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A Strong Law of Large Numbers for Stationary Fuzzy Random Variables †

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Abstract

In this paper, a strong law of large numbers for sums of stationary and ergodic fuzzy random variables is obtained.

Keywords: fuzzy number; stationary fuzzy random variables; strong law of large numbers

1. INTRODUCTION

In recent years, strong law of large numbers for sums of fuzzy random variables have received much attention by several people. A SLLN for sums of i.i.d. fuzzy random variables was obtained by Kruse(1982), and a SLLN for sums of independent fuzzy random variables was obtained by Miyakoshi and Shimbo(1984), Klement, Puri and Ralescu(1986). Also, Inoue(1991) obtained a SLLN for sums of independent tight fuzzy random sets, and Hong and Kim(1994) proved Marcinkiewicz-type law of large numbers. Recently, Joo and Kim(to appear) generalized Kolmogorov's strong law of large numbers to the case of fuzzy random variables.

Now, it seems to be natural that we ask whether Birkhoff's ergodic theorem can be generalized to the case of fuzzy random variables. The purpose of this paper is to answer such a question. Section 2 is devoted to describe some basic concepts of fuzzy random variables. Main result is given in section 3.

2. PRELIMINARIES

Let $\mathbb{R}$ denote the real line. A fuzzy number is a fuzzy set $\tilde{u} : \mathbb{R} \to [0, 1]$ with the following properties ;

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(1) $\tilde{u}$ is normal, i.e., there exists $x \in \mathbb{R}$ such that $\tilde{u}(x) = 1$.

(2) $\tilde{u}$ is upper semicontinuous.

(3) supp $\tilde{u} = cl\{x \in \mathbb{R} \mid \tilde{u}(x) > 0\}$ is compact.

(4) $\tilde{u}$ is a convex fuzzy set, i.e., $\tilde{u}(\lambda x + (1-\lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$ for $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$

Let $F(\mathbb{R})$ be the family of all fuzzy numbers. For a fuzzy set $\tilde{u}$, if we define

$$L_\alpha \tilde{u} = \begin{cases} \{x \mid \tilde{u}(x) \geq \alpha\}, & 0 < \alpha \leq 1 \\ supp\ \tilde{u}, & \alpha = 0 \end{cases}$$

then, it follows that $\tilde{u}$ is a fuzzy number if and only if $L_1\tilde{u} \neq \phi$ and $L_\alpha\tilde{u}$ is a closed bounded interval for each $\alpha \in [0,1]$. From this characterization of fuzzy numbers, a fuzzy number $\tilde{u}$ is completely determined by the end points of the intervals $L_\alpha\tilde{u} = [u_\alpha^1, u_\alpha^2]$.

The following theorem(see Goetschel and Voxman (1986)) implies that we can identify a fuzzy number $\tilde{u}$ with the parameterized representation

$$\{(u_\alpha^1, u_\alpha^2) \mid 0 \leq \alpha \leq 1\}.$$

Theorem 2.1. *For $\tilde{u} \in F(\mathbb{R})$, denote $u^1(\alpha) = u_\alpha^1$ and $u^2(\alpha) = u_\alpha^2$ by considering as functions of $\alpha \in [0,1]$. Then*

(1) *u^1 is a bounded increasing function on $[0,1]$.*

(2) *u^2 is a bounded decreasing function on $[0,1]$.*

(3) *$u^1(1) \leq u^2(1)$.*

(4) *u^1 and u^2 are left continuous on $[0,1]$ and right continuous at 0.*

(5) *If v^1 and v^2 satisfy above (1)-(4), then there exists a unique $\tilde{v} \in F(\mathbb{R})$ such that $v_\alpha^1 = v^1(\alpha), v_\alpha^2 = v^2(\alpha)$.*

