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An Adaptive Test for Ordered Interqartile Ranges among Several Distributions †

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ABSTRACT

An adaptive estimation and testing method is proposed for comparing dispersions among several ordered groups. Based upon the large sampling theory for nonparametric quartile estimators, we derive the order restricted estimators and construct a simple test statistic. This test statistic has a mixture of several chi-square distributions as its asymptotic null distribution. The proposed test is illustratively applied to survival time data for the patients with carcinoma of the oropharynx.

Keywords: Quartiles; Dispersion; Ordered hypothesis; Wald statistics; Chi-bar-square distribution.

1. INTRODUCTION

Many of statistical problems, parametric or nonparametric, arise from the need of comparing populations behind the data at our hand. If observations are modeled in some way, systematic parts of the models are of our primary concern in the comparison. In this case, our comparison is based on the measure of central tendency such as mean and median(of a random function or process). Sometimes, however, it might be more interesting to compare groups upon the measure of dispersion. Let us take an example of evaluating performances of guns. Since the direction of a gun is easily adjustable, the gun should be evaluated by the spreadness of its bullet marks rather than the location of the center of those marks. Comparing variabilities might also be necessary for parametric modeling process. For instance, it is crucial to decide whether or not to use equal variance in an one-way ANOVA model because there is a big difference in theoretical manipulation between the two cases.

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Interquartile range(IQR) is one of typical measures for dispersion which has some advantages over its colleague 'variance'. One advantage is that IQR is a more stable measure than variance in the sense that it is less sensitive to outliers in the sample. It should also be noted that variance may not exist particularly for heavy-tailed distributions such as Cauchy or many of extreme-value distributions. However, IQR always exists regardless of the underlying distributional types. Let F be a distribution function. For $0 < p < 1$, the pth quantile is defined as $\zeta_p(F) = \inf\{x : F(x) \geq p\}$. Based upon the notation of quantile, the IQR of F can be expressed as

$$\theta(F) = \zeta_{3/4}(F) - \zeta_{1/4}(F).$$

As is well known, the uncertainty in a random observation increases according to the magnitude of IQR. In the context of quality control, large IQR indicates that the corresponding system is in the state of poor control.

If several groups are ordered in a certain way, one may wonder whether the corresponding IQR's are also ordered in the same(or opposite) direction. As we see from Figure 1 in Section 4, there is a stochastic ordering tendency in survival times among those groups of patients ordered in terms of the number of tumors. In the similar context, we need to reflect the group ordering effect in some way when we make an inference on the variability of survival times. One simple formulation is to construct an ordered testing problem in which we reject

$$H_0 : \theta(F_1) = \theta(F_2) = \cdots = \theta(F_k) \tag{1.1}$$

in favor of $H_1 - H_0$ where the hypothesis H_1 is

$$H_1 : \theta(F_1) \leq \theta(F_2) \leq \cdots \leq \theta(F_k). \tag{1.2}$$

Order restricted testing problems are found in many papers. Bartholomew (1959) and Chase (1974) consider likelihood ratio tests for testing homogeneity of means against ordered alternatives in the analysis-of-variance setup. This problem is extended to repeated measurement situation by Shin et. al. (1996). Abelson and Tukey (1963) and Schaafsma and Smith (1966) propose contrast tests for the Bartholomew's problem. Mukerjee et. al. (1987) develop an optimal contrast test for comparing several treatments with a control. Another stream of researches deal with stochastic ordering among multinomial distributions, and this category of research includes Robertson and Wright (1981), Dykstra et. al. (1991), Wang (1996), Park et. al. (1998), and many others. However, only a few researches deal with order restricted testing issues on variability. Kochar and

Gupta(1986) consider distribution-free tests for testing equality of k variances against order restricted alternatives. Halperin and Gordon(1987) suggest the likelihood ratio test for comparing two variances under order restriction. Ahmed and Kochar(1988) develop a nonparametric test for dispersion ordering between two distributions.

In this paper, we propose a reasonable adaptive test for detecting the ordering tendency in IQR among several groups. No distributional assumption is made except for continuity of the underlying densities. In Section 2, quartiles are estimated under H_0 and H_1, respectively. In Section 3, we construct a test statistic and derive its asymptotic null distribution. Section 4 provides an example involving survival times for carcinoma of the oropharynx. Finally in Section 5, we discuss some other subjects in relation with the contents of this paper.

