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ON GRAM'S DETERMINANT IN 2-INNER PRODUCT SPACES

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Abstract. An analogue of Gram's inequality for 2-inner product spaces is given. Further, a number of inequalities involving Gram's determinant are stated and proved in terms of 2-inner products.

1. Introduction

The concepts of 2-inner products and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in the book [1]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let X be a linear space of dimension greater than 1 over the field $K = \mathbf{R}$ of real numbers or the field $K = \mathbf{C}$ of complex numbers. Suppose that $(\cdot,\cdot|\cdot)$ is a K-valued function defined on $X \times X \times X$ satisfying the following conditions:

- ($2I_1$) $(x,x|z) \geq 0$ and $(x,x|z) = 0$ if and only if x and z are linearly dependent,
- ($2I_2$) $(x,x|z) = (z,z|x)$,
- ($2I_3$) $(y,x|z) = \overline{(x,y|z)}$,
- ($2I_4$) $(\alpha x,y|z) = \alpha(x,y|z)$ for any scalar $\alpha \in K$,
- ($2I_5$) $(x+x',y|z) = (x,y|z) + (x',y|z)$.

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$(\cdot,\cdot|\cdot)$ is called a *2-inner product* on X and $(X,(\cdot,\cdot|\cdot))$ is called a *2-inner product space* (or *2-pre-Hilbert space*). Some basic properties of 2-inner product $(\cdot,\cdot|\cdot)$ can be immediately obtained as follows:

(1) If $K=\mathbf{R}$, then $(2I_3)$ reduces to

$$(y,x|z)=(x,y|z).$$

(2) From $(2I_3)$ and $(2I_4)$, we have

$$(0,y|z)=0, \quad (x,0|z)=0$$

and also

$$(x,\alpha y|z)=\bar{\alpha}(x,y|z). \tag{1.1}$$

(3) Using $(2I_3)$–$(2I_5)$, we have

$$(z,z|x\pm y)=(x\pm y,x\pm y|z)=(x,x|z)+(y,y|z)\pm 2\mathrm{Re}(x,y|z)$$

and

$$\mathrm{Re}(x,y|z)=\frac{1}{4}[(z,z|x+y)-(z,z|x-y)]. \tag{1.2}$$

In the real case $K=\mathbf{R}$, (1.2) reduces to

$$(x,y|z)=\frac{1}{4}[(z,z|x+y)-(z,z|x-y)] \tag{1.3}$$

and, using this formula, it is easy to see that, for any $\alpha\in\mathbf{R}$,

$$(x,y|\alpha z)=\alpha^2(x,y|z). \tag{1.4}$$

In the complex case $K=\mathbf{C}$, using (1.1) and (1.2), we have

$$\mathrm{Im}(x,y|z)=\mathrm{Re}[-i(x,y|z)]=\frac{1}{4}[(z,z|x+iy)-(z,z|x-iy)],$$

which, in combination with (1.2), yields

$$(x,y|z)=\frac{1}{4}[(z,z|x+y)-(z,z|x-y)]+\frac{i}{4}[(z,z|x+iy)-(z,z|x-iy)]. \tag{1.5}$$

Using the above formula and (1.1), we have, for any $\alpha \in \mathbf{C}$,

$$(x, y|\alpha z) = |\alpha|^2 (x, y|z). \tag{1.6}$$

However, for $\alpha \in \mathbf{R}$ (1.6) reduces to (1.4). Also, from (1.6) it follows that

$$(x, y|0) = 0.$$

(4) For any three given vectors $x, y, z \in X$, consider the vector $u = (y, y|z)x - (x, y|z)y$. By ($2\mathrm{I}_1$), we know that $(u, u|z) \geq 0$ with the equality if and only if u and z are linearly dependent. The inequality $(u, u|z) \geq 0$ can be rewritten as

$$(y, y|z)[(x, x|z)(y, y|z) - |(x, y|z)|^2] \geq 0. \tag{1.7}$$

For $x = z$, (1.7) becomes

$$-(y, y|z)|(z, y|z)|^2 \geq 0,$$

which implies that

$$(z, y|z) = (y, z|z) = 0 \tag{1.8}$$

provided y and z are linearly independent. Obviously, when y and z are linearly dependent, (1.8) holds too. Thus (1.8) is true for any two vectors $y, z \in X$. Now, if y and z are linearly independent, then $(y, y|z) > 0$ and, from (1.7), it follows

$$|(x, y|z)|^2 \leq (x, x|z)(y, y|z). \tag{1.9}$$

Using (1.8), it is easy to check that (1.9) is trivially fulfilled when y and z are linearly dependent. Therefore, the inequality (1.9) holds for any three vectors $x, y, z \in X$ and is strict unless the vectors $u = (y, y|z)x - (x, y|z)y$ and z are linearly dependent. In fact, we have the equality in (1.9) if and only if the three vectors x, y and z are linearly dependent. In any given 2-inner product space $(X, (\cdot, \cdot \mid \cdot))$, we can define a function $\| \cdot \mid \cdot \|$ on $X \times X$ by

$$\|x|z\| = \sqrt{(x, x|z)} \tag{1.10}$$

for all $x, z \in X$. It is easy to see that this function satisfies the following conditions:

($2\mathrm{N}_1$) $\|x|z\| \geq 0$ and $\|x|z\| = 0$ if and only if x and z are linearly dependent,
($2\mathrm{N}_2$) $\|z|x\| = \|x|z\|$,
($2\mathrm{N}_3$) $\|\alpha x|z\| = |\alpha|\|x|z\|$ for any scalar $\alpha \in K$,
($2\mathrm{N}_4$) $\|x + x'|z\| \leq \|x|z\| + \|x'|z\|$.

Any function $\|\cdot|\cdot\|$ defined on $X \times X$ and satisfying the conditions $(2\mathrm{N}_1)$–$(2\mathrm{N}_4)$ is called a *2-norm* on X and $(X, \|\cdot|\cdot\|)$ is called a *linear 2-normed space*. Whenever a 2-inner product space $(X, (\cdot,\cdot|\cdot))$ is given, we consider it as a linear 2-normed space $(X, \|\cdot|\cdot\|)$ with the 2-norm defined by (1.10).

A natural extension of the Cauchy-Schwarz-Bunjakowsky inequality

$$|(x,y)|^2 \le (x,x)(y,y) \tag{1.11}$$

in inner product space $(X, (\cdot,\cdot))$ is the Gram's inequality

$$\Gamma(x_1, x_2, \cdots, x_k) \ge 0, \tag{1.12}$$

which holds for any choice of vectors $x_1, x_2, \cdots, x_k \in X$ and is strict unless $x_1, x_2, \cdots, x_k$ are linearly dependent. Also, there are a number of inequalities of various types related to Gram's determinant

$$\Gamma(x_1, x_2, \cdots, x_k) = \begin{vmatrix} (x_1,x_1) & (x_1,x_2) & \dots & (x_1,x_k) \\ (x_2,x_1) & (x_2,x_2) & \dots & (x_2,x_k) \\ \vdots & \vdots & \ddots & \vdots \\ (x_k,x_1) & (x_k,x_2) & \dots & (x_k,x_k) \end{vmatrix}$$

(see, for instance, [2, pp. 381–385] or [3, Ch. XX]).

The inequality (1.9) is an analogue of the Cauchy-Schwarz-Bunjakowsky inequality (1.11) for 2-inner product spaces.

The aim of this paper is to give an analogue of Gram's inequality (1.12) for 2-inner product spaces as well as the analogues for 2-inner product spaces of some classical inequalities involving Gram's determinant.

In Section 2, we give a definition of Gram's determinant in 2-inner product space and then prove a version of Gram's inequality (1.12) for 2-inner product spaces. Also we give a versions of Parseval's identity and of Bessel's inequality in 2-inner product spaces.

In Section 3, we prove some further inequalities involving 2-inner product analogue of Gram's determinant.

Section 4 is devoted to a version for 2-inner product spaces of the well known inequality which can be regarded as a generalization via Gram's determinant of Cauchy-Schwarz inequality for sequences (see, for instance, [3, p. 599]).

In Section 5, we give a 2-inner product analogue of one well known result which can be regarded as a generalization via Gram's determinant of Bessel's inequality (see [3, pp. 396-397]). Also we give two interesting consequences of this result (Corollary 6 and Theorem 8) which are in turn 2-inner product analogues of the known classical results (see [3, pp. 603-604]).

