating with respect to $\theta$ and $\mu$ in turn and equating to zero, we obtain two equations as

$$\frac{\partial \ln L}{\partial \theta}=-\frac{n\beta}{\theta}+\frac{\beta}{\theta}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\theta}\right)^{\beta}=0 \tag{3.2}$$

and

$$\frac{\partial \ln L}{\partial \mu}=\frac{\beta}{\theta^{\beta}}\sum_{i=1}^{n}(x_i-\mu)^{\beta-1}-(\beta-1)\sum_{i=1}^{n}(x_i-\mu)^{-1}=0. \tag{3.3}$$

MMLEs may be obtained by replacing equation (3.3) with $E[F(X_{1:n})]=F(x_{1:n})$ (MMLE-I), $E(X_{1:n})=x_{1:n}$ (MMLE-II), $E(X)=\bar{x}$ (MMLE-III), $\text{Var}(X)=s^2$ (MMLE-IV), where $s^2$ is the sample variance, and $\text{Med}(X)=x_{\text{med}}$ (MMLE-V), where $\text{Med}(X)$ is the population median and $x_{\text{med}}$ is the sample median.

When $\theta$ is eliminated between the estimating equations, the equations are as follows;

$$\frac{n(x_{1:n}-\mu)^{\beta}}{-\ln[n/(n+1)]}=\sum_{i=1}^{n}(x_i-\mu)^{\beta} \quad \text{(MMLE-I)}, \tag{3.4}$$

$$n^2\left(\frac{x_{1:n}-\mu}{\Gamma(1+1/\beta)}\right)^{\beta}=\sum_{i=1}^{n}(x_i-\mu)^{\beta} \quad \text{(MMLE-II)}, \tag{3.5}$$

$$n\left(\frac{\bar{x}-\mu}{\Gamma(1+1/\beta)}\right)^{\beta}=\sum_{i=1}^{n}(x_i-\mu)^{\beta} \quad \text{(MMLE-III)}, \tag{3.6}$$

$$n\left(\frac{s^2}{\Gamma(1+2/\beta)-\Gamma(1+1/\beta)^2}\right)^{\beta/2}=\sum_{i=1}^{n}(x_i-\mu)^{\beta} \quad \text{(MMLE-IV)}, \tag{3.7}$$

$$\frac{n(x_{\text{med}}-\mu)^{\beta}}{\ln 2}=\sum_{i=1}^{n}(x_i-\mu)^{\beta} \quad \text{(MMLE-V)}, \tag{3.8}$$

and

$$\hat{\theta}_{MMLE}=\left[\frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{\mu}_{MMLE})^{\beta}\right]^{1/\beta}. \tag{3.9}$$

In this case, the MMLEs are not admit the explicit solutions. For $\beta>2$, we have to use iterative method to solve the equations. Hence, we take $\beta=2$ here for simple calculation and solve (3.4), (3.5), (3.6), (3.7), and (3.8) for $\mu$. And then $\hat{\mu}_{MMLE}$ can be derived. We compute the MSEs of these estimators and compare them with those of the AMLEs for $\theta=1(1)3$, and $n=10(10)30$ and 100 in Table 2.

From Table 2, the proposed AMLEs are much more efficient than the other modified maximum likelihood estimations. In addition, these proposed AMLEs are admit the explicit estimators. So, we can easily estimate the scale and location parameters by the approximate maximum likelihood estimation method.

**Table 1**. MSEs of $\hat{\theta}_{AMLE}$ and $\hat{\mu}_{AMLE}$ when $\mu=1,\ \theta=2,\ \beta=2$

| | | $n=10$ | | $n=20$ | | $n=30$ | |
|---|---|---|---|---|---|---|---|
| $r$ | $s$ | $\hat{\theta}_{AMLE}$ | $\hat{\mu}_{AMLE}$ | $\hat{\theta}_{AMLE}$ | $\hat{\mu}_{AMLE}$ | $\hat{\theta}_{AMLE}$ | $\hat{\mu}_{AMLE}$ |
| | 0 | 0.153428 | 0.214751 | 0.082771 | 0.085824 | 0.055127 | 0.049939 |
| | 1 | 0.163581 | 0.210680 | 0.083698 | 0.086003 | 0.056290 | 0.050028 |
| 0 | 2 | 0.225953 | 0.216558 | 0.087030 | 0.084762 | 0.057036 | 0.050039 |
| | 3 | 0.314044 | 0.232391 | 0.098546 | 0.085082 | 0.057621 | 0.049340 |
| | 4 | 0.403008 | 0.252002 | 0.115972 | 0.087474 | 0.061951 | 0.049292 |
| | 0 | 0.191277 | 0.248183 | 0.092033 | 0.087933 | 0.058404 | 0.048937 |
| | 1 | 0.215670 | 0.250057 | 0.094744 | 0.089020 | 0.060048 | 0.049226 |
| 1 | 2 | 0.308982 | 0.268506 | 0.100118 | 0.088314 | 0.061555 | 0.049569 |
| | 3 | 0.435404 | 0.300530 | 0.115199 | 0.089792 | 0.063136 | 0.049175 |
| | 4 | 0.562880 | 0.337347 | 0.136902 | 0.093750 | 0.069007 | 0.049655 |
| | 0 | 0.237716 | 0.334513 | 0.104470 | 0.103315 | 0.065490 | 0.056365 |
| | 1 | 0.271782 | 0.342755 | 0.107684 | 0.104791 | 0.067431 | 0.056806 |
| 2 | 2 | 0.401841 | 0.382623 | 0.115128 | 0.104875 | 0.069658 | 0.057540 |
| | 3 | 0.582791 | 0.445767 | 0.134456 | 0.108176 | 0.071981 | 0.057415 |
| | 4 | 0.787334 | 0.525747 | 0.161492 | 0.114596 | 0.079034 | 0.058334 |
| | 0 | 0.290812 | 0.451167 | 0.116703 | 0.122409 | 0.069738 | 0.062523 |
| | 1 | 0.334136 | 0.468140 | 0.121628 | 0.125304 | 0.072353 | 0.063399 |
| 3 | 2 | 0.511115 | 0.542276 | 0.130862 | 0.126265 | 0.074866 | 0.064346 |
| | 3 | 0.793212 | 0.675171 | 0.154565 | 0.131837 | 0.077695 | 0.064428 |
| | 4 | 1.128569 | 0.843822 | 0.187570 | 0.141466 | 0.085776 | 0.065798 |
| | 0 | 0.367371 | 0.641599 | 0.131442 | 0.146191 | 0.075193 | 0.070640 |
| | 1 | 0.424103 | 0.676636 | 0.137728 | 0.150423 | 0.078169 | 0.071805 |
| 4 | 2 | 0.673692 | 0.817830 | 0.148085 | 0.151912 | 0.081129 | 0.073095 |
| | 3 | 1.122633 | 1.093239 | 0.176130 | 0.160083 | 0.084229 | 0.073267 |
| | 4 | 1.814142 | 1.548570 | 0.215928 | 0.173905 | 0.093419 | 0.075202 |