2. ADAPTIVE LEAST SQUARES ESTMATORS

For $i = 1, 2, \cdots, k$, let $X_{i1}, X_{i2}, \cdots, X_{in_i}$ be a random sample from the ith distribution F_i having density f_i. We assume those densities are all continuous. The natural nonparametric estimators for the IQR's are easily obtained and they are

$$\theta(\hat{F}_i) = \zeta_{3/4}(\hat{F}_i) - \zeta_{1/4}(\hat{F}_i), \ i = 1, 2, \cdots, k, \tag{2.1}$$

where $\hat{F}_i$ are usual empirical distribution functions. For future convenience, we will simplify the notation by using θ_i, ζ_{i1} and ζ_{i2} for $\theta(F_i)$, $\zeta_{1/4}(F_i)$, and $\zeta_{3/4}(F_i)$, respectively. We will put $\hat{}$ on those parameters to represent the unrestricted estimators.

Since no formal procedure is available for estimating the parameters under H_0 and H_1, we will rely on adaptive least squares method based on the large sampling theory. First, we restate in the following lemma the asymptotic normality of sample quartiles(see Theorem B on p80 of Serfling(1980)).

Lemma 2.1. *Let $\hat{\zeta}_i$ be the unrestricted estimator for $\zeta_i = (\zeta_{i1}, \zeta_{i2})'$. Then, $\sqrt{n_i}(\hat{\zeta}_i - \zeta_i)$ follows asymptotically a bivariate normal distribution with mean vector 0 and covariance matrix Σ_i whose (j,l)th elements are given by*

$$\sigma_{ijl} = \frac{p_j(1-p_l)}{f_i(\zeta_{ij})f_i(\zeta_{il})} \quad \text{for } 1 \le j \le l \le 2 \tag{2.2}$$

with $p_1 = 0.25$ and $p_2 = 0.75$

Let $\hat{\Sigma}_i$ denote the estimated covariance matrix of Σ_i obtained by plugging $\hat{f}_i$ and $\hat{\zeta}_i$ into (2.2). There are plenty of methods available for estimating continuous density functions, and they are well summarized in Silverman(1986) and Wand and Jones(1995). When computing a kernel density estimator, one should not use too much small a bandwidth so that the estimated density function avoids large variability. Based upon the estimated covariance matrix, we may construct Mahalanobis distance functions such that

$$Q_i(\zeta_i) = n_i(\hat{\zeta}_i - \zeta_i)'\hat{\Sigma}_i^{-1}(\hat{\zeta}_i - \zeta_i), \quad i = 1, 2, \cdots, k. \tag{2.3}$$

Thus, we can obtain the approximate constrained estimators of ζ_i's by minimizing

$$Q(\zeta_1, \zeta_2, \cdots, \zeta_k) = \sum_{i=1}^{k} Q_i(\zeta_i)$$

under the constraints in H_0 and H_1, respectively.

Since ζ_i's are 2-dimensional vectors, $Q(\zeta_1, \zeta_2, \cdots, \zeta_k)$ is a quadratic function whose canonical form is

$$R(u, v) = \sum_{i=1}^{k} [\ \alpha_i(x_i - u_i)^2 + 2\beta_i(x_i - u_i)(y_i - v_i) + \gamma_i(y_i - v_i)^2\ ] \tag{2.4}$$

where $u = (u_1, u_2, \cdots, u_k)'$ and $v = (v_1, v_2, \cdots, v_k)'$. In (2.4) the constants satisfy $\alpha_i > 0$, $\gamma_i > 0$, and $\alpha_i\gamma_i - \beta_i^2 > 0$. The following lemma can be used to specify our constrained estimators.