The addition and scalar multiplication on $F(\mathbb{R})$ are defined as usual ;

$$\begin{aligned} (\tilde{u} + \tilde{v})(z) &= \sup_{x+y=z} \min(\tilde{u}(x), \tilde{v}(x)), \\ (\lambda\tilde{u})(z) &= \begin{cases} \tilde{u}(z/\lambda) & ,\lambda \neq 0, \\ \tilde{0} & ,\lambda = 0, \end{cases} \end{aligned}$$

for $\tilde{u}, \tilde{v} \in F(\mathbb{R})$ and $\lambda \in \mathbb{R}$, where $\tilde{0} = I_{\{0\}}$ is the indicator function of $\{0\}$. It follows that if $\tilde{u} = \{(u_\alpha^1, u_\alpha^2) \mid 0 \le \alpha \le 1\}$ and $\tilde{v} = \{(v_\alpha^1, v_\alpha^2) \mid 0 \le \alpha \le 1\}$, then

$$\begin{aligned} \tilde{u} + \tilde{v} &= \{(u_\alpha^1 + v_\alpha^1, u_\alpha^2 + v_\alpha^2) \mid 0 \le \alpha \le 1\} \\ \lambda \tilde{u} &= \{(\lambda u_\alpha^1, \lambda u_\alpha^2) \mid 0 \le \alpha \le 1\} \text{ for } \lambda \ge 0. \end{aligned}$$

Now, we define the metric d_∞ on $F(\mathbb{R})$ by

$$d_\infty(\tilde{u}, \tilde{v}) = \sup_{0 \le \alpha \le 1} h(L_\alpha \tilde{u}, L_\alpha \tilde{v}),$$

where h is Hausdorff metric defined as

$$h(L_\alpha \tilde{u}, L_\alpha \tilde{v}) = \max(|u_\alpha^1 - v_\alpha^1|, |u_\alpha^2 - v_\alpha^2|).$$

The norm of $\tilde{u} \in F(\mathbb{R})$ is defined by

$$||\tilde{u}|| = d_\infty(\tilde{u}, \tilde{0}) = \max(|u_0^1|, |u_0^2|).$$

Then it is well known that $F(\mathbb{R})$ is complete but non-separable with respect to the metric d_∞. Joo and Kim(2000) introduced a metric d_s on $F(\mathbb{R})$ which makes it a separable metric space as follows :

Definition 2.2. Let T denote the class of strictly increasing continuous mapping of $[0,1]$ onto itself. For $\tilde{u}, \tilde{v} \in F(\mathbb{R})$, we define

$$\begin{aligned} d_s(\tilde{u}, \tilde{v}) &= \inf\{\epsilon > 0 \mid \text{ there exists a } t \in T \text{ such that} \\ &\qquad \sup_{0 \le \alpha \le 1} |t(\alpha) - \alpha| \le \epsilon \text{ and } d_\infty(\tilde{u}, t(\tilde{v})) \le \epsilon\}, \end{aligned}$$

where $t(\tilde{v})$ denotes the composition of $\tilde{v}$ and t.

It follows immediately that d_s is a metric on $F(\mathbb{R})$ and $d_s(\tilde{u}, \tilde{v}) \le d_\infty(\tilde{u}, \tilde{v})$. The metric d_s will be called the Skorokhod metric. It is well-known that $F(\mathbb{R})$ is a Polish space under the topology generated by d_s (For details, see Joo and Kim(2000)).

Now, we define, for $\tilde{u} \in F(\mathbb{R})$ and $0 \le \alpha < \beta \le 1$, $0 < \delta < 1$,

$$w_{\tilde{u}}(\alpha, \beta) = h(L_{\alpha+}\tilde{u}, L_\beta \tilde{u}) = \max(u_\beta^1 - u_{\alpha+}^1, u_{\alpha+}^2 - u_\beta^2) \tag{2.1}$$

$$w'_{\tilde{u}}(\delta) = \inf \max_{1 \le i \le r} w_{\tilde{u}}(\alpha_{i-1}, \alpha_i) \tag{2.2}$$

where $L_{\alpha^+}\tilde{u}$ denotes the closed interval $[u^1_{\alpha+}, u^2_{\alpha+}]$, and the infimum is taken over all partitions $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1$ of [0,1] satisfying $\alpha_i - \alpha_{i-1} > \delta$ for all i. Then, Lemma 3.2 of Joo and Kim(2000) implies that

$$\lim_{\delta \to 0} w'_{\tilde{u}}(\delta) = 0 \text{ for each } \tilde{u} \in F(\mathbb{R}). \tag{2.3}$$

