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Bayes Estimators for Reliability of a k-Unit Standby System with Perfect Switch

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Abstract

Bayes estimators and generalized ML estimators for reliability of a k-unit hot standby system with the perfect switch based upon a complete sample of failure times observed from an exponential distribution using noninformative, generalized uniform, and gamma priors for the failure rate are proposed, and MSE's of proposed several estimators for the standby system reliability are compared numerically each other through the Monte Carlo simulation.

Keywords : Standby System, Perfect Switch, Bayes estimators, generalized ML estimators,

1. Introduction

A standby redundant system is a type of parallel system in which only a single subsystem is in operation at a time, the remaining subsystems being subsequently brought into operation upon failure of the operation subsystem. The spare tire on a automobile is a simple example of a standby system. Depending on failure characteristic, standby redundancy is classified into three types. Hot standby system, where each component has the same failure rate regardless of whether it is standby or in operation ; Cold standby system, where components do not fail when they are in standby ; Warm standby system, where a standby component can fail but it has a smaller failure rate than the principal component. In a standby system, the system reliability is extremely dependent on the reliability of the switch. A poor switch can yields a system which is worse than a single subsystem. So, we shall consider problems of estimation for a k-unit hot standby system reliability with perfect switch (i.e., switching reliability is 1) in the sense of the Bayesian analysis.

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Osaki & Nakagawa(1971) computed the reliability for a two-unit standby redundant system with constant failure rate. Fujii & Sandoh(1984) considered Bayesian estimation for reliability of a two-unit hot standby redundant system. Kaput and Garg(1990) considered the technique of Markov renewal process to obtain various reliability measures for a two-unit standby system with the perfect switch. Kim, Moon, & Lee(1997) studied problems of estimation for reliability of the two-unit hot standby using some priors under squared error loss function.

In this paper, using noninformative, generalized uniform, and gamma priors for the failure rate in the exponential model, we shall propose some Bayes estimators and generalized ML estimators for the reliability of a k-unit hot standby system with the perfect switch based upon a complete sample of failure times observed from an exponential distribution under the square error loss function, and efficiencies for proposed Bayes estimators and generalized ML estimators for standby system reliability are compared numerically in the sense of mean squared error by the Monte Carlo method in the small sample size.

2. Bayes Estimators for Standby System Reliability

In the life testing, the most widely exploited model is an exponential model with following density function ;

$$f(t \mid \lambda) = \lambda e^{-\lambda t}, \quad 0 < t < \infty,$$

where λ is the failure rate.

In the context of the lifetime of industrial equipments and components, the probability that a given system will function for a specified period of "mission" time t_0 under the certain conditions is known as a system reliability. Problems of estimation for a system reliability play an important role in many practical reliability analysis. Here we shall consider the Bayes estimators for the k-unit hot standby system reliability with perfect switch, which can be considered as analogous to an on-line banking system. In the k-unit hot standby system, we shall assume the following :

1. The system consists of k independent and identically distributed units and a switch.
2. One unit is functioning and (k-1) units are standby.
3. The switch is instantaneous when the one in use fails.
4. Times to failure of units in use and standby are independent and exponentially distributed with the constant failure rate λ.
5. The switch is completely failure free(i.e., switching reliability is 1).
6. The unit and the switch are independent.

Then the reliability for k-unit hot standby system with perfect switch at specified mission time t_0 is given by

$$
\begin{aligned}
R(t_0) &= \sum_{m=0}^{k-1} \frac{(\lambda t_0)^m}{m!} e^{-\lambda t_0}, \quad t_o > 0 \\
&= P\{N_f \le k-1\},
\end{aligned}
\tag{2.1}
$$

where N_f is the number of units that fail.

Most Bayes estimators of reliability derived so far were based upon the priors on the unknown parameters which are related to the reliability than the reliability itself. Hence we shall consider noninformative, generalized uniform, and gamma priors on the failure rate λ in the exponential distribution. Also we shall use the squared error loss function to estimate the standby system reliability.