Lemma 2.2. *Let (u^*, v^*) and (u°, v°) denote the solutions to the problem of minimizing $R(u, v)$ in (2.4) subject to $u_1 - v_1 \le u_2 - v_2 \le \cdots \le u_k - v_k$ and $u_1 - v_1 = u_2 - v_2 = \cdots = u_k - v_k$, respectively. Let $Av(i, j) = [\sum_{l=i}^{j} \frac{\alpha_l\gamma_l - \beta_l^2}{\alpha_l + 2\beta_l + \gamma_l}(x_l - y_l)]/[[\sum_{l=i}^{j} \frac{\alpha_l\gamma_l - \beta_l^2}{\alpha_l + 2\beta_l + \gamma_l}]$, $1 \le i \le j \le k$. Then, we have*

(a) $u_l^* = \frac{x_l\alpha_l + (x_l + y_l + \Delta_l^*)\beta_l + (y_l + \Delta_l^*)\gamma_l}{\alpha_l + 2\beta_l + \gamma_j}$ *and* $v_l^* = \frac{(x_l - \Delta_l^*)\alpha_l + (x_l + y_l - \Delta_l^*)\beta_l + y_l\gamma_l}{\alpha_l + 2\beta_l + \gamma_j}$

where $\Delta_l^* = \max_{i \le l} \min_{j \ge l} Av(i, j),\ l = 1, 2, \cdots, k.$

(b) $u_l^\circ = \frac{x_l\alpha_l + (x_l + y_l + \Delta_l^\circ)\beta_l + (y_l + \Delta_l^\circ)\gamma_l}{\alpha_l + 2\beta_l + \gamma_j}$ *and* $v_l^\circ = \frac{(x_l - \Delta_l^\circ)\alpha_l + (x_l + y_l - \Delta_l^\circ)\beta_l + y_l\gamma_l}{\alpha_l + 2\beta_l + \gamma_j}$

where $\Delta_l^\circ = Av(1, k),\ l = 1, 2, \cdots, k.$

Proof: Part (a): Let $\Delta_j = u_j - v_j$ and $\tau_j = v_j$ for $j = 1, 2, \cdots, k$. Then, the problem is equivalent to minimizing $\frac{1}{2}R(\tau + \Delta, \tau)$ under the constraints $\Delta_j - \Delta_{j+1} \leq 0,\ j = 1, 2, \cdots, k-1$. With slight modifications, Kuhn-Tucker conditions(see pp 314 - 315 of Luenberger(1984)) for this minimization are equivalent to

$$\sum_{l=1}^{j}[(\tau_l + \Delta_l - x_l)\alpha_l + (\tau_l - y_l)\beta_l] + \lambda_j = 0,\ j = 1, 2, \cdots, k-1, \tag{2.5}$$

$$\sum_{l=1}^{k}[(\tau_l + \Delta_l - x_l)\alpha_l + (\tau_l - y_l)\beta_l] = 0, \tag{2.6}$$

$$(\tau_j + \Delta_j - x_j)\alpha_j + (2\tau_j + \Delta_j - x_j - y_j)\beta_j + (\tau_j - y_j)\gamma_j = 0,\ j = 1, 2, \cdots, k, \tag{2.7}$$

$$\lambda_j(\Delta_{j+1} - \Delta_j) = 0,\ \lambda_j \geq 0,\ \Delta_j - \Delta_{j+1} \leq 0,\ j = 1, 2, \cdots, k-1, \tag{2.8}$$

where λ_j's are Lagrangian multipliers corresponding to the inequality constraints. Solving equations (2.7) with respect to τ_j's, we get

$$\tau_j = \frac{(x_j - \Delta_j)\alpha_j + (x_j + y_j - \Delta_j)\beta_j + y_j\gamma_j}{\alpha_j + 2\beta_j + \gamma_j},\ j = 1, 2, \cdots, k. \tag{2.9}$$

Putting (2.9) into (2.5) and (2.6) leads to

$$-\sum_{l=1}^{j}\frac{\alpha_l\gamma_l - \beta_l^2}{\alpha_l + 2\beta_l + \gamma_l}(x_l - y_l - \Delta_l) + \lambda_j = 0,\ j = 1, 2, \cdots, k-1, \tag{2.10}$$

$$-\sum_{l=1}^{k}\frac{\alpha_l\gamma_l - \beta_l^2}{\alpha_l + 2\beta_l + \gamma_l}(x_l - y_l - \Delta_l) = 0. \tag{2.11}$$

Thus, the solution to (2.9) - (2.11) satisfying (2.8) is the desired estimate. Consider a set of estimators, Δ^*, τ^* and λ^*, whose components are given by

$$\Delta_j^* = \max_{i \leq j}\min_{l \geq j} Av(i, l),\ j = 1, 2, \cdots, k,$$

$$\tau_j^* = \frac{(x_j - \Delta_j^*)\alpha_j + (x_j + y_j - \Delta_j^*)\beta_j + y_j\gamma_j}{\alpha_j + 2\beta_j + \gamma_j},\ j = 1, 2, \cdots, k,$$

$$\lambda_j^* = \sum_{l=1}^{j}\frac{\alpha_l\gamma_l - \beta_l^2}{\alpha_l + 2\beta_l + \gamma_l}(x_l - y_l - \Delta_l^*),\ j = 1, 2, \cdots, k-1.$$

