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Maximum Likelihood Estimator in Two Inverse Gaussian Populations with Unknown Common Coefficient of Variation

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ABSTRACT

This paper deals with the problem of estimating the means in two inverse Gaussian populations with equal but unknown coefficient of variation. The maximum likelihood estimators are derived by solving a cubic equation and their asymptotic variances are presented for comparative purpose. Monte-Carlo simulation is conducted to investigate the efficiency of the estimators relative to the sample means over a wide range of values for the sample size and the coefficient of variation. The effect on this efficiency under the departure from the assumption of common coefficient of variation is also studied.

Keywords: Inverse Gaussian distribution; Coefficient of variation; Maximum likelihood estimator.

1. INTRODUCTION

For the analysis of non-normal data, the inverse Gaussian distribution is widely used in applications. The inverse Gaussian distribution with a mean parameter μ and a scale parameter λ, abbreviated $IG(\mu, \lambda)$, is well known as a first passage time distribution in Brownian motion with positive drift. After the first investigation of its statistical properties by Tweedie(1957a, 1957b), extensive researches have been made into the distribution. For more detailed expositions and applications, refer to Chhikara and Folks(1978, 1989).

Estimation of the mean of a single inverse Gaussian population with known coefficient of variation has been extensively studied by a number of investigators. See, for instance, Iwase(1987) for the UMVU estimator, Hirano and Iwase(1989) for the minimum risk scale equivariant estimator based on the scale invariant

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loss functions, and Joshi and Shah(1991) for Bayes estimator. In the hypothesis testing problem, Joshi and Shah(1990) have investigated a sequential probability-ratio test of the inverse Gaussian mean with known coefficient of variation. Based on the maximum likelihood estimator, Hsieh(1990) has studied a likelihood ratio test of the coefficient of variation.

However, in contrast to the single population, there seems to be little literature on the estimation as well as the hypothesis testing of the inverse Gaussian means in two populations with common coefficient of variation. In this paper, we aim at providing maximum likelihood estimators(m.l.e.) for the means in two inverse Gaussian populations with equal but unknown coefficients of variation and investigating their performances over a range of values for the sample size and the coefficient of variation.

In Section 2 we describe the maximum likelihood estimation procedure for the means in detail. The procedure states that the m.l.e. of μ_1 can be obtained by solving a cubic equation. The equation may have multiple roots and the likelihood should be evaluated in all roots to determine the m.l.e. of μ_1. This requires the complex computation. For facilitating its computation, a result proved for the existence of the unique root which is the m.l.e. of μ_1 is given for the case of equal sample sizes because of the difficulty of revealing it theoretically for unequal sample sizes. The asymptotic variance of the m.l.e.'s is also provided for comparative purpose. In Section 3 the relative efficiency of the m.l.e.'s with respect to the sample means is investigated through Monte-Carlo simulation study. The results report that the relative efficiency is greater than one, regardless of the sample size and that it increases as the coefficient of variation increases. Finally a conclusion with brief remarks is provided in Section 4.

2. MAXIMUM LIKELIHOOD ESTIMATION

Let $X_{g1}, X_{g2}, \cdots, X_{gn}$ be a random sample of size n from an inverse Gaussian distribution with $E\left(X_{gi}\right) = \mu_g$, $Var\left(X_{gi}\right) = \gamma_g^2\mu_g^2$, $g = 1, 2$, i.e., $X_{gi} \sim IG\left(\mu_g, \mu_g/\gamma_g^2\right)$, where μ_g and γ_g respectively denote the mean and the coefficient of variation and they are positive. Given the assumption that coefficients of variation of two populations are common and unknown, i.e., $\gamma_1 = \gamma_2 = \gamma$, we are concerned with estimating the inverse Gaussian means, μ_1 and μ_2, by the maximum likelihood method.

Then the log-likelihood based on the sample is given by

$$L(\mu_1,\mu_2,\gamma) = -n\log 2\pi - 2n\log\ \gamma + \frac{n}{2}\sum_{g=1}^{2}\ \log\ \mu_g - \frac{1}{2\gamma^2}\sum_{g=1}^{2}\frac{1}{\mu_g}\sum_{i=1}^{n}\frac{(X_{gi}-\mu_g)^2}{X_{gi}} \tag{2.1}$$

Differentiating (2.1) with respect to parameters gives

$$\frac{\partial L}{\partial \mu_g} = \frac{n}{2\mu_g} - \frac{1}{2\gamma^2}\left(\sum_{i=1}^{n}\frac{1}{X_{gi}} - \frac{1}{\mu_g^2}\sum_{i=1}^{n}X_{gi}\right),\ g=1,2 \tag{2.2}$$

$$\frac{\partial L}{\partial \gamma} = \frac{1}{\gamma^3}\sum_{g=1}^{2}\frac{1}{\mu_g}\sum_{i=1}^{n}\frac{(X_{gi}-\mu_g)^2}{X_{gi}} - \frac{2n}{\gamma} \tag{2.3}$$

Setting these equations equal to zero and simplifying them lead to the following likelihood equations:

$$\mu_g^2 - \gamma^2\overline{X}_{H_g}\mu_g - \overline{X}_{H_g}\overline{X}_g = 0,\ g=1,2 \tag{2.4}$$

$$\gamma^2 = \frac{1}{2n}\sum_{g=1}^{2}\frac{1}{\mu_g}\sum_{i=1}^{n}\frac{(X_{gi}-\mu_g)^2}{X_{gi}} \tag{2.5}$$

where $\overline{X}_g = \sum_{i=1}^{n} X_{gi}\ /\ n$ and $\overline{X}_{H_g} = n\ /\ \sum_{i=1}^{n} X_{gi}^{-1}$. Solving these equations simultaneously gives m.l.e.'s and the results are given in Theorem 2.1.