2. Gram's inequality

Let $(X, (\cdot, \cdot|\cdot))$ be 2-inner product space over the field of real numbers $K = \mathbf{R}$ or the field of complex numbers $K = \mathbf{C}$. For any given vectors $x_1, x_2, \cdots, x_k \in X$ and $z \in X$, define a matrix $G(x_1, x_2, \cdots, x_k|z)$ by

$$G(x_1, x_2, \cdots, x_k|z) = \begin{pmatrix} (x_1, x_1|z) & (x_1, x_2|z) & \cdots & (x_1, x_k|z) \\ (x_2, x_1|z) & (x_2, x_2|z) & \cdots & (x_2, x_k|z) \\ \vdots & \vdots & \ddots & \vdots \\ (x_k, x_1|z) & (x_k, x_2|z) & \cdots & (x_k, x_k|z) \end{pmatrix}$$

and Gram's determinant $\Gamma(x_1, x_2, \cdots, x_k|z)$ of the vectors $x_1, x_2, \cdots, x_k$ with respect to the vector z by

$$\begin{aligned} \Gamma(x_1, x_2, \cdots, x_k|z) &= \det G(x_1, x_2, \cdots, x_k|z) \\ &= \begin{vmatrix} (x_1, x_1|z) & (x_1, x_2|z) & \cdots & (x_1, x_k|z) \\ (x_2, x_1|z) & (x_2, x_2|z) & \cdots & (x_2, x_k|z) \\ \vdots & \vdots & \ddots & \vdots \\ (x_k, x_1|z) & (x_k, x_2|z) & \cdots & (x_k, x_k|z) \end{vmatrix}. \end{aligned} \tag{2.1}$$

THEOREM 1. *Let $x_1, x_2, \cdots, x_k \in X$ and $z \in X$ be given vectors in 2-inner product space X. Then we have*

$$\Gamma(x_1, x_2, \cdots, x_k|z) \geq 0. \tag{2.2}$$

Moreover, the equality holds in (2.2) if and only if the vectors $x_1, x_2, \cdots, x_k, z$ are linearly dependent.

Proof. First we consider the case of the equality in (2.2). Suppose that the vectors $x_1, x_2, \cdots, x_k, z$ are linearly dependent. Then we have

$$\alpha_1 x_1 + \cdots + \alpha_k x_k + \beta z = 0 \tag{2.3}$$

for some scalars $\alpha_1, \cdots, \alpha_k, \beta \in K$ and at least one of them is different from zero. From (2.3), it follows that

$$(\alpha_1 x_1 + \cdots + \alpha_k x_k + \beta z, x_j|z) = 0$$

for $j = 1, \cdots, k$, that is, since $(z, x_j|z) = 0$,

$$\alpha_1(x_1, x_j|z) + \cdots + \alpha_k(x_k x_j|z) = 0 \tag{2.4}$$

for $j = 1, \cdots, k$. If $\alpha_1 = \cdots = \alpha_k = 0$ and $\beta \neq 0$, then, from (2.3), we have $z = 0$ and obviously $\Gamma(x_1, x_2, \cdots, x_k|0) = 0$. If $\alpha_j \neq 0$ for at least one $j \in \{1, \cdots, k\}$, then the system (2.4) has a nontrivial solution $(\alpha_1, \cdots, \alpha_k)$, which means that the matrix of the system $G(x_1, x_2, \cdots, x_k|z)$ must be singular and hence $\Gamma(x_1, x_2, \cdots, x_k|z) = 0$. So, if the vectors $x_1, x_2, \cdots, x_k, z$ are linearly dependent, then $\Gamma(x_1, x_2, \cdots, x_k|z) = 0$.

Conversely, suppose that $\Gamma(x_1, x_2, \cdots, x_k|z) = 0$. Then the system (2.4) has a nontrivial solution $(\alpha_1, \cdots, \alpha_k)$. But (2.4) can be rewritten as

$$(\alpha_1 x_1 + \cdots + \alpha_k x_k, x_j|z) = 0 \tag{2.5}$$

for $j = 1, \cdots, k$. Multiplying the j^{th} equation in (2.5) by $\bar{\alpha}_j$ and then summing over $j = 1, \cdots, k$, we have

$$(\alpha_1 x_1 + \cdots + \alpha_k x_k, \alpha_1 x_1 + \cdots + \alpha_k x_k|z) = 0.$$

This means that the vectors $\alpha_1 x_1 + \cdots + \alpha_k x_k$ and z are linearly dependent so that there are the scalars $\alpha, \beta \in K$ such that $\alpha \neq 0$ or $\beta \neq 0$ and

$$\alpha(\alpha_1 x_1 + \cdots + \alpha_k x_k) + \beta z = 0.$$

Since $\alpha_j \neq 0$ for at least one $j \in \{1, \cdots, k\}$, we conclude that the vectors $x_1, x_2, \cdots, x_k, z$ are linearly dependent.

Suppose that the vectors $x_1, x_2, \cdots, x_k, z$ are linearly independent. Then, for $r \in \{1, \cdots, k\}$, the vectors $x_1, x_2, \cdots, x_r, z$ are linearly independent and

$$\Gamma(x_1, x_2, \cdots, x_r|z) \neq 0$$

for $r = 1, \cdots, k$. Define the vectors $y_1, \cdots, y_k$ as

$$y_1 = x_1$$

and

$$y_r = \left| \begin{array}{c|c} G(x_1, \cdots, x_{r-1}|z) & \begin{matrix} x_1 \\ \vdots \\ x_{r-1} \end{matrix} \\ \hline (x_r, x_1|z) \cdots (x_r, x_r - 1|z) & x_r \end{array} \right| \tag{2.6}$$

for $r = 2, \cdots, k$. Expanding the determinant in (2.6) over the last column, we have

$$y_r = \alpha_1 x_1 + \cdots + \alpha_{r-1} x_{r-1} + \Gamma(x_1, \cdots, x_{r-1}|z) x_r \tag{2.7}$$

and

$$\begin{aligned}(y_r, x_s|z) &= \alpha_1(x_1, x_s|z) + \cdots + \alpha_{r-1}(x_{r-1}, x_s|z) \\ &\quad + \Gamma(x_1, \cdots, x_{r-1}|z)(x_r, x_s|z) \\ &= \left|\begin{array}{c|c} G(x_1, \cdots, x_{r-1}|z) & \begin{matrix} x_1 \\ \vdots \\ x_{r-1} \end{matrix} \\ \hline (x_r, x_1|z) \cdots (x_r, x_r - 1|z) & x_r \end{array}\right|\end{aligned} \tag{2.8}$$

for $r = 2, \cdots, k$ and $1 \leq s \leq r$. If $1 \leq s < r$, then the determinant in (2.8) has two equal columns and hence

$$(y_r, x_s|z) = 0, \quad 1 \leq s < r.$$

For $s = r$, it follows from (2.8) that

$$(y_r, x_r|z) = \Gamma(x_1, \cdots, x_r|z).$$

Now, using the expansion (2.7) and the above equalities, we have

$$\begin{aligned}(y_r, y_r|z) &= \Gamma(x_1, \cdots, x_{r-1}|z)(x_r, y_r|z) \\ &= \Gamma(x_1, \cdots, x_{r-1}|z)\overline{\Gamma(x_1, \cdots, x_r|z)} \\ &\neq 0\end{aligned}$$

and hence

$$(y_r, y_r|z) > 0$$

for $r = 2, \cdots, k$. In fact, we have

$$\Gamma(x_1, \cdots, x_r|z) = \frac{\overline{(y_r, y_r|z)}}{\overline{\overline{\Gamma(x_1, \cdots, x_{r-1}|z)}}} = \frac{(y_r, y_r|z)}{\overline{\overline{\Gamma(x_1, \cdots, x_{r-1}|z)}}} \tag{2.9}$$

for $r = 2, \cdots, k$. Now

$$\Gamma(x_1|z) = (x_1, x_1|z) > 0,$$

by the assumed independence of x_1 and z. Using this and (2.9) with $r = 2$, we have further

$$\Gamma(x_1, x_2|z) = \frac{(y_2, y_2|z)}{\overline{\Gamma(x_1|z)}} = \frac{(y_2, y_2|z)}{\Gamma(x_1|z)} > 0.$$

Continuing in this way, we conclude that

$$\Gamma(x_1, \cdots, x_r|z) > 0$$

for all $r \in \{1, \cdots, k\}$. This completes the proof. □

REMARK 1. The inequality (2.2) is an analogue of the Gram's inequality for 2-inner product spaces. In the case when $k = 2$, (2.2) reduces to

$$(x_1, x_1|z)(x_2, x_2|z) - |(x_1, x_2|z)|^2 \geq 0$$

with the equality if and only if the vectors x_1, x_2, z are linearly dependent. This is just the inequality (1.9) so that Gram's inequality can be regarded as a generalization of Cauchy–Schwarz-Bunjakowsky inequality.

Note that, in the case when the vectors $x_1, x_2, \cdots, x_k, z$ are linearly independent, we can define the vectors $y_1, y_2, \cdots, y_k$ as in the proof above and, from (2.7), it follows that

$$L(y_1, y_2, \cdots, y_r) = L(x_1, x_2, \cdots, x_r)$$

for $r = 1, \cdots, k$. Moreover, from the proof above, we see that

$$(y_r, y_s|z) = 0, \quad 1 \leq s < r \leq k.$$

Also we have

$$\|y_1|z\|^2 = (y_1, y_1|z) = \Gamma(x_1|z)$$

and

$$\|y_r|z\|^2 = (y_r, y_r|z) = \Gamma(x_1, \cdots, x_{r-1}|z)\Gamma(x_1, \cdots, x_r|z)$$

for $r = 2, \cdots, k$.