Theorem 2.3. *Let K be a subset of $F(\mathbb{R})$. Then K is relatively compact in the d_s-topology if and only if*

$$\sup\{||\tilde{u}|| : \tilde{u} \in K\} < \infty \tag{2.4}$$

and

$$\lim_{\delta \to 0} \sup\{w'_{\tilde{u}}(\delta) : \tilde{u} \in K\} = 0 \tag{2.5}$$

Proof. See Ghil et al.(to appear).

3. MAIN RESULT

Let $(\Omega, \mathcal{A}, P)$ be a probability space. A function $\tilde{X} : \Omega \to F(\mathbb{R})$ is called a fuzzy random variable if it is measurable when $F(\mathbb{R})$ is considered as a metric space endowed with the metric d_s. If we denote $\{(X^1_\alpha, X^2_\alpha) \mid 0 \le \alpha \le 1\}$ by $\tilde{X}$, then it is well-known that $\tilde{X}$ is a fuzzy random variable if and only if for each $\alpha \in [0,1]$, X^1_α, and X^2_α are random variable in the usual sense. If $E||\tilde{X}|| < \infty$, then the expectation of $\tilde{X}$ is defined by $E\tilde{X} = \{(EX^1_\alpha, EX^2_\alpha) \mid 0 \le \alpha \le 1\}$.

Let $F^\infty(\mathbb{R})$ be the countable product of the space $F(\mathbb{R})$ endowed with the metric d_s. Then $F^\infty(\mathbb{R})$ is a separable metric space which is topologically complete. Let $\mathcal{B}(F^\infty(\mathbb{R}))$ be the Borel σ-field of $F^\infty(\mathbb{R})$.

Definition 3.1. Let $\{\tilde{X}_n\}$ be a sequence of fuzzy random variables.

(1) $\{\tilde{X}_n\}$ is called stationary if

$$P((\tilde{X}_1, \tilde{X}_2, \cdots) \in B) = P((\tilde{X}_2, \tilde{X}_3, \cdots) \in B)$$

for all $B \in \mathcal{B}(F^\infty(\mathbb{R}))$.

(2) A set $A \in \mathcal{A}$ is called invariant with respect to $\{\tilde{X}_n\}$ if there is a set $B \in \mathcal{B}(F^\infty(\mathbb{R}))$ such that for all $n \ge 1$

$$A = \{\omega \mid (\tilde{X}_n(\omega), \tilde{X}_{n+1}(\omega), \cdots) \in B\}.$$

(3) A stationary sequence $\{\tilde{X}_n\}$ is called ergodic if the probability of every invariant set is either 0 or 1.

Lemma 3.2. *Let $E||\tilde{X}|| < \infty$ and $\epsilon > 0$ be given.*

(1) *There exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1$ of $[0,1]$ such that*

$$E(X^1_{\alpha_i} - X^1_{\alpha^+_{i-1}}) \leq \epsilon \text{ for all } i = 1, 2, \cdots, r.$$

(2) *Similar statement holds for X^2_α.*

Proof. For $0 \leq \alpha < \beta \leq 1$, let us write

$$\theta(\alpha, \beta) = E(X^1_\beta - X^1_{\alpha+}).$$

Let

$$\tau_1 = \begin{cases} 1, & \text{if } \theta(0,1) \leq \epsilon \\ \inf\{\alpha : \theta(0, \alpha) > \epsilon\}, & \text{otherwise.} \end{cases}$$