Theorem 2.1. *The maximum likelihood estimator, $\widehat{\mu}_1$, of μ_1 is a positive root satisfying the cubic equation*

$$f(\mu_1) = 2(C_1+1)\mu_1^3 - (C_1+C_2-2)\overline{X}_1\mu_1^2 - 6\overline{X}_1^2\mu_1 + 2\overline{X}_1^3 = 0 \tag{2.6}$$

where $C_g = V_g\overline{X}_g$ and $V_g = 1\ /\ \overline{X}_{H_g} - 1\ /\ \overline{X}_g$, $g=1,2$. Also the maximum likelihood estimators for μ_2 and γ^2 are given by

$$\widehat{\mu}_2 = \frac{\widehat{\mu}_1\overline{X}_2}{2\widehat{\mu}_1 - \overline{X}_1} \tag{2.7}$$

$$\widehat{\gamma}^2 = \frac{\widehat{\mu}_1}{\overline{X}_{H_1}} - \frac{\overline{X}_1}{\widehat{\mu}_1} \tag{2.8}$$

Proof: Solving (2.4) for γ^2 leads to

$$\gamma^2 = \frac{\mu_g}{\overline{X}_{H_g}} - \frac{\overline{X}_g}{\mu_g},\ g=1,2 \tag{2.9}$$

and summing up (2.9) for g again yields

$$\gamma^2 = \frac{1}{2}\left(\sum_{g=1}^{2}\frac{\mu_g}{\overline{X}_{H_g}} - \sum_{g=1}^{2}\frac{\overline{X}_g}{\mu_g}\right) \tag{2.10}$$

Then substituting (2.10) into (2.5) gives, after some arrangement, a restriction for μ_1 and μ_2:

$$\sum_{g=1}^{2}\frac{\overline{X}_g}{\mu_g} = 2 \tag{2.11}$$

Since γ^2 of (2.9) can be expressed as

$$\gamma^2 = V_g\mu_g + \frac{\mu_g}{\overline{X}_{H_g}} - \frac{\overline{X}_g}{\mu_g},\ g = 1, 2 \tag{2.12}$$

equating (2.12) for g yields the following equation:

$$(C_2+1)\,\overline{X}_1\mu_1\mu_2^2 - \left[(C_1+1)\,\overline{X}_2\mu_1^2 - \overline{X}_1^2\overline{X}_2\right]\mu_2 - \overline{X}_1\overline{X}_2^2\mu_1 = 0 \tag{2.13}$$

Therefore, by expressing (2.11) in terms of μ_1 and substituting it into (2.13), a cubic equation for μ_1 is obtained:

$$f(\mu_1) = 2(C_1+1)\,\mu_1^3 - (C_1+C_2-2)\,\overline{X}_1\mu_1^2 - 6\overline{X}_1^2\mu_1 + 2\overline{X}_1^3 = 0 \tag{2.14}$$

Thus a positive root of (2.14) is the m.l.e. of μ_1 because $\mu_1 > 0$. In case of μ_2 and γ^2, expressing (2.11) and (2.9) in terms of μ_1 and substituting $\widehat{\mu}_1$ into them lead to (2.7) and (2.8). □

The cubic equation (2.14) has three roots. Among them, in order to determine the number of real roots occurring at the positive part, we use a result known as Descartes' rule of signs (Johnson and Riess, 1977, pp. 154-155). The rule states as follows: Let $f(x)$ be a polynomial of degree n with real coefficients. Further let n_p be the number of positive real roots of $f(x)$ and v be the number of sign changes in the nonzero coefficients of $f(x)$. Then $n_p \leq v$ and $v - n_p$ is a nonnegative even integer. Similarly the number of negative real roots of $f(x)$ is at most equal to the number of sign changes in the coefficients of $f(-x)$.

Since there are two sign changes in the coefficients of $f(\mu_1)$ regardless of the sign of $C_1 + C_2$, there are either zero or two positive real roots. Also since there are one sign change in $f(-\mu_1)$ regardless of the sign of $C_1 + C_2$ there must be one negative real root. By combining these results we see that $f(\mu_1)$ has either two positive real roots and one negative real root or one negative root and a pair of complex conjugate roots. But the value of $f(\mu_1)$ at certain positive point, for example at $\mu_1 = \overline{X}_1/2$, is negative and thus we must have two positive real roots and one negative real root.

For determining the unique m.l.e. for μ_1 among the positive roots, we should evaluate the likelihood function at all two. But this does not facilitate the computation of the m.l.e. In the following theorem we prove that there is only one positive real root greater than $max\left(\overline{X}_1/2, \sqrt{\overline{X}_{H_1}\overline{X}_1}\right)$ and it is the unique m.l.e. of μ_1.