Now, the following result is evident:

COROLLARY 1. *Let $x_1, x_2, \cdots, x_k, z \in X$ be given linearly independent vectors in 2-inner product space X. Let the vectors $y_1, y_2, \cdots, y_k$ be defined as in the proof of Theorem 1. Define the vectors $e_1, e_2, \cdots, e_k$ as*

$$e_1 = \frac{y_1}{\|y_1|z\|} = \frac{x_1}{\Gamma(x_1|z)^{1/2}}$$

and

$$e_r = \frac{y_r}{\|y_r|z\|} = \frac{y_r}{[\Gamma(x_1, \cdots, x_{r-1}|z)\Gamma(x_1, \cdots, x_r|z)]^{1/2}}$$

for $r = 2, \cdots, k$. Then we have

(i) *For $r, s \in \{1, 2, \cdots, k\}$,*

$$(e_r, e_s|z) = \delta_{rs} = \begin{cases} 0 & \text{for } r \neq s, \\ 1 & \text{for } r = s. \end{cases}$$

(ii) *For $r \in \{1, 2, \cdots, k\}$,*

$$L(e_1, e_2, \cdots, e_r) = L(x_1, x_2, \cdots, x_r).$$

REMARK 2. Note that, in the case when there is an infinite sequence $x_1, x_2, \cdots$ of linearly independent vectors in the space X, we can take any vector $z \in X$ such that $z \notin L(x_1, x_2, \cdots)$ and then construct an infinite sequence of vectors $e_1, e_2, \cdots$ such that the conclusions of Corollary 1 are valid for all $r, s \in \mathbf{N}$.

Suppose now that Y is a finite-dimensional linear subspace of 2-inner product space $(X, (\cdot, \cdot|\cdot))$ and that $z \in X$ is such that $z \notin Y$. If $\dim Y = n$, then, by the Corollary 1, we can construct the base $\{e_1, \cdots, e_n\}$ for Y such that

$$(e_i, e_j|z) = \delta_{ij} \tag{2.10}$$

for all $i, j \in \{1, \cdots, n\}$.

Any vector $x \in Y$ has unique representation of the form $x = \sum_{i=1}^{n} \alpha_i e_i$. Using (2.10), we have $(x, e_j|z) = \alpha_j$ for all $j = 1, \cdots, n$ so that

$$x = \sum_{i=1}^{n} (x, e_i|z) e_i$$

for all $x \in Y$. Therefore, if $x, y \in Y$ are two given vectors from the subspace Y, then, using (2.10), we have

$$
\begin{aligned}
(x, y|z) &= \Big(\sum_{i=1}^{n}(x, e_i|z)e_i, \sum_{j=1}^{n}(y, e_j|z)e_j|z\Big) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}(x, e_i|z)\overline{(y, e_j|z)}\delta_{ij} \qquad (2.11)\\
&= \sum_{i=1}^{n}(x, e_i|z)(e_i, y|z),
\end{aligned}
$$

which is an analogue of Parseval's identity for 2-inner product spaces. Especially, for any $x \in Y$, (2.11) with $y = x$ becomes

$$\|x|z\|^2 = (x, x|z) = \sum_{i=1}^{n} |(x, e_i|z)|^2.$$

Further, for any $x \in X$, define the vectors $u \in Y$ and $v \in X$ as

$$u = \sum_{i=1}^{n}(x, e_i|z)e_i, \quad v = x - u.$$

For $j = 1, \cdots, n$, we have

$$(v, e_j|z) = \left(x - \sum_{i=1}^{n}(x, e_i|z)e_i, e_j|z\right) = (x, e_j|z) - \sum_{i=1}^{n}(x, e_i|z)\delta_{ij} = 0,$$

which implies that $(v, y|z) = 0$ for every $y \in Z$.

THEOREM 2. *Let Y be a finite-dimensional linear subspace of 2-inner product space X and let $z \in X$ is such that $z \notin Y$. Then every $x \in X$ can be uniquely represented as*

$$x = u + v,$$

where $u \in Y$ and $v \in X$ with $(v, y|z) = 0$ for all $y \in Y$.

Proof. The existence of the proposed representation for $x \in X$ is already proved. It remains to prove the uniqueness. So, suppose that

$$x = u + v = u' + v',$$

where $u, u' \in Y$ and $(v, y|z) = (v', y|z) = 0$ for all $y \in Y$. Then we have

$$v - v' = u' - u \in Y$$

and

$$(u - u', u - u'|z) = (v - v', u - u'|z) = (v, u - u'|z) - (v', u - u'|z) = 0.$$

This implies that $u - u'$ and z are linearly dependent, which is possible only when $u - u' = 0$ since $z \notin Y$. Thus, we must have $v - v' = u' - u = 0$, that is $v = v'$ and $u = u'$. This completes the proof. □

COROLLARY 2. *Let Y be a finite-dimensional linear subspace of 2-inner product space X and let $z \in X$ is such that $z \notin Y$. If $\{e_1, \cdots, e_n\}$ is the base for Y such that* (2.10) *holds, then, for any $x \in X$,*

$$\sum_{i=1}^{n} |(x, e_i|z)|^2 \leq \|x|z\|^2. \tag{2.12}$$

The equality in (2.12) *holds if and only if $x = u + \gamma z$ for some $u \in Y$ and some scalar $\gamma \in K$.*

Proof. By Theorem 2, every $x \in X$ can be represented as $x = u + v$, where $u \in Y$ and $(v, y|z) = 0$ for all $y \in Y$. More precisely, we have

$$x = u + v, \quad u = u = \sum_{i=1}^{n} (x, e_i|z) e_i, \quad (v, u|z) = 0,$$

which yields that

$$\begin{aligned}
\|x|z\|^2 &= (u + v, u + v|z) \\
&= (u, u|z) + (u, v|z) + (v, u|z) + (v, v|z) \\
&= \|u|z\|^2 + \|v|z\|^2 \\
&\geq \|u|z\|^2 \\
&= \sum_{i=1}^{n} |(x, e_i|z)|^2.
\end{aligned}$$

Thus (2.12) is valid. Further, it is evident that we have the equality if and only if $\|v|z\|^2 = 0$, which is equivalent to the requirement that v and z are linearly dependent, that is, $\alpha v + \beta z = 0$ for some scalars α and β with $\alpha \neq 0$ or $\beta \neq 0$. Now, $\alpha = 0$ would imply that $\beta \neq 0$ and $z = 0$, which can not be true. So we have $\alpha \neq 0$ and $v = x - u = -\frac{\beta}{\alpha} z$, that is, $x = u - \frac{\beta}{\alpha} z$. This completes the proof. □

REMARK 3. The inequality (2.12) is an analogue of Bessel's inequality for 2-inner product spaces. It is easy to see that it is also valid for an infinite sequence of vectors. Namely, if $e_1, e_2, \cdots$ is an infinite sequence of vectors from X and $z \in X$ such that

$$z \notin L(e_1, e_2, \cdots)$$

and

$$(e_i, e_j|z) = \delta_{ij}$$

for all $i, j = 1, 2, \cdots$, then we can apply Corollary 2 to the subspace $Y = L(e_1, \cdots, e_n)$ to obtain the inequality (2.12) for any fixed $n \in \mathbf{N}$. When $n \to \infty$, we have that

$$\sum_{i=1}^{\infty} |(x, e_i|z)|^2 \leq \|x|z\|^2$$

for any $x \in X$.

3. Some inequalities involving Gram's determinant

Throughout this section, we assume the notation from the previous two sections. We prove some inequalities involving Gram's determinant in 2-inner product spaces defined by (2.1). First we need one technical result.

LEMMA 1. *Let Y be any linear subspace of 2-inner product space X and let $z \in X$ such that $z \notin Y$. Suppose that $x \in X$ can be represented as*

$$x = u + v,$$

where $u \in Y$ and $(v, y|z) = 0$ for all $y \in Y$. Then, for arbitrarily chosen vectors $x_1, \cdots, x_m \in Y$, we have

$$\Gamma(x, x_1, \cdots, x_m|z) = \Gamma(u, x_1, \cdots, x_m|z) + \|v|z\|^2 \Gamma(x_1, \cdots, x_m|z). \tag{3.1}$$

Especially, if $u \in L(x_1, \cdots, x_m)$*, then*

$$\Gamma(x, x_1, \cdots, x_m|z) = \|v|z\|^2 \Gamma(x_1, \cdots, x_m|z). \tag{3.2}$$

Proof. Under given assumptions, we have, for all $j = 1, \cdots, m$,
$(x_j, x|z) = (x_j, u + v|z) = (x_j, u|z)$, $(x, x_j|z) = (u + v, x_j|z) = (u, x_j|z)$.
Also it follows that

$$(x, x|z) = (u + v, u + v|z) = (u, u|z) + (v, v|z) = (u, u|z) + \|v|z\|^2.$$

Using this and the elementary properties of determinant, we have

$$\begin{aligned}
\Gamma(x, x_1, \cdots, x_m|z) &= \begin{vmatrix} (u, u|z) + \|v|z\|^2 & (u, x_1|z) & \cdots & (u, x_m|z) \\ (x_1, u|z) & (x_1, x_1|z) & \cdots & (x_1, x_m|z) \\ \vdots & \vdots & \ddots & \vdots \\ (x_m, u|z) & (x_m, x_1|z) & \cdots & (x_m, x_m|z) \end{vmatrix} \\
&= \begin{vmatrix} (u, u|z) & (u, x_1|z) & \cdots & (u, x_m|z) \\ (x_1, u|z) & (x_1, x_1|z) & \cdots & (x_1, x_m|z) \\ \vdots & \vdots & \ddots & \vdots \\ (x_m, u|z) & (x_m, x_1|z) & \cdots & (x_m, x_m|z) \end{vmatrix} \\
&\quad + \begin{vmatrix} \|v|z\|^2 & (u, x_1|z) & \cdots & (u, x_m|z) \\ 0 & (x_1, x_1|z) & \cdots & (x_1, x_m|z) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (x_m, x_1|z) & \cdots & (x_m, x_m|z) \end{vmatrix} \\
&= \Gamma(u, x_1, \cdots, x_m|z) + \|v|z\|^2 \Gamma(x_1, \cdots, x_m|z),
\end{aligned}$$

which is just the identity (3.1). The identity (3.2) follows directly from (3.1) since $u \in L(x_1, \cdots, x_m)$ implies that $u, x_1, \cdots, x_m, z$ are linearly dependent and hence $\Gamma(u, x_1, \cdots, x_m|z) = 0$. This completes the proof. □

Now, we can prove some inequalities involving Gram's determinant.