Note that $Av(i,j), i \le j$ is the solution of Δ_l's to

$$\sum_{l=i}^{j} \frac{\alpha_l\gamma_l - \beta_l^2}{\alpha_l + 2\beta_l + \gamma_l}(x_l - y_l - \Delta_l) = 0.$$

with constraints $\Delta_i = \Delta_{i+1} = \cdots = \Delta_j$. Scrutinizing into Δ^*, its components can be grouped into several level sets, say, $I_1 = \{1, \cdots, i_1\}$, $I_2 = \{i_1 + 1, \cdots, i_2\}, \cdots$, $I_M = \{i_{M-1} + 1, \cdots, i_M\}$ with $i_M = k$ such that

$$\Delta_1^* = \cdots = \Delta_{i_1}^* < \Delta_{i_1+1}^* = \cdots = \Delta_{i_2}^* < \cdots < \Delta_{i_{M-1}+1}^* = \cdots = \Delta_{i_M}^*.$$

Equations (2.9) and (2.10) are obviously satisfied by the expressions for τ^* and λ^*. From the notion of level sets, we have

$$\sum_{l \in I_j} \frac{\alpha_l\gamma_l - \beta_l^2}{\alpha_l + 2\beta_l + \gamma_l}(x_l - y_l - \Delta_l^*) = 0, \ j = 1, 2, \cdots, M, \tag{2.12}$$

which proves the equation in (2.11). The first and third parts of constraints in (2.8) can be proved from the fact that

$$\lambda_{i_j}^* = \sum_{m=1}^{j} \sum_{l \in I_m} \frac{\alpha_l\gamma_l - \beta_l^2}{\alpha_l + 2\beta_l + \gamma_l}(x_l - y_l - \Delta_l^*) \ = \ 0, \ \ j = 1, 2, \cdots, M-1.$$

From the same fact, we can say that

$$\lambda_l^* = \sum_{m=i_{j-1}+1}^{l} \frac{\alpha_m\gamma_m - \beta_m^2}{\alpha_m + 2\beta_m + \gamma_m}(x_m - y_m - \Delta_m^*), \ l \in I_j$$

for any level set $I_j = \{i_{j-1} + 1, i_{j-1} + 2, \cdots, i_j\}$. Now, the last one to prove is the second part of (2.8). For this, consider a set of functions

$$b_l(z) = \sum_{m=i_{j-1}+1}^{l} \frac{\alpha_m\gamma_m - \beta_m^2}{\alpha_m + 2\beta_m + \gamma_m}(x_m - y_m - z), \ l \in I_j.$$

Since $\alpha_m\gamma_m - \beta_m^2 > 0$, we have $-\sqrt{\alpha_m\gamma_m} < \beta_m < \sqrt{\alpha_m\gamma_m}$, and hence

$$\alpha_m + 2\beta_m + \gamma_m > \alpha_m + \gamma_m - 2\sqrt{\alpha_m\gamma_m} = (\sqrt{\alpha_m} - \sqrt{\gamma_m})^2 > 0$$

from the positivity of α_m and γ_m. Thus, $b_l(z)$'s are decreasing functions of z. Since $\Delta_l^*, l \in I_j$ are the same and $Av(i_{j-1} + 1, l) \ge Av(i_{j-1} + 1, i_j) = \Delta_{i_j}, l \in I_j$, it follows that $\Delta_l^* \le Av(i_{j-1} + 1, l), \ l \in I_j$, and consequently

$$\lambda_l^* = b_l(\Delta_l^*) \ge b_l(Av(i_{j-1} + 1, l)) = 0, \ l \in I_j.$$

This completes proving the optimality of Δ^*, τ^* and λ^*. Taking the inverse transformation, we get the optimal solution for the original parameters as given in the lemma.