By induction, we define

$$\tau_j = \begin{cases} 1, & \text{if } \theta(\tau_{j-1}, 1) \leq \epsilon \\ \inf\{\alpha > \tau_{j-1} : \theta(\tau_{j-1}, \alpha) > \epsilon\}, & \text{otherwise.} \end{cases}$$

The lemma will be proved if we show that $\tau_j = 1$ for some j. Suppose that $\tau_j < 1$ for all j. Since $|X^1_\alpha| \leq ||\tilde{X}||$ for all α , it follows from the dominated convergence theorem that there exists a point $\alpha_j \in (\tau_{j-1}, \tau_j]$ such that

$$E(X^1_{\tau_j} - X^1_{\alpha_j}) > \frac{\epsilon}{2} \text{ for all } j.$$

Since τ_j increases to a limit as $j \to \infty$, it follows that

$$X^1_{\tau_j} - X^1_{\alpha_j} \to 0 \text{ as } j \to \infty.$$

By dominated convergence theorem,

$$E(X^1_{\tau_j} - X^1_{\alpha_j}) \to 0 \text{ as } j \to \infty,$$

which is a contradiction. This completes the proof.

Lemma 3.3. *Let K be a compact subset of $F(\mathbb{R})$ with respect to the d_s-topology.*

(1) *If* $\epsilon > 0$ *is given, there exists a* $\delta > 0$ *such that for all* $\tilde{u} \in K$ *and* $0 \leq a < \alpha \leq b \leq 1$ *with* $b - a < \delta$,

$$\min(u^1_\alpha - u^1_{a+},\ u^1_b - u^1_\alpha) < \epsilon.$$

(2) *Similar statement holds for* u^2_α .

Proof. Let $\epsilon > 0$ be given. Then by Theorem 2.3, there exists a $\delta > 0$ such that

$$w'_{\tilde{u}}(\delta) < \epsilon \text{ for all } \tilde{u} \in K.$$

This implies that for each $\tilde{u} \in K$, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1$ of [0,1] satisfying $\alpha_i - \alpha_{i-1} > \delta$ for all i such that

$$u^1_{\alpha_i} - u^1_{\alpha^+_{i-1}} \leq w_{\tilde{u}}(\alpha_{i-1}, \alpha_i) < \epsilon \text{ for all } i.$$

Thus, if $0 \leq a < \alpha \leq b \leq 1$ and $b - a < \delta$, then for some i, either $(a, \alpha) \subset (\alpha_{i-1}, \alpha_i)$ or $[\alpha, b) \subset (\alpha_{i-1}, \alpha_i)$, which implies

$$\min(u^1_\alpha - u^1_{a+}, u^1_b - u^1_\alpha) < \epsilon.$$

Theorem 3.4. *Let* $\{\tilde{X}_n\}$ *be a sequence of stationary fuzzy random variables. If* $\{\tilde{X}_n\}$ *is ergodic and* $E||\tilde{X}_1|| < \infty$, *then*

$$\lim_{n\to\infty} d_\infty(\frac{\tilde{S}_n}{n}, E\tilde{X}_1) = 0 \qquad \text{a.s.}$$

where $\tilde{S}_n = \sum_{i=1}^n \tilde{X}_i$.