THEOREM 3. *Let* $x_1, \cdots, x_m \in X$ *be given vectors in 2-inner product space* X *all different from null vector* 0 *and let* $z \in X$ *such that* $z \notin L(x_1, \cdots, x_m)$*. Then we have*

$$\Gamma(x_1, \cdots, x_m|z) \le \|x_1|z\|^2 \cdots \|x_m|z\|^2. \tag{3.3}$$

For $m \ge 2$*, the equality in* (3.3) *holds if and only if*

$$(x_i, x_j|z) = 0, \quad 1 \le i < j \le m.$$

Proof. For $m = 1$, we have $\Gamma(x_1|z) = \|x_1|z\|^2$ and (3.3) is trivially satisfied. So, take $m \geq 2$ and first suppose that $x_1, \cdots, x_m$ are linearly dependent. Then the left-hand side in (3.3) is equal to zero, while the right-hand side is strictly positive. Namely, the equality $\|x_j|z\|^2 = 0$ for some j is possible only with $x_j = 0$, which is excluded. Thus (3.3) holds with strict inequality in this case.

Next, suppose that $x_1, \cdots, x_m$ are linearly independent and define $Y = L(x_2, \cdots, x_m)$. By Theorem 2, the vector x_1 can be represented as

$$x_1 = u + v, \quad u \in Y, \quad (v, y|z) = 0$$

for all $y \in Y$. Applying Lemma 1, we have

$$\Gamma(x_1, x_2, \cdots, x_m|z) = \|v|z\|^2 \Gamma(x_2, \cdots, x_m|z). \tag{3.4}$$

On the other side, $(v, u|z) = 0$ implies that

$$\|x_1|z\|^2 = \|u|z\|^2 + \|v|z\|^2 \geq \|v|z\|^2. \tag{3.5}$$

Since $\Gamma(x_2, \cdots, x_m|z) > 0$, from (3.4) and (3.5), it follows that

$$\Gamma(x_1, x_2, \cdots, x_m|z) \leq \|x_1|z\|^2 \Gamma(x_2, \cdots, x_m|z). \tag{3.6}$$

Moreover, the equality in (3.6) holds if and only if $\|u|z\|^2 = 0$, which is possible only with $u = 0$. But $u = 0$ is equivalent with $x_1 = v$, that is, $(x_1, y|z) = 0$ for all $y \in Y$, that is,

$$(x_1, x_i|z) = 0$$

for all $i = 2, \cdots, m$. Applying analogous observations to $\Gamma(x_2, \cdots, x_m|z), \cdots, \Gamma(x_{m-1}, x_m|z)$, we easily get the proposed conclusions. This completes the proof. □

THEOREM 4. *Let $x_1, \cdots, x_m \in X$ be linearly independent vectors in 2-inner product space X and*

$$Y := L(x_1, \cdots, x_m).$$

Let $z \in X$ be such that $z \notin Y$. Then, for any $x \in X$, we have

$$\inf_{y \in Y} \|x - y|z\| = \min_{y \in Y} \|x - y|z\| = [\frac{\Gamma(x, x_1, \cdots, x_m|z)}{\Gamma(x_1, \cdots, x_m|z)}]^{1/2}.$$

Proof. Let $x \in X$ be given. By Theorem 2, x can be uniquely represented as

$$x = u + v, \quad u \in Y, \quad (v, y|z) = 0$$

for all $y \in Y$. Now, if $y \in Y$ is arbitrarily chosen, then we have

$$x - y = u - y + v, \quad u - y \in Y, \quad (v, u - y|z) = 0.$$

Therefore, we have

$$\|x - y|z\|^2 = \|u - y|z\|^2 + \|v|z\|^2 \geq \|v|z\|^2$$

and the equality occurs when $y = u$. We conclude that

$$\inf_{y \in Y} \|x - y|z\| = \min_{y \in Y} \|x - y|z\| = \|v|z\|. \tag{3.7}$$

On the other hand, by Lemma 1, we have

$$\Gamma(x, x_1, \cdots, x_m|z) = \|v|z\|^2 \Gamma(x_1, \cdots, x_m|z).$$

Also, $\Gamma(x_1, \cdots, x_m|z) > 0$ since $x_1, \cdots, x_m$ are linearly independent and $z \notin Y$, so that

$$\|v|z\|^2 = \frac{\Gamma(x, x_1, \cdots, x_m|z)}{\Gamma(x_1, \cdots, x_m|z)},$$

which, in combination with (3.7), proves our assertion. This completes the proof. □

Corollary 3. *Let $x_1, \cdots, x_m \in X$ be linearly independent vectors in 2-inner product space X, $m \geq 2$, and let $z \in X$ be such that $z \notin L(x_1, \cdots, x_m)$. Then we have*

$$\begin{aligned} \frac{\Gamma(x_1, \cdots, x_m|z)}{\Gamma(x_1, \cdots, x_k|z)} &\leq \frac{\Gamma(x_2, \cdots, x_m|z)}{\Gamma(x_2, \cdots, x_k|z)} \leq \cdots \\ &\leq \frac{\Gamma(x_k, \cdots, x_m|z)}{\Gamma(x_k|z)} \leq \Gamma(x_{k+1}, \cdots, x_m|z) \end{aligned} \tag{3.8}$$

for $1 \leq k < m$. Moreover, the equality

$$\frac{\Gamma(x_{r-1}, \cdots, x_m|z)}{\Gamma(x_{r-1}, \cdots, x_k|z)} = \frac{\Gamma(x_r, \cdots, x_m|z)}{\Gamma(x_r, \cdots, x_k|z)}$$

occurs for some $r \in \{2, \cdots, k\}$ if and only if

$$x_{r-1} = u_r + v_r, \quad u_r \in L(x_r, \cdots, x_k), \quad (v_r, x_i|z) = 0$$

for all $i = r, \cdots, m$. The equality

$$\frac{\Gamma(x_k, \cdots, x_m|z)}{\Gamma(x_k|z)} = \Gamma(x_{k+1}, \cdots, x_m|z)$$

occurs if and only if $(x_k, x_i|z) = 0$ for all $i = k+1, \cdots, m$.

Proof. First take $k = 1$. Then (3.8) reduces to

$$\frac{\Gamma(x_1, \cdots, x_m|z)}{\Gamma(x_1|z)} \leq \Gamma(x_2, \cdots, x_m|z), \tag{3.9}$$

which is in fact the inequality (3.6). Also, the equality in this inequality occurs if and only if $(x_1, x_i|z) = 0$ for all $i = 2, \cdots, m$, as we proved for the inequality (3.6). Further, suppose that $1 < k < m$. Replacing $x_1, \cdots, x_m$ in (3.9) by $x_k, \cdots, x_m$, we obtain the last inequality in (3.8) and obviously the assertion on the equality case is true. Next, for $r \in \{2, \cdots, k\}$, define the subspaces Y_r and Y_r' as

$$Y_r = L(x_r, \cdots, x_m), \quad Y_r' = L(x_r, \cdots, x_k).$$

By Theorem 2, the vector x_{r-1} can be uniquely represented in the following two forms

$$x_{r-1} = u_r + v_r, \quad u_r \in Y_r, \quad (v_r, x_i|z) = 0$$

for all $i = r, \cdots, m$ and

$$x_{r-1} = u_r' + v_r', \quad u_r' \in Y_r', \quad (v_r', x_i|z) = 0$$

for all $i = r, \cdots, k$. Applying Theorem 4, we have

$$\inf_{y \in Y_r} \|x_{r-1} - y|z\|^2 = \|v_r|z\|^2 = \frac{\Gamma(x_{r-1}, x_r, \cdots, x_m|z)}{\Gamma(x_r, \cdots, x_m|z)}$$

and

$$\inf_{y \in Y_r'} \|x_{r-1} - y|z\|^2 = \|v_r'|z\|^2 = \frac{\Gamma(x_{r-1}, x_r, \cdots, x_k|z)}{\Gamma(x_r, \cdots, x_k|z)}.$$