**Table 2.** MSEs of the AMLEs and the MMLEs when $\mu=1$, $\beta=2$

| MSEs of the estimators of $\theta$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\theta$ | $n$ | $\hat{\theta}_{AMLE}$ | $\hat{\theta}_{MMLE-I}$ | $\hat{\theta}_{MMLE-II}$ | $\hat{\theta}_{MMLE-III}$ | $\hat{\theta}_{MMLE-IV}$ | $\hat{\theta}_{MMLE-V}$ |
| 1 | 10 | 0.038357 | 0.145232 | 0.138178 | 0.039096 | 0.042393 | 0.133215 |
| | 20 | 0.020693 | 0.089660 | 0.083986 | 0.023587 | 0.024501 | 0.105281 |
| | 30 | 0.013782 | 0.066034 | 0.061408 | 0.016181 | 0.016564 | 0.097136 |
| | 100 | 0.004103 | 0.024471 | 0.022414 | 0.005239 | 0.005272 | 0.087978 |
| 2 | 10 | 0.153428 | 0.580930 | 0.552714 | 0.156385 | 0.169571 | 0.532858 |
| | 20 | 0.082771 | 0.358638 | 0.335944 | 0.094349 | 0.098003 | 0.421125 |
| | 30 | 0.055127 | 0.264136 | 0.245630 | 0.064724 | 0.066257 | 0.388543 |
| | 100 | 0.016413 | 0.097886 | 0.089656 | 0.020956 | 0.021087 | 0.351911 |
| 3 | 10 | 0.345214 | 1.307092 | 1.243606 | 0.351867 | 0.381534 | 1.198931 |
| | 20 | 0.186235 | 0.806936 | 0.755873 | 0.212284 | 0.220507 | 0.947531 |
| | 30 | 0.124035 | 0.594306 | 0.552668 | 0.145629 | 0.149077 | 0.874222 |
| | 100 | 0.036929 | 0.220243 | 0.201727 | 0.047151 | 0.047446 | 0.791800 |
| MSEs of the estimators of $\mu$ | | | | | | | |
| $\theta$ | $n$ | $\hat{\mu}_{AMLE}$ | $\hat{\mu}_{MMLE-I}$ | $\hat{\mu}_{MMLE-II}$ | $\hat{\mu}_{MMLE-III}$ | $\hat{\mu}_{MMLE-IV}$ | $\hat{\mu}_{MMLE-V}$ |
| 1 | 10 | 0.053688 | 0.261114 | 0.246641 | 3.306912 | 3.526114 | 2.542017 |
| | 20 | 0.021456 | 0.140762 | 0.130832 | 3.228637 | 3.333382 | 2.271949 |
| | 30 | 0.012485 | 0.098038 | 0.090441 | 3.201594 | 3.270323 | 2.182967 |
| | 100 | 0.002694 | 0.032499 | 0.029514 | 3.159506 | 3.179683 | 2.075152 |
| 2 | 10 | 0.214751 | 1.044456 | 0.986564 | 13.227649 | 14.104456 | 10.168066 |
| | 20 | 0.085824 | 0.563047 | 0.523328 | 12.914550 | 13.333527 | 9.087797 |
| | 30 | 0.049939 | 0.392153 | 0.361763 | 12.806377 | 13.081292 | 8.731869 |
| | 100 | 0.010776 | 0.129997 | 0.118055 | 12.638024 | 12.718733 | 8.300608 |
| 3 | 10 | 0.483189 | 2.350025 | 2.219769 | 29.762209 | 31.735026 | 22.878149 |
| | 20 | 0.193104 | 1.266856 | 1.177489 | 29.057737 | 30.000435 | 20.447543 |
| | 30 | 0.112364 | 0.882344 | 0.813966 | 28.814349 | 29.432907 | 19.646706 |
| | 100 | 0.024247 | 0.292494 | 0.265624 | 28.435555 | 28.617149 | 18.676369 |

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