Part (b): We can apply the Lagrangian method under the equality constraints similarly to Part(a). In the Kuhn-Tucker conditions for this problem, no restrictions are imposed on the Lagrangian multipliers, and thus we need simply the equality constraints on Δ_j's in the place of (2.8). Since these Kuhn-Tucker conditions are easy to solve, we will omit the detailed procedures. □

Comparing equations (2.3) and (2.4), Lemma 2.2 can be directly applied to find our estimators by setting variables appropriately as follows:

$$\alpha_i = \frac{n_i\hat{\sigma}_{i22}}{\hat{\sigma}_{i11}\hat{\sigma}_{i22} - \hat{\sigma}_{i12}^2}, \ \beta_i = -\frac{n_i\hat{\sigma}_{i12}}{\hat{\sigma}_{i11}\hat{\sigma}_{i22} - \hat{\sigma}_{i12}^2}, \ \gamma_i = \frac{n_i\hat{\sigma}_{i11}}{\hat{\sigma}_{i11}\hat{\sigma}_{i22} - \hat{\sigma}_{i12}^2},$$
$$x_i = \hat{\zeta}_{i2}, \quad y_i = \hat{\zeta}_{i1}, \quad u_i = \zeta_{i2}, \quad v_i = \zeta_{i1}. \tag{2.13}$$

When constructing the Mahalanobis distance given in (2.3), we used $\hat{\Sigma}_i$ obtained by plugging $\hat{f}_i(\hat{\zeta}_{ij})$ and $\hat{f}_i(\hat{\zeta}_{il})$ into (2.2). Thus, the restricted estimators obtained by minimizing $Q(\zeta_1, \zeta_2, \cdots, \zeta_k)$ under each hypothesis are adaptive ones in the sense that we used sample information on Σ_i's to estimate ζ_i's which are of our primary concern. This adaptive method is advocated by the fact that the estimators are much less sensitive to $\hat{\Sigma}_i$'s than $\hat{\zeta}_i$'s.

3. TEST STATISTIC AND ITS ASYMPTOTICS

In the previous section, we proposed adaptive least squares estimators for quartiles. Based upon these estimators, we can construct a statistic for our testing problem similarly to Wald's test. Thus, using the same notation as in Lemma 2.2 with variables in (2.13), the test statistic is written by

$$T_{01} = R(u^\circ, v^\circ) - R(u^*, v^*), \tag{3.1}$$

and we reject H_0 for large values of T_{01} in favor of $H_1 - H_0$. Since finite sample behavior of T_{01} is completely unknown, the test should be based on the critical values from the asymptotic distribution of T_{01}. The following lemma is crucial for deriving the asymptotic null distribution of our test statistic.

Lemma 3.1. *The test statistic, T_{01}, in (3.1) is simplified as*

$$T_{01} = \sum_{i=1}^{k} (\Delta_i^* - \Delta_i^\circ)^2 \frac{\alpha_i\gamma_i - \beta_i^2}{\alpha_i + 2\beta_i + \gamma_i}. \tag{3.2}$$

Proof: After several simple computations, we get

$$x_i - u_i^\circ = \frac{\beta_i + \gamma_i}{\alpha_i + 2\beta_i + \gamma_i}(x_i - y_i - \Delta_i^\circ)$$

and

$$y_i - v_i^\circ = -\frac{\alpha_i + \beta_i}{\alpha_i + 2\beta_i + \gamma_i}(x_i - y_i - \Delta_i^\circ).$$

Putting these terms into R gives

$$R(u^\circ, v^\circ) = \sum_{i=1}^{k} (x_i - y_i - \Delta_i^\circ)^2 \frac{\alpha_i\gamma_i - \beta_i^2}{\alpha_i + 2\beta_i + \gamma_i}.$$

Similarly, we can show that

$$R(u^*, v^*) = \sum_{i=1}^{k} (x_i - y_i - \Delta_i^*)^2 \frac{\alpha_i\gamma_i - \beta_i^2}{\alpha_i + 2\beta_i + \gamma_i}.$$