But $Y_r' \subseteq Y_r$, which implies that $\inf_{y \in Y_r} \|x_{r-1} - y|z\|^2 \leq \inf_{y \in Y_r'} \|x_{r-1} - y|z\|^2$, that is,

$$\frac{\Gamma(x_{r-1}, x_r, \cdots, x_m|z)}{\Gamma(x_r, \cdots, x_m|z)} \leq \frac{\Gamma(x_{r-1}, x_r, \cdots, x_k|z)}{\Gamma(x_r, \cdots, x_k|z)}$$

or, equivalently,

$$\frac{\Gamma(x_{r-1}, x_r, \cdots, x_m|z)}{\Gamma(x_{r-1}, x_r, \cdots, x_k|z)} \leq \frac{\Gamma(x_r, \cdots, x_m|z)}{\Gamma(x_r, \cdots, x_k|z)}. \tag{3.10}$$

Moreover, the equality in (3.10) occurs if and only if

$$\|v_r|z\|^2 = \|v_r'|z\|^2. \tag{3.11}$$

Now, from $x_{r-1} = u_r + v_r = u_r' + v_r'$, it follows that

$$v_r' = u_r - u_r' + v_r, \quad u_r - u_r' \in Y_r, \quad (v_r, u_r - u_r'|z) = 0,$$

which implies

$$\|v_r'|z\|^2 = \|u_r - u_r'|z\|^2 + \|v_r|z\|^2.$$

From the above inequality and (3.11), we get $\|u_r - u_r'|z\|^2 = 0$ which is possible only with $u_r - u_r' = 0$. This means that $u_r = u_r'$ and $v_r = v_r'$. In fact (3.11) is equivalent to the requirement

$$x_{r-1} = u_r + v_r, \quad u_r = u_r' \in L(x_r, \cdots, x_k), \quad (v_r, x_i|z) = 0$$

for all $i = r, \cdots, m$. This completes the proof. □

COROLLARY 4. *Let $x_1, \cdots, x_m \in X$ be arbitrarily chosen vectors in 2-inner product space X, $m \geq 2$, and let $z \in X$ be such that $z \notin L(x_1, \cdots, x_m)$. Then we have*

$$\begin{aligned} &\Gamma(x_1, \cdots, x_k, x_{k+1}, \cdots, x_m|z) \\ &\leq \Gamma(x_1, \cdots, x_k|z)\Gamma(x_{k+1}, \cdots, x_m|z) \end{aligned} \tag{3.12}$$

for $1 \leq k < m$. Moreover, the equality in (3.12) can occur only in one of the following three cases:

(i) *The vectors $x_1, \cdots, x_k$ are linearly dependent.*
(ii) *The vectors $x_{k+1}, \cdots, x_m$ are linearly dependent.*
(iii) *The vectors $x_1, \cdots, x_m$ are linearly independent and $(x_i, x_j|z) = 0$ for all $i \in \{1, \cdots, k\}$ and $j \in \{k+1, \cdots, m\}$.*

Proof. If $x_1, \cdots, x_m$ are linearly dependent, then (3.12) trivially holds since the left hand side is zero and the right hand side is nonnegative. Also, the equality in this case occurs in (3.12) if and only if the right hand side is zero, which is equivalent with the requirement that either the vectors $x_1, \cdots, x_k$ are linearly dependent, or the vectors $x_{k+1}, \cdots, x_m$ are linearly dependent. Further, if the vectors $x_1, \cdots, x_m$ are linearly independent, then we can apply the first and the last inequality from (3.8) to obtain the inequality

$$\frac{\Gamma(x_1, \cdots, x_m|z)}{\Gamma(x_1, \cdots, x_k|z)} \leq \Gamma(x_{k+1}, \cdots, x_m|z),$$

which is equivalent to (3.12). Also, the equality occurs in (3.12) if and only if we have the equalities throughout in (3.8), that is,

$$\frac{\Gamma(x_k, \cdots, x_m|z)}{\Gamma(x_k|z)} = \Gamma(x_{k+1}, \cdots, x_m|z) \tag{3.13}$$

and

$$\frac{\Gamma(x_{r-1}, \cdots, x_m|z)}{\Gamma(x_{r-1}, \cdots, x_k|z)} = \frac{\Gamma(x_r, \cdots, x_m|z)}{\Gamma(x_r, \cdots, x_k|z)} \tag{3.14}$$

for all $r \in \{2, \cdots, k\}$. Now, (3.13) is equivalent with

$$(x_k, x_i|z) = 0 \tag{3.15}$$

for all $i = k+1, \cdots, m$. Next, (3.14) with $r = k$ is equivalent to

$$x_{k-1} = u_k + v_k, \quad u_k \in L(x_k), \quad (v_k, x_i|z) = 0 \tag{3.16}$$

for all $i = k, \cdots, m$. It is easy to see that (3.15) and (3.16) together are equivalent with

$$(x_i, x_j|z) = 0$$

for $i \in \{k-1, k\}$ and $j \in \{k+1, \cdots, m\}$. Continuing the argument in this way for $r = k-1, \cdots, 2$, then we have the equalities throughout in (3.8) if and only if

$$(x_i, x_j|z) = 0$$

for all $i \in \{1, \cdots, k\}$ and $j \in \{k+1, \cdots, m\}$. This completes the proof.□

4. A generalization of Cauchy-Schwarz inequality

Let $(X, (\cdot, \cdot|\cdot))$ be 2-inner product space over the field of real numbers $K = \mathbf{R}$ or the field of complex numbers $K = \mathbf{C}$. For given $m \in \mathbf{N}$, consider two sequences of vectors $x_1, \cdots, x_m \in X$ and $y_1, \cdots, y_m \in X$. Then, for any given $z \in X$, we can define the matrix A of order m by

$$(4.1) \qquad A = \begin{bmatrix} (x_1, y_1|z) & (x_1, y_2|z) & \cdots & (x_1, y_m|z) \\ (x_2, y_1|z) & (x_2, y_2|z) & \cdots & (x_2, y_m|z) \\ \vdots & \vdots & \ddots & \vdots \\ (x_m, y_1|z) & (x_m, y_2|z) & \cdots & (x_m, y_m|z) \end{bmatrix}.$$

If we define

$$Y = L(x_1, \cdots, x_m, y_1, \cdots, y_m),$$

then Y is a finite-dimensional linear subspace of X of dimension $\dim Y = n$. If z is such that $z \notin Y$, then we can choose the base $\{e_1, \cdots, e_n\}$ for Y such that

$$(e_i, e_j|z) = \delta_{ij}, \quad i, j \in \{1, \cdots, n\}.$$

Using the Parseval's identity (2.11), it is easy to see that A can be represented as

(4.2)

$$A = \begin{bmatrix} (x_1, e_1|z) & \cdots & (x_1, e_n|z) \\ \vdots & \ddots & \vdots \\ (x_m, e_1|z) & \cdots & (x_m, e_n|z) \end{bmatrix} \begin{bmatrix} (e_1, y_1|z) & \cdots & (e_1, y_m|z) \\ \vdots & \ddots & \vdots \\ (e_n, y_1|z) & \cdots & (e_n, y_m|z) \end{bmatrix}.$$

If $m > n$, then obviously the vectors $x_1, \cdots, x_m$ must be linearly dependent (the same is true for the vectors $(y_1, \cdots, y_m)$ which implies that the rows (columns) of the matrix A are linearly dependent and hence

$$\det A = 0.$$

LEMMA 2. *If $m \leq n$, then we have the identity*

$$(4.3) \qquad \det A = \sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq n} \xi(j_1, j_2, \cdots, j_m) \eta(j_1, j_2, \cdots, j_m),$$

where

$$(4.4) \qquad \xi(j_1, j_2, \cdots, j_m) = \begin{vmatrix} (x_1, e_{j_1}|z) & \cdots & (x_1, e_{j_m}|z) \\ \vdots & \ddots & \vdots \\ (x_m, e_{j_1}|z) & \cdots & (x_m, e_{j_m}|z) \end{vmatrix}$$

and

$$(4.5) \qquad \eta(j_1, j_2, \cdots, j_m) = \begin{vmatrix} (e_{j_1}, y_1|z) & \cdots & (e_{j_1}, y_m|z) \\ \vdots & \ddots & \vdots \\ (e_{j_m}, y_1|z) & \cdots & (e_{j_m}, y_m|z) \end{vmatrix}.$$

Proof. Applying Binet-Cauchy's theorem, we get (4.3) directly from (4.2). □

THEOREM 5. *Let $x_1, \cdots, x_m$ be given vectors in 2-inner product space X. Set $Y = L(x_1, \cdots, x_m)$ and take any $z \in X$ such that $z \notin Y$. If $\{e_1, \cdots, e_n\}$, $n = \dim Y \geq m$ is the base for Y such that*

$$(e_i, e_j | z) = \delta_{ij}$$

for all $i, j \in \{1, \cdots, n\}$, then

$$\Gamma(x_1, \cdots, x_m | z) = \sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq n} |\xi(j_1, j_2, \cdots, j_m)|^2, \tag{4.6}$$

where $\xi(j_1, j_2, \cdots, j_m)$ is defined by (4.4).