Let $T_1, \cdots, T_n$ be a simple random sample from an exponential distribution with failure rate λ and $T = \sum T_i$ be the total test time. Then it is well known that T has a gamma distribution with a shape parameter n and a scale parameter $1/\lambda$.

Bayesian analysis with noninformative prior is very common when little or no prior information is available. From the likelihood function and Fisher's information(Martz & Waller), the noninformative prior distribution for the failure rate Λ is given by

$$g(\lambda) \propto \frac{1}{\lambda}.$$

Then the posterior distribution of the failure rate Λ given total test time $T = t$ is

$$f(\lambda \mid t) = \frac{t^n}{\Gamma(n)} \lambda^{n-1} e^{-t\lambda}, \quad 0 < \lambda < \infty, \tag{2.2}$$

which is a $GAM(n, t)$ distribution.

Since the Bayes estimator of system reliability is found by taking the expectation of reliability function (2.1) with respect to the posterior distribution (2.2) under the squared error loss function, the Bayes estimator $\widehat{R_N}(t_0)$ for the k-unit standby system reliability $R(t_0)$ based on a noninformative prior for failure rate Λ is given by

$$\widehat{R_N}(t_0) = \sum_{k=0}^{n-1} \frac{t_0^k \, \Gamma(n+k)}{\Gamma(k+1)\Gamma(n)} \frac{T^n}{(T+t_0)^{n+k}}. \tag{2.3}$$

Maximizing the posterior distribution for the failure rate Λ in (2.2) with respect to λ, we can obtain the generalized ML estimator for the failure rate λ as follows :

$$\widehat{\lambda}_N = \frac{n}{T}.$$

Therefore, the generalized ML estimator $\widehat{R_{NG}}(t_0)$ for k-unit standby system reliability $R(t_0)$ under noninformative prior distribution on Λ is

$$\widehat{R_{NG}}(t_0) = \sum_{k=0}^{n-1} \frac{t_0^k}{k!}\left(\frac{n}{T}\right)^k \exp\left\{-\frac{nt_0}{T}\right\}. \tag{2.4}$$

Now, we suppose that the prior distribution for the failure rate Λ is a $GUNIF(\alpha, \beta)$, where $GUNIF$ denotes a generalized uniform distribution with the density function (see Tiwari et al(1996) ;

$$g(\lambda;\alpha,\beta) = \frac{\alpha+1}{\beta^{\alpha+1}}\lambda^{\alpha}, \quad 0<\lambda<\beta, \ -1<\alpha.$$

According to Bayes theorem, the posterior distribution of the failure rate Λ given the total test time $T=t$ is given as follows :

$$f(\lambda \mid t;\alpha,\beta) = \frac{t^{n+\alpha+1}}{\Gamma(n+\alpha+1, \beta t)}\lambda^{n+\alpha}e^{-t\lambda}, \quad 0<\lambda<\beta, \tag{2.5}$$

which is a truncated $GAM(n+\alpha, t\beta)$ and $\Gamma(\alpha, z)$ represents the standard incomplete gamma function.

Therefore, under the squared error loss function, the Bayes estimator $\widehat{R_U}(t_0)$ for the system reliability $R(t_0)$ on a generalized uniform prior of the failure rate Λ is given by

$$\widehat{R_U}(t_0) = \sum_{k=0}^{n-1} \frac{t_0^k}{k!} \frac{\Gamma[n+\alpha+k+1, \beta(t+t_0)]}{\Gamma[n+\alpha+1, \beta t]} \frac{T^{n+\alpha+1}}{(T+t_0)^{n+\alpha+k+1}}. \tag{2.6}$$

Maximizing the posterior distribution for the failure rate Λ in (2.5) with respect to λ, we can obtain the generalized MLE for the failure rate λ as follows :

$$\widehat{\lambda}_U = \frac{n+\alpha}{T}.$$

Therefore, under the generalized uniform prior distribution on λ, the generalized MLE $\widehat{R_{UG}}(t_0)$ for the system reliability $R(t_0)$ is

$$\widehat{R_{UG}}(t_0) = \sum_{k=0}^{n-1} \frac{t_0^k}{k!}\left(\frac{n+\alpha}{T}\right)^k \exp\left\{-\frac{(n+\alpha)t_0}{T}\right\}. \tag{2.7}$$