Theorem 2.2. *The cubic equation, $f(\mu_1)$ has exactly one positive real root greater than $max\left(\overline{X}_1/2, \sqrt{\overline{X}_{H_1}\overline{X}_1}\right)$. This positive root becomes the unique maximum likelihood estimator for μ_1.*

Proof: Let m denote the maximum value of $\overline{X}_1/2$ and $\sqrt{\overline{X}_{H_1}\overline{X}_1}$. Let α_1 be the small root and α_2 be the large root of $f(\mu_1)$. Because of the positiveness of $\widehat{\mu}_2$ and $\widehat{\gamma}^2$ we have the following condition that $\widehat{\mu}_1 > m$. To prove the existence of the unique root satisfying the condition, it suffices to show that $\alpha_1 < m$ and $\alpha_2 > m$ for two different cases.

Case(i). Let $m = \overline{X}_1/2$. Then, evaluating $f(\widehat{\mu}_1)$ at $\widehat{\mu}_1 = 0$ and m, we have

$$f(0) = 2\overline{X}_1^3 > 0$$
$$f(m) = -\frac{\overline{X}_1^3(1 + C_2)}{4} < 0$$

Since $f(0)\,f(m) < 0$ we see that α_1 lies between 0 and m. This implies that $\alpha_1 < m$ and $\alpha_2 > m$. Thus α_2 is the unique m.l.e. of μ_1.

Case(ii). Let $m = \sqrt{\overline{X}_{H_1}\overline{X}_1}$. But in this case it is not easy to show whether $\alpha_2 > m$ because there is a difficulty in determining the sign of $f(m)$. But fortunately, using the relation that $\overline{X}_1/2 < m < \left(\overline{X}_1 + \overline{X}_{H_1}\right)/2$, we can show that $\alpha_2 > m$. Evaluating $f(\widehat{\mu}_1)$ at $\widehat{\mu}_1 = \left(\overline{X}_1 + \overline{X}_{H_1}\right)/2$ gives

$$f\left(\frac{\overline{X}_1 + \overline{X}_{H_1}}{2}\right) = \left(\overline{X}_1 + \overline{X}_{H_1}\right)^2\left[(C_1+1)\,\overline{X}_{H_1} - (C_2+1)\,\overline{X}_1\right] - 6\overline{X}_1\overline{X}_{H_1}\left(\overline{X}_1 - \overline{X}_{H_1}\right)$$

When the assumption for two samples is true we can expect C_1 and C_2 to be nearly equal and thus $f\left(\frac{\overline{X}_1+\overline{X}_{H_1}}{2}\right) < 0$. Accordingly, since $f\left(\frac{\overline{X}_1}{2}\right) < 0$, $f\left(\frac{\overline{X}_1+\overline{X}_{H_1}}{2}\right) < 0$ and moreover, there is only one root in the interval $\left(\frac{\overline{X}_1}{2}, \infty\right)$, we see that $f(m) < 0$. By observing that $f(0)f(m) < 0$ it turns out that $\alpha_1 < m$ and $\alpha_2 > m$ and so α_2 is the unique m.l.e. of μ_1. □

Since Theorem 2.2 states that the larger one of two positive roots becomes the unique m.l.e. for μ_1, we do not need neither to worry about the multiple m.l.e.'s of μ_1 nor to evaluate the likelihood function for selecting the proper m.l.e.

The asymptotic variances of the estimators can be obtained by use of the information matrix. To get them we need the negative second order partial derivatives:

$$-\frac{\partial^2 L}{\partial \mu_g^2} = \frac{n}{2\mu_g^2} + \frac{n\overline{X}_g}{\gamma^2 \mu_g^3}, \; g = 1, 2 \tag{2.15}$$

$$-\frac{\partial^2 L}{\partial \mu_g \partial \gamma} = \frac{1}{\gamma^3}\left(\frac{n}{\overline{X}_{H_g}} - \frac{n\overline{X}_g}{\mu_g^2}\right) \tag{2.16}$$

$$-\frac{\partial^2 L}{\partial \gamma^2} = \frac{3}{\gamma^4}\sum_{g=1}^{2}\frac{1}{\mu_g}\sum_{i=1}^{n}\frac{(X_{gi}-\mu_g)^2}{X_{gi}} - \frac{2n}{\gamma^2} \tag{2.17}$$

Taking expectations to these derivatives we can get the following information matrix for μ_1, μ_2, and γ:

$$I(\mu_1, \mu_2, \gamma) = \begin{pmatrix} \frac{n(2+\gamma^2)}{2\gamma^2\mu_1^2} & 0 & -\frac{n}{\gamma\mu_1} \\ 0 & \frac{n(2+\gamma^2)}{2\gamma^2\mu_2^2} & -\frac{n}{\gamma\mu_2} \\ -\frac{n}{\gamma\mu_1} & -\frac{n}{\gamma\mu_2} & \frac{4n}{\gamma^2} \end{pmatrix} \tag{2.18}$$

By inverting (2.18), the asymptotic variances are given by

$$Asym.\ var(\hat{\mu}_g) = \frac{\gamma^2\mu_g^2\left(2+\gamma^2/2\right)}{n\left(2+\gamma^2\right)}, \; g = 1, 2 \tag{2.19}$$

$$Asym.\ var(\hat{\gamma}) = \frac{\gamma^2\left(2+\gamma^2\right)}{8n} \tag{2.20}$$

The asymptotic relative efficiency of $\widehat{\mu}_g$ with respect to the sample means measured in terms of the variance is, as a function of γ, given by

$$Ref\left(\hat{\mu}_g, \overline{X}_g\right) = \frac{2+\gamma^2}{2+\gamma^2/2}, \; g = 1, 2 \tag{2.21}$$