Proof. Set $y_j = x_j$ for $j = 1, \cdots, m$. Then, for the matrix A defined by (4.1), we have

$$A = G(x_1, \cdots, x_m | z), \quad \det A = \Gamma(x_1, \cdots, x_m | z).$$

Also, for $\xi(j_1, j_2, \cdots, j_m)$ and $\eta(j_1, j_2, \cdots, j_m)$ respectively given by (4.4) and (4.5), we have

$$\eta(j_1, j_2, \cdots, j_m) = \overline{\xi(j_1, j_2, \cdots, j_m)}.$$

Therefore, (4.3) reduces to (4.6) in this case. □

THEOREM 6. *Let $\{x_1, \cdots, x_m\}$ and $\{y_1, \cdots, y_m\}$ be two sets of linearly independent vectors in 2-inner product space X. Set $Y = L(x_1, \cdots, x_m, y_1, \cdots, y_m)$ and take any $z \in X$ such that $z \notin Y$. If A is defined by (4.1), then*

$$|\det A|^2 \leq \Gamma(x_1, \cdots, x_m | z)\Gamma(y_1, \cdots, y_m | z). \tag{4.7}$$

The equality occurs in (4.7) if and only if $\{x_1, \cdots, x_m\}$ spans the same subspace as $\{y_1, \cdots, y_m\}$ does, that is, if and only if $L(x_1, \cdots, x_m) = L(y_1, \cdots, y_m) = Y$.

Proof. Set $n = \dim Y$. Obviously $n \geq m$ under given assumptions. Take any base $\{e_1, \cdots, e_n\}$ for Y such that

$$(e_i, e_j | z) = \delta_{ij}$$

for all $i, j \in \{1, \cdots, n\}$. Then the identity (4.3) is valid and we can apply Cauchy's inequality for sequences to obtain the inequality

$$\begin{aligned} &|\det A|^2 \\ &\leq \sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq n} |\xi(j_1, j_2, \cdots, j_m)|^2 \sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq n} |\eta(j_1, j_2, \cdots, j_m)|^2. \end{aligned}$$

By (4.6), the first sum on the right hand side of the above inequality is equal to $\Gamma(x_1, \cdots, x_m | z)$, while the second sum is equal to $\Gamma(y_1, \cdots, y_m | z)$ since, for the transpose M^τ of a square matrix M, we have $\det M^\tau = \det M$. Thus, the above inequality is equivalent to (4.7). It remains the question on the equality case in (4.7).

The orthonormal base $\{e_1, \cdots, e_n\}$ for Y can always be chosen so that the first m vectors are obtained by applying the procedure of getting orthonormal vectors described in Corollary 1 to the vectors $x_1, \cdots, x_m$. It is easy to see that, in this case, we can express the vectors $x_1, \cdots, x_m$ in the form

$$\begin{aligned} x_1 &= \gamma_1^{1/2} e_1, \\ x_r &= \left(\frac{\gamma_r}{\gamma_{r-1}}\right)^{1/2} [\alpha_{r,1} e_1 + \cdots + \alpha_{r,r-1} e_{r-1} + e_r] \end{aligned}$$

for $r = 2, \cdots, m$, where

$$\gamma_r = \Gamma(x_1, \cdots, x_r | z)$$

for $r = 1, 2, \cdots, m$. Therefore, for $j = 1, \cdots, m$, we have

$$(x_1, y_j | z) = \gamma_1^{1/2} (e_1, y_j | z)$$

and

$$(x_r, y_j | z) = \left(\frac{\gamma_r}{\gamma_{r-1}}\right)^{1/2} [\alpha_{r,1}(e_1, y_j | z) + \cdots + \alpha_{r,r-1}(e_{r-1}, y_j | z) + (e_r, y_j | z)]$$

for $r = 2, \cdots, m$. Using this and the elementary properties of determinant, we get

$$\det A = \Gamma(x_1, \cdots, x_m|z)^{1/2} \det B,$$

where

$$B = \begin{bmatrix} (e_1, y_1|z) & (e_1, y_2|z) & \cdots & (e_1, y_m|z) \\ (e_2, y_1|z) & (e_2, y_2|z) & \cdots & (e_2, y_m|z) \\ \vdots & \vdots & \ddots & \vdots \\ (e_m, y_1|z) & (e_m, y_2|z) & \cdots & (e_m, y_m|z) \end{bmatrix}.$$

This means that

$$|\det A|^2 = \Gamma(x_1, \cdots, x_m|z) |\det B|^2.$$

Note that actually we have

$$\det B = \eta(1, 2, \cdots, m),$$

where $\eta(j_1, j_2, \cdots, j_m)$ for $1 \leq j_1 < j_2 < \cdots < j_m \leq n$ is given by (4.5). Now, the equality in (4.7) is equivalent to the requirement that

$$\Gamma(y_1, \cdots, y_m|z) = |\det B|^2 = |\eta(1, 2, \cdots, m)|^2. \tag{4.8}$$

On the other side, by Theorem 5, we have

$$\Gamma(y_1, \cdots, y_m|z) = \sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq n} |\eta(j_1, j_2, \cdots, j_m)|^2. \tag{4.9}$$

From the equalities (4.8) and (4.9), it follows that the equality in (4.7) holds if and only if

$$\eta(j_1, j_2, \cdots, j_m) = 0 \tag{4.10}$$

for all $(j_1, j_2, \cdots, j_m) \neq (1, 2, \cdots, m)$. Further, consider the vectors

$$v_i = \left| \begin{array}{ccc|c} (e_1, y_1|z) & \cdots & (e_1, y_m|z) & e_1 \\ \vdots & \ddots & \vdots & \vdots \\ (e_m, y_1|z) & \cdots & (e_m, y_m|z) & e_m \\ \hline (e_i, y_1|z) & \cdots & (e_i, y_m|z) & e_i \end{array} \right| \qquad (i = m+1, \cdots, n).$$

Expanding the above determinant over the last column and using (4.10), we have

$$v_i = \eta(1, 2, \cdots, m) e_i$$

for $i = m+1, \cdots, n$. On the other side, we have, for all $j = 1, 2, \cdots, m$ and $i = m+1, \cdots, n$,

$$(v_i, y_j|z) = \begin{vmatrix} (e_1, y_1|z) & \cdots & (e_1, y_m|z) & (e_1, y_j|z) \\ \vdots & \ddots & \vdots & \vdots \\ (e_m, y_1|z) & \cdots & (e_m, y_m|z) & (e_m, y_j|z) \\ (e_i, y_1|z) & \cdots & (e_i, y_m|z) & (e_i, y_j|z) \end{vmatrix} = 0$$

since two columns in this determinant are identical. This implies that

$$(e_i, y_j|z) = \frac{(v_i, y_j|z)}{\eta(1, 2, \cdots, m)} = 0$$

for all $i = m+1, \cdots, n$ and $j = 1, 2, \cdots, m$. Using this and the fact that any $y \in Y$ is uniquely represented as $y = \sum_{i=1}^{n}(y, e_i|z)e_i$, we see that, for all $j = 1, 2, \cdots, m$,

$$y_j = \sum_{i=1}^{m}(y_j, e_i|z)e_i \in L(e_1, \cdots, e_m) = L(x_1, \cdots, x_m).$$

This means that $y_1, \cdots, y_m$ span the same subspace as the one spanned by $x_1, \cdots, x_m$, since $y_1, \cdots, y_m$ are linearly independent. Therefore, (4.10) is equivalent to the requirement that $L(x_1, \cdots, x_m) = L(y_1, \cdots, y_m)$. This completes the proof. □

COROLLARY 5. *Let $x_1, \cdots, x_n$ and y_1 be given vectors in 2-inner product space X. Suppose $x_1, \cdots, x_n$ are linearly independent and take any $z \in X$ such that $z \notin L(x_1, \cdots, x_n, y_1)$. Then*

$$\begin{aligned} &\Gamma(x_1 + y_1, x_2, \cdots, x_n|z)^{1/2} \\ &\leq \Gamma(x_1, x_2, \cdots, x_n|z)^{1/2} + \Gamma(y_1, x_2, \cdots, x_n|z)^{1/2}. \end{aligned} \tag{4.11}$$

The equality occurs in (4.11) if and only if

$$y_1 = \lambda x_1 + u, \quad \lambda \geq 0, \quad u \in L(x_2, \cdots, x_n). \tag{4.12}$$

Proof. Using the elementary properties of determinant, we easily have the following identity

$$\Gamma(x_1+y_1,x_2,\cdots,x_n|z)=\Gamma(x_1,x_2,\cdots,x_n|z)+\det A+\overline{\det A} \\ +\Gamma(y_1,x_2,\cdots,x_n|z),$$

where

$$A=\begin{bmatrix} (x_1,y_1|z) & (x_1,x_2|z) & \cdots & (x_1,x_n|z) \\ (x_2,y_1|z) & (x_2,x_2|z) & \cdots & (x_2,x_n|z) \\ \vdots & \vdots & \ddots & \vdots \\ (x_n,y_1|z) & (x_n,x_2|z) & \cdots & (x_n,x_n|z) \end{bmatrix}.$$

Applying Theorem 6 to the sets of vectors $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, x_2, \cdots, x_n\}$, we have

$$|\det A|\le\Gamma(x_1,x_2,\cdots,x_n|z)^{1/2}\Gamma(y_1,x_2,\cdots,x_n|z)^{1/2}.$$