Now,

$$\begin{aligned} R(u^\circ, v^\circ) &= \sum_{i=1}^{k} (x_i - y_i - \Delta_i^* + \Delta_i^* - \Delta_i^\circ)^2 \frac{\alpha_i\gamma_i - \beta_i^2}{\alpha_i + 2\beta_i + \gamma_i} \\ &= R(u^*, v^*) + \sum_{i=1}^{k} (\Delta_i^* - \Delta_i^\circ)^2 \frac{\alpha_i\gamma_i - \beta_i^2}{\alpha_i + 2\beta_i + \gamma_i} \\ &\quad + 2\sum_{i=1}^{k} (x_i - y_i - \Delta_i^*)(\Delta_i^* - \Delta_i^\circ) \frac{\alpha_i\gamma_i - \beta_i^2}{\alpha_i + 2\beta_i + \gamma_i}. \end{aligned}$$

Since Δ^* is the same as the isotonic regression of $\hat{\Delta} = x - y$ with weights $[\alpha_i\gamma_i - \beta_i^2]/[\alpha_i + 2\beta_i + \gamma_i]$ and Δ_i°'s are all equal, the last term vanishes by Theorems 1.3.2 and 1.3.3, and the lemma holds by (3.1). □

The theorem below gives the asymptotic distribution of our test statistic under H_0. Here, our sample sizes, $n_1, n_2, \cdots, n_k$, are assumed to increase to infinity in such a way that $n_i/n_1 \to r_i(> 0), i = 1, 2, \cdots, k$. This situation will be denoted by $n \to \infty$.

Theorem 3.1. *Let* $d_i = r_i/[\sigma_{i11} - 2\sigma_{i12} + \sigma_{i22}]$ *where* σ_{ijl} *are those in (2.2). Let* $d = \{d_1, d_2, \cdots, d_k\}'$ *and* $D = diag\{d_1, d_2, \cdots, d_k\}$. *Then, we have for any* $t > 0$

$$\lim_{n\to\infty} P[T_{01} > t] = \sum_{j=1}^{k} p(j,k;d) P[\chi^2_{j-1} > t], \tag{3.3}$$

where $p(j,k;d)$ *is the probability that the isotonic regression of a* $N(0, D^{-1})$ *- random vector has exactly* j *distinct component values.*

Proof: Let $\hat{w}_i = n_i^{-1}[\alpha_i\gamma_i - \beta_i^2]/[\alpha_i + 2\beta_i + \gamma_i]$. From the relations in (2.13), it follows that $\hat{w}_i = 1/[\hat{\sigma}_{i11} - 2\hat{\sigma}_{i12} + \hat{\sigma}_{i22}]$, and hence, $n_i\hat{w}_i/n_1 \to d_i$ by the strong consistency of $\hat{\sigma}_{ijl}$'s. Since $\hat{\Delta}_i = \hat{\zeta}_{i2} - \hat{\zeta}_{i1}, i = 1, 2, \cdots, k$ are independent, it can be verified by Lemma 2.1 that $\sqrt{n_1}(\hat{\Delta} - \Delta) \xrightarrow{L} Z \sim N(0, D^{-1})$. This argument together with Lemma 3.1 implies that

$$T_{01} \xrightarrow{L} \sum_{i=1}^{k} (Z_i^* - Z_i^\circ)^2 d_i$$

under H_0, where Z^* is the isotonic regression of the random vector Z following $N(0, D^{-1})$ and $Z_i^\circ = \sum_{l=1}^{k} d_l Z_l / \sum_{l=1}^{k} d_l$, $i = 1, 2, \cdots, k$. Thus, the theorem holds immediately by the Corollary of Theorem 2.3.1 of Robertson et al.(1988). □

The null distribution of type (3.3) is called a chi-bar-square distribution, and its level probabilities, $p(j,k;d), j = 1, 2, \cdots, k$, are generally unknown except for very limited cases. If there is no serious deviation among d_j's, then one may use equal weights approximation which gives the recursive form

$$p(j,k) = \frac{1}{k} p(j-1, k-1) + \frac{k-1}{k} p(j,k-1) \quad \text{for } j = 2, 3, \cdots, k-1 \tag{3.4}$$

with $p(1,k) = \frac{1}{k}$ and $p(k,k) = \frac{1}{k!}$. If those weights are seriously deviated, the level probabilities are usually estimated by simulating the distribution of the number of levels in the isotonic regression of the random vector generated from $N(0, D^{-1})$. Here, we use $\hat{d} = n_i\hat{w}_i/n_1$ adaptively for the unknown parameter vector d involved in the level probabilities.