Next, we suppose that a prior distribution the failure rate Λ is a $GAM(\alpha, \beta)$ distribution given by

$$g(\lambda : \gamma, \theta) = \frac{1}{\Gamma(\gamma)} \theta^\gamma \lambda^{\gamma-1} e^{-\lambda/\theta}, \quad 0 < \lambda < \infty .$$

The most widely used prior distribution for λ is the gamma distribution. The main reason for this acceptability is the mathematical tractability resulting from the fact that the gamma distribution is the nature conjugate prior distribution. Such authors as Apostolakis and Mosleh (1979) and Grohowski, Hausman, and Lamberson (1976), and others have found the gamma distribution to be sufficiently versatile for practical reliability applications.
Then the posterior distribution of Λ given $T=t$ is given by

$$g(\lambda | t ; \gamma, \theta) = \frac{(t+1/\theta)^{n+\gamma}}{\Gamma(n+\gamma)} \lambda^{n+\gamma-1} \exp\{-(t+1/\theta)\lambda\}, \quad 0 < \lambda < \infty, \tag{2.8}$$

which is recognized as a $GAM(n+\gamma, (t+1/\theta)^{-1})$ distribution.
Under the squared error loss function, the Bayes estimator $\hat{R}_G(t_0)$ for the system reliability $R(t_0)$ based on gamma prior for the failure rate Λ is given by

$$\widehat{R_G}(t_0) = \sum_{k=0}^{n-1} \frac{t_0^k}{k!} \frac{\Gamma(n+\gamma+k)}{\Gamma(n+\gamma)} \frac{(T+1/\theta)^{n+\alpha}}{(T+t_0+1/\theta)^{n+\alpha+k}} . \tag{2.9}$$

Maximizing the posterior distribution of Λ in (2.8) with respect to λ, we can obtain the generalized MLE for λ as follows :

$$\widehat{\lambda_G} = \frac{(n+\gamma-1)}{T+1/\theta} .$$

Hence, under the gamma prior distribution on λ, the generalized MLE $\widehat{R_{GG}}(t_0)$ for the system reliability $R(t_0)$ is

$$\widehat{R_{GG}}(t_0) = \sum_{k=0}^{n-1} \frac{t_0^k}{k!}\left(\frac{n+\gamma-1}{T+1/\theta}\right)^k \exp\left\{-\frac{(n+\gamma-1)t_0}{T+1/\theta}\right\}. \tag{2.10}$$

To compare the performances of the proposed estimators for the reliability of k-unit hot standby system, the Monte Carlo simulations were carried out for the exponential distribution with failure rate λ. Tables 1 through 3 show the simulated values for the MSE of the proposed system reliability estimators for the k-unit hot standby system with perfect switch when the sample size n = 10(5)25, the failure rate $\lambda=1$, reliability $R(t)$=0.1 and number of unit k=2,3. From the tables, Bayes estimator of system reliability using generalized uniform prior for failure rate is more efficient than other Bayes estimators when α=0 and 0.5. But generalized ML estimator of system reliability using noninformative prior for failure rate is more efficient than other Bayes estimators when α=-0.5. The reasons for this results are that the generalized uniform density function is decreasing of x if $-1 < \alpha < 0$, and constant if $\alpha=0$ and increasing if $\alpha > 0$.