Therefore, we have

$$\begin{aligned}&\Gamma(x_1+y_1,x_2,\cdots,x_n|z)\\ &=\Gamma(x_1,x_2,\cdots,x_n|z)+\det A+\overline{\det A}+\Gamma(y_1,x_2,\cdots,x_n|z)\\ &=\Gamma(x_1,x_2,\cdots,x_n|z)+2\mathrm{Re}[\det A]+\Gamma(y_1,x_2,\cdots,x_n|z)\\ &\le\Gamma(x_1,x_2,\cdots,x_n|z)+2|\det A|+\Gamma(y_1,x_2,\cdots,x_n|z)\\ &\le\Gamma(x_1,x_2,\cdots,x_n|z)+2\Gamma(x_1,x_2,\cdots,x_n|z)^{1/2}\Gamma(y_1,x_2,\cdots,x_n|z)^{1/2}\\ &\quad+\Gamma(y_1,x_2,\cdots,x_n|z)\\ &=\left[\Gamma(x_1,x_2,\cdots,x_n|z)^{1/2}+\Gamma(y_1,x_2,\cdots,x_n|z)^{1/2}\right]^2,\end{aligned}$$

which yields (4.11). Obviously we have equality in (4.11) if and only if

$$\begin{aligned}\mathrm{Re}[\det A]&=|\det A|\\ &=\Gamma(x_1,x_2,\cdots,x_n|z)^{1/2}\Gamma(y_1,x_2,\cdots,x_n|z)^{1/2}.\end{aligned}\tag{4.13}$$

The first equality in (4.12) is equivalent with $\det A\ge 0$, while the second one holds if and only if $y_1,x_2,\cdots,x_n$ are linearly dependent or $L(y_1, x_2, \cdots, x_n)=L(x_1, x_2, \cdots, x_n)$. In the case when $y_1,x_2,\cdots,x_n$ are linearly dependent, we have $y_1=u\in L(x_2,\cdots,x_n)$ and $\det A=0$, while, in the case when $L(y_1,x_2,\cdots,x_n)=L(x_1,x_2,\cdots,x_n)$, we have $y_1=\lambda x_1+u$ for some $\lambda\ne 0$ and some $u\in L(x_2,\cdots, x_n)$. In this case, we get $\det A=\overline{\lambda}\Gamma(x_1,x_2,\cdots,x_n|z)$ so that the condition $\det A\ge 0$ is equivalent with the condition $\lambda\ge 0$. This proves that the equality occurs in (4.11) if and only if (4.12) holds. This completes the proof. □

5. A generalization of Bessel's inequality

Let $(X, (\cdot, \cdot|\cdot))$ be 2-inner product space over the field of real numbers $K = \mathbf{R}$ or the field of complex numbers $K = \mathbf{C}$. In this section, we give a generalization of Bessel's inequality

$$\sum_{i=1}^{n} |(x, e_i|z)|^2 \leq \|x|z\|^2, \tag{5.1}$$

which holds for any $x \in X$ whenever $e_1, \cdots, e_n, z \in X$ are the vectors such that

$$z \notin L(e_1, \cdots, e_n)$$

and

$$(e_i, e_j|z) = \delta_{ij}, \quad i, j \in \{1, \cdots, n\}.$$

Also, we know that equality occurs in (5.1) if and only if $x = u + \gamma z$ for some $u \in L(e_1, \cdots, e_n)$ and some scalar $\gamma \in K$.

THEOREM 7. *Let X be 2-inner product space and let $x_1, \cdots, x_n, z \in X$ be the vectors such that $x_1, \cdots, x_n$ are linearly independent and $z \notin L(x_1, \cdots, x_n)$. For any given vector $x \in X$, define*

$$\lambda_i = (x, x_i|z)$$

for $i = 1, \cdots, n$. If $\Delta = \Gamma(x_1, \cdots, x_n|z)$ and Δ_i is equal to the determinant obtained from Δ by replacing the i^{th} row of $\Gamma(x_1, \cdots, x_n|z)$ with $(\lambda_1, \cdots, \lambda_n)$ for $i = 1, \cdots, n$, then we have

$$\Big\| \sum_{i=1}^{n} \Delta_i x_i | z \Big\| \leq \Delta \|x|z\|. \tag{5.2}$$

The equality in (5.2) occurs if and only if there is a scalar $\lambda \in K$ such that

$$x = \frac{1}{\Delta} \sum_{i=1}^{n} \Delta_i x_i + \lambda z.$$

Proof. Note that $\Delta > 0$ and consider the vector $y \in X$ defined as

$$y = \frac{1}{\Delta} \sum_{i=1}^{n} \delta_i x_i \quad (\delta_i \in K, \ i = 1, \cdots, n).$$

We have

$$(y,x|z)=\frac{1}{\Delta}\sum_{i=1}^{n}\delta_i\overline{(x,x_i|z)}=\frac{1}{\Delta}\sum_{i=1}^{n}\delta_i\overline{\lambda_i}$$

and

$$(y,y|z)=\frac{1}{\Delta^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_i\overline{\delta_j}(x_i,x_i|z).$$

The requirement that $(y,x|z)=(y,y|z)$ is therefore equivalent to

$$\sum_{i=1}^{n}\delta_i\overline{\lambda_i}=\frac{1}{\Delta}\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_i\overline{\delta_j}(x_i,x_i|z). \tag{5.3}$$

Obviously (5.3) will be satisfied if $\delta_1,\cdots,\delta_n$ are chosen so that

$$\overline{\lambda_i}=\frac{1}{\Delta}\sum_{j=1}^{n}\overline{\delta_j}(x_i,x_i|z) \qquad (i=1,\cdots,n),$$

that is,

$$\sum_{j=1}^{n}(x_j,x_i|z)\delta_j=\Delta\lambda_i \qquad (i=1,\cdots,n). \tag{5.4}$$

The matrix of the above system of linear equations has determinant equal to $\Gamma(x_1,\cdots,x_n|z)=\Delta>0$. Therefore, the system (5.4) has unique solution given as

$$\begin{aligned}\delta_j&\\ =&\frac{1}{\Delta}\begin{vmatrix}(x_1,x_1|z) & \cdots & (x_{j-1},x_1|z) & \Delta\lambda_1 & (x_{j+1},x_1|z) & \cdots & (x_n,x_1|z)\\ \vdots & & \vdots & \vdots & \vdots & & \vdots\\ (x_1,x_n|z) & \cdots & (x_{j-1},x_n|z) & \Delta\lambda_n & (x_{j+1},x_n|z) & \cdots & (x_n,x_n|z)\end{vmatrix}\\ =&\frac{1}{\Delta}\Delta\Delta_j\\ =&\Delta_j \qquad (j=1,\cdots,n).\end{aligned}$$

We conclude that, for the vector $y\in X$ defined as

$$y=\frac{1}{\Delta}\sum_{i=1}^{n}\Delta_i x_i,$$

we have

$$(y, x|z) = (y, y|z) \quad \text{and} \quad (x, y|z) = \overline{(y, x|z)} = (y, y|z).$$

Using this, we have

$$\begin{aligned} 0 &\le \|x - y|z\|^2 \\ &= (x - y, x - y|z) \\ &= \|x|z\|^2 - 2(y, y|z) + \|y|z\|^2 \\ &= \|x|z\|^2 - \|y|z\|^2 \end{aligned}$$

or, equivalently,

$$\|y|z\| = \frac{1}{\Delta}\Big\| \sum_{i=1}^{n} \Delta_i x_i|z \Big\| \le \|x|z\|,$$

which is equivalent to (5.2). Moreover, the equality occurs in the above inequality and hence in (5.2) if and only if $x - y$ and z are linearly dependent, that is,

$$\alpha(x - y) + \beta z = 0$$

for some $\alpha, \beta \in K$ with $\alpha \neq 0$ or $\beta \neq 0$. Evidently we must have $\alpha \neq 0$ or, otherwise, we will have $z = 0$, which is not true. Therefore, $x = y - (\beta/\alpha)z$. We conclude that the equality in (5.2) occurs if and only $x = y + \lambda z$ for some $\lambda \in K$. This completes the proof. □

REMARK 4. (1) The inequality (5.2) can be regarded as a generalization of Cauchy-Schwarz-Bunjakowsky inequality. Namely, for $n = 1$, we have

$$\Delta = (x_1, x_1|z), \quad \Delta_1 = \lambda_1 = (x, x_1|z)$$

and (5.2) reduces to

$$|(x, x_1|z)|\sqrt{(x_1, x_1|z)} \le (x_1, x_1|z)\sqrt{(x, x|z)}$$

or, equivalently,

$$|(x, x_1|z)|^2 \le (x_1, x_1|z)(x, x|z).$$

The equality occurs if and only if

$$x = \frac{(x, x_1|z)}{(x_1, x_1|z)} x_1 + \lambda z$$

for some $\lambda \in K$. This is just the Cauchy-Schwarz-Bunjakowsky inequality stated for three vectors $x, x_1, z \in X$ such that $x_1 \neq 0$ and $z \notin L(x_1)$.