4. AN EXAMPLE

The testing method developed in earlier sections is illustrated with a data set presented in Table 1 of Dykstra et. al.(1991). The table contains survival times

Table 1. Survival time for carcinoma of the oropharynx

Group	Survival time in days									
	38	107	130	167	172	191	238	243	296	324
1	336	343	351	372	374	376	404	432	446	446
	498	525	541	545	560	561	714	755	1574	
	81	105	128	154	170	184	216	222	228	230
	254	275	279	291	301	310	324	328	338	346
2	347	382	395	407	465	477	518	532	546	553
	575	599	608	631	661	666	763	915	929	1064
	1092	1317	1317							
	11	11	15	74	89	94	99	99	112	112
	112	127	134	144	147	147	159	162	172	173
	174	177	192	205	208	209	213	219	219	235
3	245	255	256	262	264	266	270	270	272	274
	293	307	308	317	327	334	363	369	370	407
	414	459	461	480	494	513	517	526	544	548
	637	637	696	726	757	782	785	800	805	911
	916	1565								

for the patients with carcinoma of the oropharynx classified into four groups according to the amount of lymph node deterioration. Thus, those groups are ordered in terms of the number of serious tumors. These data support the hypothesis that their survival distributions are ordered in the sense of uniform stochastic ordering.

Now, one may have interest in testing whether those groups of survival times are also ordered in the sense of dispersion measured as IQR. Since the second and third groups in the original table are rather small, those two groups are combined and thereby three groups of survival times are given here in Table 1. Note that we consider only complete observations as we assume in the development of our methodology. Figure 1 presents the survival functions estimated from the data in Table 1. Estimated median survival times for those groups are 374.0, 382.0, and 268.0. These data support the descending ordering tendency rather than the equality of the median survival times. However, estimates of IQR's are 271.0, 361.5, and 320.0 which seem to be ordered reversely to median survival times. So, it is interesting to test whether patients with more serious symptoms show

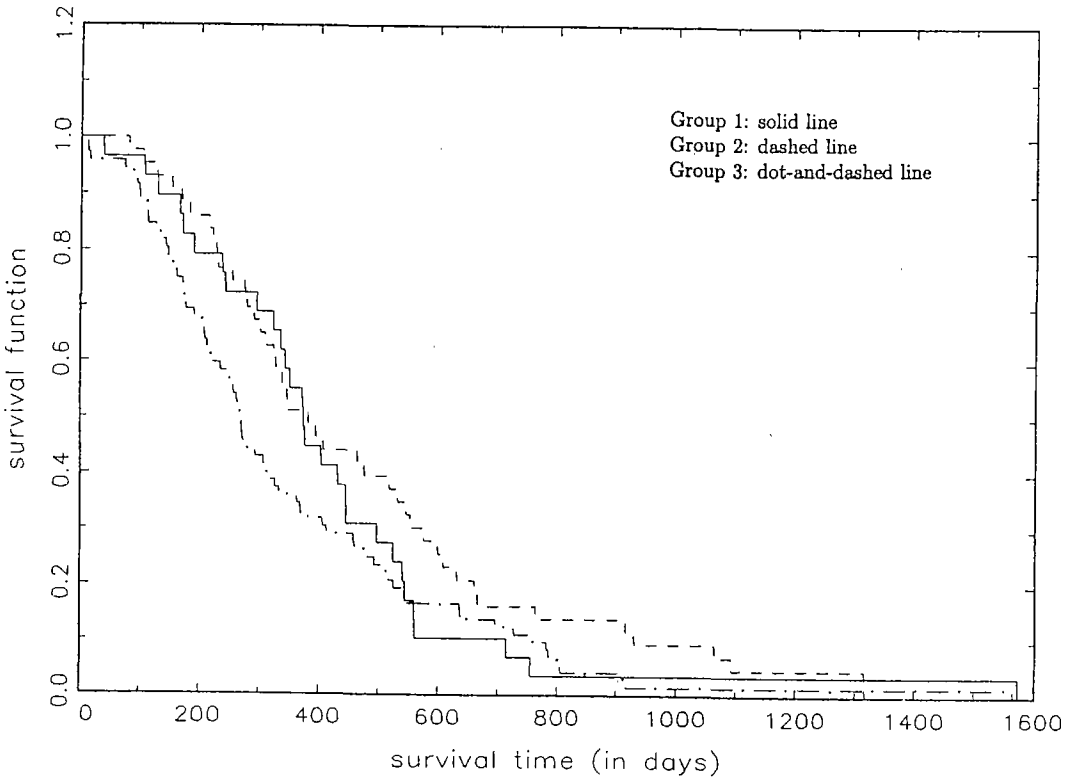

Figure 1. Estimates of Survival Functions

more dispersion in survival time. This is why we consider the ordered hypothesis on IQR's.