Table 3.1: Monte-Carlo simulation results ($n = 10, 30$ and $\mu_1 = 1, \mu_2 = 3$)

		n = 10			n = 30		
γ	Estimator	Variance	Squared Bias	Relative Efficiency	Variance	Squared Bias	Relative Efficiency
0.1	$\overline{X}_1$	0.00101	0.164E-06	1.003	0.00033	0.485E-07	1.003
	$\hat{\mu}_1$	0.00101	0.189E-06		0.00033	0.472E-07	
	$\overline{X}_2$	0.00900	0.104E-05	1.003	0.00295	0.502E-06	1.001
	$\hat{\mu}_2$	0.00898	0.121E-05		0.00295	0.510E-05	
0.5	$\overline{X}_1$	0.02439	0.645E-06	1.047	0.00824	0.503E-06	1.053
	$\hat{\mu}_1$	0.02329	0.165E-06		0.00783	0.144E-06	
	$\overline{X}_2$	0.23125	0.392E-04	1.050	0.07399	0.176E-04	1.051
	$\hat{\mu}_2$	0.22026	0.292E-04		0.07042	0.256E-04	
1.0	$\overline{X}_1$	0.09975	0.524E-06	1.140	0.03286	0.230E-05	1.185
	$\hat{\mu}_1$	0.08752	0.455E-05		0.02773	0.126E-05	
	$\overline{X}_2$	0.90465	0.242E-04	1.154	0.29829	0.685E-05	1.180
	$\hat{\mu}_2$	0.78399	0.110E-04		0.25274	0.454E-05	
1.5	$\overline{X}_1$	0.21420	0.123E-03	1.198	0.07633	0.253E-06	1.330
	$\hat{\mu}_1$	0.17875	0.447E-05		0.05738	0.199E-06	
	$\overline{X}_2$	1.99244	0.154E-03	1.232	0.65367	0.937E-04	1.325
	$\hat{\mu}_2$	1.61684	0.277E-03		0.49323	0.679E-04	
2.0	$\overline{X}_1$	0.42005	0.121E-03	1.295	0.13204	0.154E-05	1.457
	$\hat{\mu}_1$	0.32449	0.125E-03		0.09062	0.914E-05	
	$\overline{X}_2$	3.57243	0.121E-03	1.229	1.23341	0.245E-03	1.429
	$\hat{\mu}_2$	2.90756	0.240E-02		0.86331	0.148E-03	
2.5	$\overline{X}_1$	0.63176	0.769E-04	1.212	0.20613	0.757E-04	1.539
	$\hat{\mu}_1$	0.52110	0.688E-03		0.13395	0.166E-04	
	$\overline{X}_2$	5.69085	0.392E-03	1.342	1.85444	0.108E-04	1.532
	$\hat{\mu}_2$	4.24127	0.409E-02		1.21043	0.294E-04	
3.0	$\overline{X}_1$	0.85057	0.174E-03	1.304	0.29834	0.118E-04	1.560
	$\hat{\mu}_1$	0.65232	0.117E-03		0.19124	0.263E-04	
	$\overline{X}_2$	7.56261	0.757E-03	1.325	2.77417	0.383E-03	1.650
	$\hat{\mu}_2$	5.70744	0.129E-02		1.68112	0.224E-03	

It tells us that $Ref(\hat{\mu}_g, \overline{X}_g)$ approaches one as $\gamma \to 0$ and two as $\gamma \to \infty$ and that $Ref(\hat{\mu}_g, \overline{X}_g)$ increases with γ but can not exceed two.

3. SIMULATION STUDY

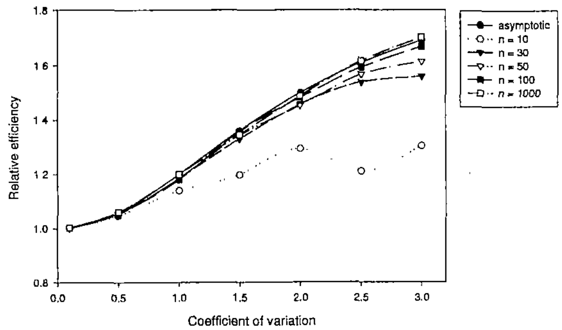
In order to investigate the performance of the m.l.e.'s, Monte-Carlo simulation study was conducted over a wide range of the sample size and the coefficient of variation. In the study, true means of two populations were set equal to $\mu_1 = 1$, $\mu_2 = 3$ and $\mu_1 = 5$, $\mu_2 = 10$, respectively. 10000 random samples of size

Table 3.2: Monte-Carlo simulation results ($n = 50, 100$ and $\mu_1 = 1, \mu_2 = 3$)