Proof. Let $\tilde{X}_n = \{(X^1_{n\alpha}, X^2_{n\alpha}) \mid 0 \leq \alpha \leq 1\}$ and $\tilde{S}_n = \{(S^1_{n\alpha}, S^2_{n\alpha}) \mid 0 \leq \alpha \leq 1\}$. For a fixed compact subset K of $F(\mathbb{R})$ in the d_s-topology, we write

$$T_{n\alpha} = \sum_{i=1}^n I_{\{\tilde{X}_1 \in K\}} X^1_{i\alpha}.$$

Then

$$|\frac{1}{n}S^1_{n\alpha} - EX^1_{1\alpha}| \leq |\frac{1}{n}T_{n\alpha} - \int_{\{\tilde{X}_1\in K\}} X^1_{1\alpha} dP| + \frac{1}{n}\sum_{i=1}^n I_{\{\tilde{X}_1 \notin K\}}|X^1_{i\alpha}| + \int_{\{\tilde{X}_1 \notin K\}} ||\tilde{X}_1|| dP \quad (3.1)$$

Let $\epsilon > 0$ be arbitrary. Then by Lemma 3.3, there exist a $\delta > 0$ such that for all $\tilde{u} \in K$ and $0 \le a < \alpha \le b \le 1$ with $b - a < \delta$,

$$u^1_\alpha - u^1_{a+} < u^1_b - u^1_{a+} + \frac{\epsilon}{2}.$$

Consequently, we have

$$\sup_{a<\alpha\le b} |\frac{1}{n} T_{n\alpha} - \int_{\{\tilde{X}_1 \in K\}} X^1_{1\alpha} dP| \le |\frac{1}{n} T_{na+} - \int_{\{\tilde{X}_1 \in K\}} X^1_{1a+} dP| + \frac{1}{n} |T_{na+} - T_{nb}| + \int_{\{\tilde{X}_1 \in K\}} |X^1_{1a+} - X^1_{1b}| dP + \epsilon. \quad (3.2)$$

Letting $n \to \infty$, we have from the Birkhoff's ergodic theorem for real valued random variables(See Theorem V.2 of Shiryayev (1984)),

$$\lim_{n\to\infty} \sup_{a<\alpha\le b} |\frac{1}{n} T_{n\alpha} - \int_{\{\tilde{X}_1 \in K\}} X^1_{1\alpha} dP| \le 2 \int |X^1_{1a+} - X^1_{1b}| dP + \epsilon \quad (3.3)$$

with probability one.

Now by Lemma 3.2, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r < 1$ satisfying $\alpha_i - \alpha_{i-1} > \delta$ for all i such that

$$E(X^1_{\alpha_i} - X^1_{\alpha^+_{i-1}}) \le \frac{\epsilon}{2} \text{ for all } i = 1, 2, \cdots, r. \quad (3.4)$$

It follows from (3.3) and (3.4) that with probability one

$$\lim_{n\to\infty} \sup_{a<\alpha\le b} |\frac{1}{n} T_{n\alpha} - \int_{\{\tilde{X}_1 \in K\}} X^1_{1\alpha} dP| \le 2\epsilon$$

This inequality, (3.1), (3.2) and (3.3) together with Birkhoff's ergodic theorem imply that with probability one

$$\lim_{n\to\infty} \sup_{0\le\alpha\le 1} |\frac{1}{n} S^1_{n\alpha} - EX^1_{1\alpha}| \le 2\epsilon + 2 \int_{\{\tilde{X}_1 \notin K\}} ||\tilde{X}_1|| dP. \quad (3.5)$$

Since $F(\mathbb{R})$ is a polish space, every probability measure on $F(\mathbb{R})$ is tight. Thus we can take a compact subset K of $F(\mathbb{R})$ such that

$$\int_{\{\tilde{X}_1 \notin K\}} ||\tilde{X}_1|| dP < \epsilon.$$

Then (3.5) implies that with probability one

$$\lim_{n\to\infty} \sup_{0\le\alpha\le 1} |\frac{1}{n} S^1_{n\alpha} - EX^1_{1\alpha}| \le 4\epsilon.$$

Similarily, it can be proved that with probability one

$$\lim_{n\to\infty} \sup_{0\leq\alpha\leq 1} |\frac{1}{n} S_{n\alpha}^2 - EX_{1\alpha}^2| \leq 4\epsilon.$$

Letting $\epsilon \to 0$, we complete the proof.

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