(2) Suppose that $x_1, \cdots, x_n$ satisfy additional condition

$$(x_i, x_j|z) = \delta_{ij}$$

for all $i, j \in \{1, \cdots, n\}$. In this case, we have

$$\Delta = 1, \qquad \Delta_i = \lambda_i = (x, x_i|z) \qquad (i = 1, \cdots, n),$$

so that (5.2) becomes

$$\|\sum_{i=1}^{n} (x, x_i|z) x_i|z\| \leq \|x|z\|$$

or, equivalently,

$$\sum_{i=1}^{n} |(x, x_i|z)|^2 \leq \|x|z\|^2,$$

which is just the Bessel's inequality (5.1), where $e_1, \cdots, e_n$ are replaced with $x_1, \cdots, x_n$. The equality holds in this case if and only if

$$x = \sum_{i=1}^{n} (x, x_i|z) x_i + \lambda z$$

for some scalar $\lambda \in K$.

COROLLARY 6. *Let X be 2-inner product space and let $a, b, z \in X$ be the vectors such that a, b are linearly independent and $z \notin L(a, b)$. For any given vector $x \in X$, define*

$$\mu = (x, a|z), \ \nu = (x, b|z).$$

Then we have

$$\Gamma(a, b|z)\|x|z\|^2 \geq \|\overline{\nu} a - \overline{\mu} b|z\|^2. \tag{5.5}$$

The equality in (5.5) occurs if and only if

$$x = \frac{(a, \overline{\nu} a - \overline{\mu} b|z) b - (b, \overline{\nu} a - \overline{\mu} b|z) a}{\Gamma(a, b|z)} + \lambda z$$

for some scalar $\lambda \in K$.

Proof. We apply Theorem 7 with $n = 2$, $x_1 = a$ and $x_2 = b$. Then we have

$$\lambda_1 = \mu,\ \lambda_2 = \nu,\ \Delta = \Gamma(a, b|z)$$

and

$$\Delta_1 = \begin{vmatrix} \mu & \nu \\ (b, a|z) & (b, b|z) \end{vmatrix} = \mu(b, b|z) - \nu(b, a|z) = -(b, \overline{\nu}a - \overline{\mu}b|z),$$

$$\Delta_2 = \begin{vmatrix} (a, a|z) & (a, b|z) \\ \mu & \nu \end{vmatrix} = \nu(a, a|z) - \mu(a, b|z) = (a, \overline{\nu}a - \overline{\mu}b|z).$$

Consider the vector

$$\tilde{y} = \Delta_1 a + \Delta_2 b = (a, \overline{\nu}a - \overline{\mu}b|z)b - (b, \overline{\nu}a - \overline{\mu}b|z)a.$$

We know, by (5.2), that

$$\|\tilde{y}|z\| \leq \Delta\|x|z\| = \Gamma(a, b|z)\|x|z\|,$$

or, equivalently,

$$(\tilde{y}, \tilde{y}|z) \leq \Gamma(a, b|z)^2(x, x|z).$$

This is a consequence of the equality $(y, y|z) = (y, x|z)$, which holds for the vector

$$y = \frac{\tilde{y}}{\Delta} = \frac{\tilde{y}}{\Gamma(a, b|z)}.$$

Note that the equality $(y, y|z) = (y, x|z)$ is equivalent to

$$\frac{(\tilde{y}, \tilde{y}|z)}{\Gamma(a, b|z)} = (\tilde{y}, x|z).$$

Therefore, we have

$$\begin{aligned} \Gamma(a, b|z)\|x|z\|^2 &\geq \frac{(\tilde{y}, \tilde{y}|z)}{\Gamma(a, b|z)} \\ &= (\tilde{y}, x|z) \\ &= ((a, \overline{\nu}a - \overline{\mu}b|z)b - (b, \overline{\nu}a - \overline{\mu}b|z)a, x|z) \\ &= (a, \overline{\nu}a - \overline{\mu}b|z)\overline{\nu} - (b, \overline{\nu}a - \overline{\mu}b|z)\overline{\mu} \\ &= (\overline{\nu}a - \overline{\mu}b, \overline{\nu}a - \overline{\mu}b|z) \\ &= \|\overline{\nu}a - \overline{\mu}b\|^2, \end{aligned}$$

which is just the inequality (5.5). Also, we know, by Theorem 7, that the equality occurs if and only if

$$x = y + \lambda z = \frac{\tilde{y}}{\Gamma(a, b|z)} + \lambda z \qquad (\lambda \in K).$$

This completes the proof. □

THEOREM 8. *Let X be 2-inner product space over the field $K = \mathbf{R}$ of real numbers. Let $a, b, z \in X$ and $e_1, \cdots, e_m \in X$ be the vectors from X such that $z \notin L(a,b)$ and, for all $i, j \in \{1, \cdots, m\}$,*

$$(e_i, e_j|z) = \delta_{ij}$$

and

$$(a, e_j|z)(b, e_i|z) \neq (a, e_i|z)(b, e_j|z) \tag{5.6}$$

for $i \neq j$. If $p_{ij} \in \mathbf{R}$ $(i, j \in \{1, \cdots, m\},\ i \neq j)$ are given real numbers which satisfy the conditions

$$p_{ij} = p_{ji} \tag{5.7}$$

for all $i \neq j,\ i, j \in \{1, \cdots, m\}$ and

$$P = \sum_{1 \leq i < j \leq m} p_{ij} \neq 0,$$

then, for any two scalars $\mu, \nu \in \mathbf{R}$,

$$P^2 \frac{\|\nu a - \mu b|z\|^2}{\Gamma(a,b|z)} \leq \sum_{i=1}^{m} \left| \sum_{i \neq j = 1}^{m} \frac{p_{ij}(\nu a - \mu b, e_j|z)}{(a, e_j|z)(b, e_i|z) - (a, e_i|z)(b, e_j|z)} \right|^2 .$$

Proof. First note that the condition (5.6) implies that a, b are linearly independent. Next, for any two scalars $\mu, \nu \in \mathbf{R}$, consider the vector

$$\tilde{x} = \sum_{i=1}^{m} \sum_{i \neq j = 1}^{m} \frac{p_{ij}(\nu a - \mu b, e_j|z)}{(a, e_j|z)(b, e_i|z) - (a, e_i|z)(b, e_j|z)} e_i .$$

Using the properties of 2-inner product $(\cdot, \cdot|\cdot)$ in the real case, we have

$$\begin{aligned}(\tilde{x}, a|z) = \nu \sum_{i=1}^{m} \sum_{i \neq j = 1}^{m} \frac{p_{ij}(a, e_j|z)(a, e_i|z)}{(a, e_j|z)(b, e_i|z) - (a, e_i|z)(b, e_j|z)} \\ - \mu \sum_{i=1}^{m} \sum_{i \neq j = 1}^{m} \frac{p_{ij}(b, e_j|z)(a, e_i|z)}{(a, e_j|z)(b, e_i|z) - (a, e_i|z)(b, e_j|z)} .\end{aligned}$$

On the other side, using the condition (5.7), we easily see that

$$\sum_{i=1}^{m}\sum_{i\neq j=1}^{m}\frac{p_{ij}(a,e_j|z)(a,e_i|z)}{(a,e_j|z)(b,e_i|z)-(a,e_i|z)(b,e_j|z)}=0$$

and

$$\sum_{i=1}^{m}\sum_{i\neq j=1}^{m}\frac{p_{ij}(b,e_j|z)(a,e_i|z)}{(a,e_j|z)(b,e_i|z)-(a,e_i|z)(b,e_j|z)}=-\sum_{1\leq i<j\leq m}p_{ij}=-P.$$

Therefore, we have

$$(\tilde{x},a|z)=\mu P.$$

Analogously, we have also

$$(\tilde{x},b|z)=\nu P.$$

Therefore, we can apply Corollary 6 to the vector $x=\tilde{x}/P$ so that, by (5.5), we have

$$\frac{\Gamma(a,b|z)\|\tilde{x}|z\|^2}{P^2}=\Gamma(a,b|z)\|\frac{\tilde{x}}{P}|z\|^2\geq\|\overline{\nu}a-\overline{\mu}b|z\|^2$$

or, equivalently,

$$P^2\frac{\|\nu a-\mu b|z\|^2}{\Gamma(a,b|z)}\leq\|\tilde{x}|z\|^2.$$

To complete the proof, it is enough to apply Parseval's identity to the term $\|\tilde{x}|z\|^2$. This completes the proof. □

COROLLARY 7. *Under assumptions of Theorem 8, we have*

$$\binom{m}{2}^2\frac{\|\nu a-\mu b|z\|^2}{\Gamma(a,b|z)}\leq\sum_{i=1}^{m}\left|\sum_{i\neq j=1}^{m}\frac{(\nu a-\mu b,e_j|z)}{(a,e_j|z)(b,e_i|z)-(a,e_i|z)(b,e_j|z)}\right|^2$$

for any two scalars $\mu,\nu\in\mathbf{R}$.

Proof. Set $p_{ij}=1$ for all $i\neq j$, $i,j\in\{1,\cdots,m\}$ and note that $P=\sum_{1\leq i<j\leq m}p_{ij}=\binom{m}{2}$. Then, applying Theorem 8, we have the conclusion. □

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