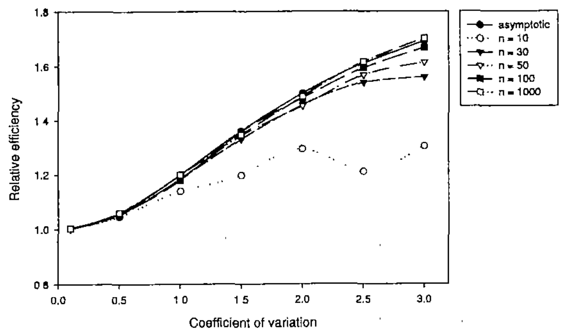
		$n = 50$			$n = 100$		
γ	Estimator	Variance	Squared Bias	Relative Efficiency	Variance	Squared Bias	Relative Efficiency
0.1	$\overline{X}_1$	0.00020	0.566E-09	1.001	0.00010	0.338E-08	1.004
	$\widehat{\mu}_1$	0.00020	0.497E-09		0.00010	0.419E-08	
	$\overline{X}_2$	0.00178	0.673E-08	1.003	0.00091	0.283E-06	1.001
	$\widehat{\mu}_2$	0.00178	0.495E-08		0.00091	0.259E-06	
0.5	$\overline{X}_1$	0.00506	0.248E-07	1.057	0.00245	0.465E-07	1.058
	$\widehat{\mu}_1$	0.00479	0.638E-10		0.00231	0.338E-07	
	$\overline{X}_2$	0.04428	0.429E-05	1.065	0.02284	0.180E-05	1.071
	$\widehat{\mu}_2$	0.04159	0.236E-05		0.02133	0.204E-05	
1.0	$\overline{X}_1$	0.02026	0.411E-09	1.185	0.01008	0.147E-05	1.180
	$\widehat{\mu}_1$	0.01710	0.452E-06		0.00854	0.174E-05	
	$\overline{X}_2$	0.18236	0.601E-04	1.200	0.08987	0.656E-07	1.211
	$\widehat{\mu}_2$	0.15195	0.350E-04		0.07420	0.392E-09	
1.5	$\overline{X}_1$	0.04570	0.848E-06	1.343	0.02248	0.473E-06	1.357
	$\widehat{\mu}_1$	0.03402	0.154E-05		0.01656	0.830E-06	
	$\overline{X}_2$	0.40571	0.345E-06	1.316	0.20046	0.150E-04	1.335
	$\widehat{\mu}_2$	0.30839	0.166E-05		0.15013	0.927E-05	
2.0	$\overline{X}_1$	0.08033	0.566E-05	1.453	0.03932	0.295E-05	1.482
	$\widehat{\mu}_1$	0.05530	0.427E-05		0.02653	0.148E-07	
	$\overline{X}_2$	0.71434	0.450E-04	1.443	0.35744	0.713E-04	1.472
	$\widehat{\mu}_2$	0.49502	0.183E-04		0.24281	0.760E-06	
2.5	$\overline{X}_1$	0.12695	0.103E-05	1.567	0.06381	0.106E-06	1.593
	$\widehat{\mu}_1$	0.08102	0.198E-05		0.04005	0.355E-06	
	$\overline{X}_2$	1.15419	0.529E-04	1.584	0.56014	0.395E-07	1.567
	$\widehat{\mu}_2$	0.72872	0.591E-04		0.35750	0.256E-05	
3.0	$\overline{X}_1$	0.18138	0.183E-04	1.614	0.09026	0.408E-05	1.668
	$\widehat{\mu}_1$	0.11238	0.845E-05		0.05411	0.361E-05	
	$\overline{X}_2$	1.59668	0.197E-03	1.625	0.81603	0.488E-04	1.683
	$\widehat{\mu}_2$	0.98228	0.696E-10		0.48478	0.549E-04	

$n = 10, 30, 50, 100$ and 1000 were drawn independently from the inverse Gaussian distribution $IG\left(\mu_g, \mu_g/\gamma^2\right)$, $g = 1, 2$ for the common coefficient of variation γ ranging from 0.1 to 3. The algorithm of Michael, et al.(1976) was used to generate the inverse Gaussian samples. Using the ZPORC subroutine of the International Mathematics and Statistics Library, we found out a positive root satisfying the condition as the unique m.l.e. of μ_1 and $\widehat{\mu}_2$ was obtained by use of $\widehat{\mu}_1$. The Monte-Carlo variance and squared bias of the m.l.e.'s were calculated from the sample. For comparative purpose, we measured the efficiency of the m.l.e.'s relative to the corresponding sample mean in terms of the variance.

Summarized results are presented in the tabular form. Table 3.1-3.2 contain the simulation results for $\mu_1 = 1$ and $\mu_2 = 3$ when $n = 10, 30, 50$ and 100. In the table, column 1 displays the value of the common population coefficient of variation, column 2 the estimator of the mean, column 3 the Monte-Carlo variance of the estimator, and column 4 the squared bias of the estimator and column 5 the relative efficiency of $\hat{\mu}_g$ relative to $\overline{X}_g$, $g = 1, 2$ calculated from the Monte-Carlo variance. The results report that no bias appears in the estimators on the whole. The Monte-Carlo variances were larger than the asymptotic variances calculated from the formula given by (2.19), not presented in the tables, for larger values of



(a)



(b)

Figure 3.1: Plot of (a) $Ref(\hat{\mu}_1, \overline{X}_1)$ and (b) $Ref(\hat{\mu}_2, \overline{X}_2)$ when $\mu_1 = 1$ and $\mu_2 = 3$

Table 3.3: Monte-Carlo simulation results ($n = 10, 30$ and $\mu_1 = 5, \mu_2 = 10$)