Unconstrained quartiles are easily estimated for each group by an usual quantile estimation method. For the constrained estimators, Lemma 2.2 can be directly used by replacing the variables with those in (2.13). The estimates of the parameters under each hypothesis are provided in Table 2. Since $n_1\hat{d}_i = [\alpha_i\gamma_i - \beta_i^2]/[\alpha_i + 2\beta_i + \gamma_i]$, we have by Lemma 3.1

$$T_{01} = \sum_{i=1}^{k} n_1(\Delta_i^* - \Delta_i^\circ)^2\hat{d}_i = 0.8201.$$

Let $\rho_{12} = -[\{\hat{d}_1\hat{d}_3\}/\{(\hat{d}_1 + \hat{d}_2)(\hat{d}_2 + \hat{d}_3)\}]^{1/2} = -0.5246$. Then, from the discussions in Section 2.4 of Robertson et. al.(1988), level probabilities are computed

Table 2. Estimates of parameters

Group	n_i	Constraints	ζ_{i1} $(\times 10^3)$	ζ_{i2} $(\times 10^3)$	Δ_i $(\times 10^3)$	d_i $(\times 10^{-5})$
1	29	No	.2405	.5115	.2710	.9012
		H_1	.2405	.5115	.2710	
		H_0	.2200	.5371	.3171	
2	43	No	.2420	.6035	.3615	.8693
		H_1	.2605	.5996	.3391	
		H_0	.2787	.5958	.3171	
3	72	No	.1670	.4870	.3200	1.0230
		H_1	.1484	.4874	.3391	
		H_0	.1698	.4869	.3171	

as $p(1,3;\hat{d}) = \frac{1}{4} - \frac{1}{2\pi}\sin^{-1}\rho_{12} = 0.3192, p(2,3;\hat{d}) = \frac{1}{2} = 0.5$, and $p(1,3;\hat{d}) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho_{12} = 0.1808$. Thus, the p-value of our test statistic is 0.1516 by Theorem 3.1. This result implies that there is no strong statistical evidence even at 10% level for rejecting the equality of IQR's in favor of IQR ordering specified in $H_1 - H_0$.

5. DISCUSSION

One of referees suggested to include discussions on the likelihood ratio test (LRT) for our problem. Since we allow no parametric model and assume simply the continuity of the underlying densities, it is impossible to construct likelihood functions based on the observations. However, if we discretize appropriately the variables, we may consider the LRT for dispersion ordering (more general than IQR ordering) in a multinomial setup. But, even in this case, the maximum likelihood estimators are completely unknown for multi-sample cases.

Censored observations have not been allowed through out the paper. If the proportion of censored observations is negligible, we may estimate quartiles from Kaplan-Meier estimators of survival functions and follow the same procedures with no change in asymptotic results. However, if the proportion is nonnegligibly large (that is, converges to a positive value as sample size goes to infinity), the consistency of the Kaplan-Meier estimator is not guaranteed. Moreover, it is difficult to estimate $f_i(\zeta_{ij})$ and $f_i(\zeta_{il})$ in (2.2) unless particular model assumptions

are made for censored observations.

The level probabilities, $p(j,k;d)$, $j = 1,2,\cdots,k$, have no closed expressions for general weight vector $d = (d_1, d_2, \cdots, d_k)'$ if $k \geq 4$. In such case, estimated level probabilities are usually suggested. First, generate sufficiently large number, say, M, of vectors from the k-dimensional multivariate normal distribution $N(0, D^{-1})$ where $D = diag\{d_1, d_2, \cdots, d_k\}$, and denote those sample vectors by $X_1, X_2, \cdots, X_M$. Compute the isotonic regression, X_i^*, of X_i for each i using the pool-adjacent violators algorithm (PAVA) discussed in Chapter 1 of Robertson et al.(1988). Let m_j be the number of X_i^*'s having exactly j distinct component values. Then, the estimates of $p(j,k;d)$ are given by $\frac{m_j}{M}$, $j = 1,2,\cdots,k$.

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