Table 1. Simulated MSE's for system reliability estimators under the noninformative prior distribution for the failure rate Λ

k	n	MSE	
		$\widehat{R_N}(t)$	$\widehat{R_{NG}}(t)$
2	10	0.011177	0.008480
	15	0.007087	0.005718
	20	0.005540	0.004615
	25	0.004204	0.003613
3	10	0.016841	0.012420
	15	0.010698	0.008287
	20	0.008359	0.006701
	25	0.006325	0.005235

Table 2. Simulated MSE's for system reliability estimators under the generalized uniform prior distribution for the failure rate Λ

k	α	β	n	MSE	
				$\widehat{R_U}$	$\widehat{R_{UG}}$
2	-0.5	3	10	0.008880	0.011178
			15	0.005978	0.007088
			20	0.004830	0.005540
			25	0.003750	0.004204
	0	2	10	0.006901	0.011178
			15	0.004983	0.007088
			20	0.004200	0.005540
			25	0.003359	0.004204
	0.5	5/3	10	0.005217	0.011178
			15	0.003979	0.007088
			20	0.003516	0.005540
			25	0.002911	0.004204

3	-0.5	3	10	0.013221	0.016842
			15	0.008884	0.010698
			20	0.007187	0.008359
			25	0.005565	0.006326
	0	2	10	0.010246	0.016842
			15	0.007335	0.010698
			20	0.006180	0.008359
			25	0.004919	0.006326
	0.5	5/3	10	0.007804	0.016842
			15	0.005881	0.010698
			20	0.005179	0.008359
			25	0.004254	0.006326

Table 3. Simulated MSE's for system reliability estimators under the gamma prior distribution for the failure rate Λ

k	θ	γ	n	MSE	
				$\widehat{R_G}$	$\widehat{R_{GG}}$
2	2	0.5	10	0.008057	0.009114
			15	0.005611	0.006077
			20	0.004626	0.004892
			25	0.003626	0.003788
	1	1	10	0.009434	0.010852
			15	0.006287	0.006864
			20	0.005052	0.005369
			25	0.003899	0.004086
	0.5	2	10	0.010252	0.011905
			15	0.006670	0.007314
			20	0.005288	0.005632
			25	0.004047	0.004248
3	2	0.5	10	0.012240	0.013886
			15	0.008489	0.009144
			20	0.006984	0.007325
			25	0.005455	0.005643
	1	1	10	0.014279	0.016549
			15	0.009501	0.010342
			20	0.007627	0.008048
			25	0.005867	0.006093
	0.5	2	10	0.015485	0.018160
			15	0.010074	0.011027
			20	0.007981	0.008448
			25	0.006090	0.006338

where simulations were repeated 5000 times when $R(t) = 0.1$.

References

[1] Apostolakis, G. and Mosleh, A.(1979), Expert Opinion and Statistical Evidence ; An Application to Reactor Core Melt Frequency, *Nuclear Sciences and Engineering,* Vol. 70, 135-149.

[2] Fujii, S. and Sandoh, H.(1984), Bayes Reliability Assessment of a 2-unit Hot Standby Redundant System, *IEEE Transactions on Reliability*, Vol. 33, 297-300.

[3] Grohowski, G., Hausman, W.C. and Lamberson, L.R.(1976), A Bayesian Statistical Inference Approach to Automative Reliability Estimation, *Journal of Quality Technology*, Vol. 8, 197-208.

[4] Kapur, P.K. and Garg, R.B.(1990), Compound Availability Measures for a Two-unit Standby System, *Microelectronics and Reliability*, Vol. 30, 425-429.

[5] Kim, H.J., Moon, Y.G, and Lee, C.S.(1997), Bayesian Reliability Estimation for a Two-unit Hot Standby System, *Journal of Statistical Theory & Methods*, Vol. 8, 31-39.

[6] Martz, H.F. and Waller, R.A.(1982), *Bayesian Reliability Analysis*, John Woley, New York.

[7] Osaki, S. and Nakagawa, N.(1971), On a Two-unit Standby Redundant System with Standby Failure, *Operations Research,* Vol. 19, 510-523.

[8] Tiwari, R.C., Yang, Y. and Zalkikar, J.N.(1996), Bayes Estimation for the Pareto Failure-Model Using Gibbs Sampling, *IEEE Transactions on Reliability*, Vol. 45,(3), 471-476.