		$n = 10$			$n = 30$		
γ	Estimator	Variance	Squared Bias	Relative Efficiency	Variance	Squared Bias	Relative Efficiency
0.1	$\overline{X}_1$	0.02506	0.516E-06	1.000	0.00833	0.196E-06	1.002
	$\hat{\mu}_1$	0.02506	0.546E-06		0.00831	0.133E-06	
	$\overline{X}_2$	0.09975	0.324E-06	1.002	0.03274	0.576E-05	1.003
	$\hat{\mu}_2$	0.09953	0.299E-06		0.03265	0.635E-05	
0.5	$\overline{X}_1$	0.61635	0.189E-03	1.045	0.20870	0.588E-04	1.054
	$\hat{\mu}_1$	0.58993	0.153E-03		0.19798	0.509E-04	
	$\overline{X}_2$	2.53082	0.231E-05	1.048	0.82230	0.583E-05	1.064
	$\hat{\mu}_2$	2.41591	0.110E-07		0.77270	0.184E-05	
1.0	$\overline{X}_1$	2.50986	0.392E-03	1.153	0.84718	0.197E-03	1.218
	$\hat{\mu}_1$	2.17620	0.690E-03		0.69541	0.312E-04	
	$\overline{X}_2$	0.13672	0.258E-03	1.175	3.35443	0.308E-03	1.181
	$\hat{\mu}_2$	8.62924	0.290E-03		2.83975	0.230E-06	
1.5	$\overline{X}_1$	5.55052	0.170E-03	1.217	1.91893	0.857E-03	1.337
	$\hat{\mu}_1$	4.56053	0.147E-03		1.43555	0.772E-03	
	$\overline{X}_2$	22.40350	0.195E-02	1.256	7.44825	0.610E-03	1.312
	$\hat{\mu}_2$	17.83020	0.897E-07		5.67850	0.168E-02	
2.0	$\overline{X}_1$	9.89056	0.106E-04	1.257	3.34390	0.696E-04	1.467
	$\hat{\mu}_1$	7.86628	0.442E-02		2.27885	0.179E-04	
	$\overline{X}_2$	40.93097	0.140E-02	1.276	12.96052	0.362E-04	1.425
	$\hat{\mu}_2$	32.08504	0.130E-01		9.09559	0.101E-03	
2.5	$\overline{X}_1$	15.40312	0.154E-02	1.241	5.18193	0.447E-04	1.502
	$\hat{\mu}_1$	12.41097	0.265E-01		3.44973	0.910E-06	
	$\overline{X}_2$	67.79441	0.326E-01	1.290	19.95951	0.436E-03	1.477
	$\hat{\mu}_2$	52.56171	0.978E-01		13.51659	0.756E-03	
3.0	$\overline{X}_1$	23.06845	0.217E-02	1.271	7.33494	0.160E-03	1.582
	$\hat{\mu}_1$	18.15131	0.196E-01		4.63686	0.572E-05	
	$\overline{X}_2$	92.36459	0.127E-03	1.276	29.30820	0.125E-02	1.540
	$\hat{\mu}_2$	72.36800	0.894E-01		19.03323	0.788E-03	

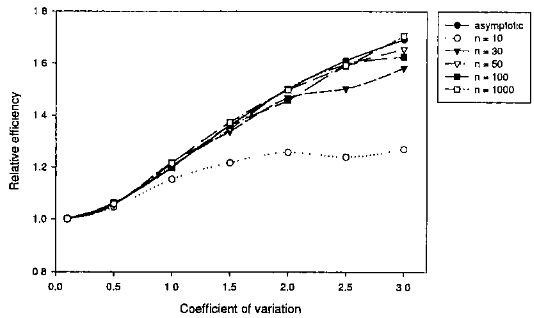
γ and smaller values of n. But we could see satisfactory agreement between the Monte-Carlo variances and the asymptotic ones as n increases. Similar results were observed when $n = 1000$. For all of the sample sizes the relative efficiency of the m.l.e.'s is shown to be greater than one and to increase with γ. The results say that $\hat{\mu}_1$ and $\hat{\mu}_2$ have higher efficiency than $\overline{X}_1$ and $\overline{X}_2$, particularly, for large values of γ and n. Figure 3.1 displays the pattern of the relative efficiency of $\hat{\mu}_1$ and $\hat{\mu}_2$ for n and γ. Also the results for $\mu_1 = 5$ and $\mu_2 = 10$, which is provided in Table 3.3-3.4 and Figure 3.2, also show that $\hat{\mu}_1$ and $\hat{\mu}_2$ outperform their counterparts.

Table 3.4: Monte-Carlo simulation results ($n = 50, 100$ and $\mu_1 = 5, \mu_2 = 10$)

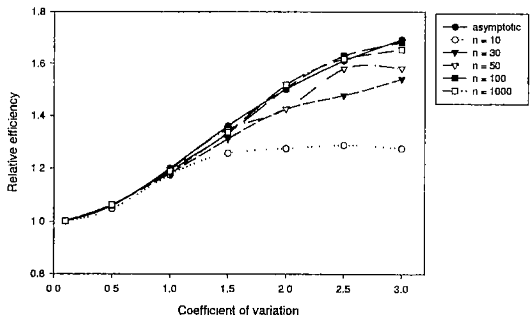
		$n = 50$			$n = 100$		
γ	Estimator	Variance	Squared Bias	Relative Efficiency	Variance	Squared Bias	Relative Efficiency
0.1	$\overline{X}_1$	0.00488	0.230E-06	1.002	0.00249	0.665E-07	1.002
	$\hat{\mu}_1$	0.00487	0.224E-06		0.00248	0.101E-06	
	$\overline{X}_2$	0.02041	0.609E-05	1.002	0.00994	0.187E-06	1.002
	$\hat{\mu}_2$	0.02038	0.628E-05		0.00992	0.914E-07	
0.5	$\overline{X}_1$	0.12664	0.700E-05	1.055	0.06300	0.538E-05	1.062
	$\hat{\mu}_1$	0.12000	0.766E-05		0.05932	0.600E-05	
	$\overline{X}_2$	0.50217	0.334E-04	1.061	0.25740	0.130E-04	1.060
	$\hat{\mu}_2$	0.47349	0.360E-04		0.24288	0.144E-04	
1.0	$\overline{X}_1$	0.50552	0.872E-04	1.205	0.24818	0.903E-05	1.198
	$\hat{\mu}_1$	0.41945	0.454E-04		0.20721	0.948E-05	
	$\overline{X}_2$	2.01334	0.633E-03	1.196	1.01300	0.137E-03	1.195
	$\hat{\mu}_2$	1.68383	0.965E-03		0.84802	0.130E-03	
1.5	$\overline{X}_1$	1.11691	0.621E-04	1.346	0.57277	0.115E-03	1.360
	$\hat{\mu}_1$	0.83003	0.324E-04		0.42127	0.420E-04	
	$\overline{X}_2$	4.47716	0.277E-04	1.348	2.20166	0.754E-03	1.332
	$\hat{\mu}_2$	3.32250	0.141E-04		1.65297	0.281E-03	
2.0	$\overline{X}_1$	1.99694	0.106E-04	1.503	0.99085	0.301E-04	1.457
	$\hat{\mu}_1$	1.32840	0.163E-03		0.67993	0.818E-05	
	$\overline{X}_2$	7.87637	0.495E-02	1.424	3.99516	0.125E-03	1.504
	$\hat{\mu}_2$	5.53087	0.731E-03		2.65605	0.109E-04	
2.5	$\overline{X}_1$	3.15990	0.517E-04	1.595	1.52988	0.271E-04	1.589
	$\hat{\mu}_1$	1.98101	0.653E-04		0.96291	0.182E-03	
	$\overline{X}_2$	12.64476	0.948E-03	1.580	6.33158	0.418E-02	1.629
	$\hat{\mu}_2$	8.00256	0.152E-03		3.88745	0.200E-02	
3.0	$\overline{X}_1$	4.38055	0.240E-07	1.655	2.26314	0.320E-03	1.624
	$\hat{\mu}_1$	2.64749	0.271E-04		1.39316	0.602E-04	
	$\overline{X}_2$	17.21369	0.174E-02	1.581	9.14687	0.367E-05	1.681
	$\hat{\mu}_2$	10.88676	0.327E-04		5.44116	0.179E-04	

As shown in the simulation results, whenever the samples come from the inverse Gaussian distribution with common coefficient of variation, the efficiency of the m.l.e.'s relative to the sample mean is always greater than one. In order to investigate to what extent this efficiency gain holds up under the departure from the assumption, we considered the situation where two populations have different coefficients of variation.

To evaluate the performance of the m.l.e.'s, Monte-Carlo simulation was carried out for sample sizes $n = 20$ and 100 and for the cases $\gamma_2 = 1.1\gamma_1$ and $\gamma_2 = 1.3\gamma_1$. Two population means were fixed at $\mu_1 = \mu_2 = 1$ and the parameter



(a)



(b)

Figure 3.2: Plot of (a) $Ref(\hat{\mu}_1, \overline{X}_1)$ and (b) $Ref(\hat{\mu}_2, \overline{X}_2)$ when $\mu_1 = 5$ and $\mu_2 = 10$

γ_1 was varied from 0.1 to 2. For considering the effect on the bias which might be appeared in the estimation of the parameters, the relative efficiency of $\widehat{\mu}_1$ was measured in terms of the mean-squared error. Summarized results are given in Table 3.5. Table 3.5a, which is the results for the case $\gamma_2 = 1.1\gamma_1$, shows that no bias is present in the estimation of the means. The relative efficiency of $\widehat{\mu}_1$, shown in column 4, is around 1.0 in all cases. While that of $\widehat{\mu}_2$ is greater than one and shows a tendency to increase as γ_1 increases. However, increasing sample sizes leads to a reduction of the efficiency of $\widehat{\mu}_2$. On the whole, the efficiency of the m.l.e.'s is good.

For the case $\gamma_2 = 1.3\gamma_1$, as shown in Table 3.5b, all estimators have a moderate

Table 3.5: Monte-Carlo simulation results when $\gamma_1 \neq \gamma_2$

		a. $\gamma_2 = 1.1\gamma_1$					
		$n = 20$			$n = 100$		
γ_1	Estimator	Variance	Squared Bias	RE	Variance	Squared Bias	RE
0.1	$\overline{X}_1$	0.00049	0.282E-06	0.998	0.00009	0.154E-06	0.995
	$\hat{\mu}_1$	0.00049	0.833E-06		0.00009	0.877E-06	
	$\overline{X}_2$	0.00071	0.127E-06	1.000	0.00014	0.297E-06	1.002
	$\hat{\mu}_2$	0.00071	0.669E-09		0.00014	0.118E-05	
0.5	$\overline{X}_1$	0.01328	0.165E-05	1.000	0.00254	0.560E-06	1.009
	$\hat{\mu}_1$	0.01324	0.400E-04		0.00239	0.127E-03	
	$\overline{X}_2$	0.01636	0.855E-05	1.096	0.00342	0.524E-05	1.046
	$\hat{\mu}_2$	0.01483	0.105E-03		0.00307	0.198E-03	
1.0	$\overline{X}_1$	0.05136	0.269E-03	0.991	0.01017	0.134E-04	0.993
	$\hat{\mu}_1$	0.05191	0.200E-03		0.00879	0.146E-02	
	$\overline{X}_2$	0.05625	0.840E-04	1.174	0.01376	0.426E-04	1.191
	$\hat{\mu}_2$	0.04662	0.134E-02		0.01008	0.151E-02	
1.5	$\overline{X}_1$	0.10510	0.767E-03	1.008	0.02183	0.727E-04	0.997
	$\hat{\mu}_1$	0.10410	0.900E-03		0.01812	0.386E-02	
	$\overline{X}_2$	0.12437	0.748E-04	1.473	0.02691	0.848E-04	1.338
	$\hat{\mu}_2$	0.08254	0.192E-02		0.01695	0.322E-02	
2.0	$\overline{X}_1$	0.17058	0.549E-04	1.020	0.04034	0.195E-03	1.072
	$\hat{\mu}_1$	0.16367	0.355E-02		0.03187	0.594E-02	
	$\overline{X}_2$	0.23356	0.150E-03	1.773	0.04617	0.253E-03	1.488
	$\hat{\mu}_2$	0.12945	0.233E-02		0.02631	0.490E-02	
		b. $\gamma_2 = 1.3\gamma_1$					
		$n = 20$			$n = 100$		
γ_1	Estimator	Variance	Squared Bias	RE	Variance	Squared Bias	RE
0.1	$\overline{X}_1$	0.00008	0.104E-02	0.941	0.00009	0.912E-07	0.974
	$\hat{\mu}_1$	0.00007	0.111E-02		0.00009	0.193E-05	
	$\overline{X}_2$	0.00007	0.569E-03	0.929	0.00019	0.911E-07	0.990
	$\hat{\mu}_2$	0.00007	0.620E-03		0.00019	0.195E-05	
0.5	$\overline{X}_1$	0.00371	0.443E-01	0.795	0.00254	0.541E-06	0.628
	$\hat{\mu}_1$	0.00347	0.570E-01		0.00260	0.144E-02	
	$\overline{X}_2$	0.00140	0.270E-03	0.638	0.00475	0.192E-05	0.895
	$\hat{\mu}_2$	0.00113	0.148E-02		0.00392	0.139E-02	
1.0	$\overline{X}_1$	0.01042	0.163E+00	0.850	0.01013	0.127E-04	0.403
	$\hat{\mu}_1$	0.00839	0.196E+00		0.01116	0.140E-01	
	$\overline{X}_2$	0.00360	0.208E-02	0.777	0.01867	0.347E-04	0.895
	$\hat{\mu}_2$	0.00224	0.506E-02		0.01114	0.975E-02	
1.5	$\overline{X}_1$	0.00997	0.896E-02	1.044	0.02198	0.735E-04	0.352
	$\hat{\mu}_1$	0.01509	0.304E-02		0.02725	0.355E-01	
	$\overline{X}_2$	0.02637	0.110E+00	1.297	0.03626	0.210E-03	0.967
	$\hat{\mu}_2$	0.01073	0.947E-01		0.01702	0.207E-01	
2.0	$\overline{X}_1$	0.00916	0.302E-01	1.468	0.04020	0.171E-03	0.368
	$\hat{\mu}_1$	0.01600	0.108E-01		0.05291	0.569E-01	
	$\overline{X}_2$	0.02727	0.109E+00	1.494	0.05863	0.259E-03	1.115
	$\hat{\mu}_2$	0.00967	0.816E-01		0.02507	0.277E-01	

amount of bias, particularly, when the sample size is small. When $n = 20$, the efficiency of the m.l.e.'s is low for the cases $\gamma_1 = 0.5$ and 1.0 but it becomes high for large values of γ_1. The results for $n = 100$ show noticeable features. The efficiency of $\widehat{\mu}_1$ is under one and decreases rapidly as γ_1 increases. Thus the performance of $\widehat{\mu}_1$ is poor except in the case of $\gamma_1 = 0.1$. Whereas that of $\widehat{\mu}_2$ is not bad overall, though its efficiency is also less than one.

4. CONCLUDING REMARKS

In this paper we dealt with the problem of estimating the means in two inverse Gaussian populations with equal but unknown coefficients of variation. We derived the maximum likelihood estimators of the inverse Gaussian means by solving a cubic equation and provided their asymptotic variances. In Monte-Carlo simluation study, we investigated the performance of the m.l.e.'s by measuring the relative efficiency with respect to the sample mean. From the results we observed that the efficiency of the m.l.e.'s is greater than one regardless of the sample size and it shows a tendency to increase as the coefficient of variation increases. It turns out that maximum likelihood estimators outperform their counterparts, that is, sample means for large values of the coefficient of variation. Even in the case that the samples are departed from the assumption of common coefficient of variation, the relative efficiency of the m.l.e.'s is good overall.

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