J. Korean Math. Soc. **38** (2001), No. 6, pp. 1157–1166

A NOTE ON APPROXIMATE SIMILARITY

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Dedicated to my loving and wonderful parents.

ABSTRACT. This paper answers some old questions about approximate similarity and raises new ones. We provide positive evidence and a technique for finding negative evidence on the question of whether approximate similarity is the equivalence relation generated by approximate equivalence and similarity.

In [4] the author introduced and studied the approximate similarity of Hilbert space operators. This paper includes some new results, answers some old questions, and raises new questions. Suppose $B(H)$ is the set of all operators on a separable infinite-dimensional Hilbert space H, and $\mathcal{K}(H)$ is the set of compact operators on H.

A sequence $\{A_n\}$ in $B(H)$ is *invertibly bounded* if

$$\sup_{n\geq 1} \|A_n\| \left\|A_n^{-1}\right\| < \infty.$$

Two operators $S, T \in B(H)$ are *approximately similar*, denoted by $S \sim_{as} T$ if there is an invertibly bounded sequence $\{A_n\}$ such that $\left\|A_n S A_n^{-1} - T\right\| \to 0$. If all of the A_n's can be chosen to be unitary, we say that S and T are *approximately equivalent*, denoted by $S \sim_a T$.

It is clear that approximate equivalence and approximate similarity are equivalence relations. It is also clear that similarity and approximate equivalence each implies approximate similarity. The most important result on approximate equivalence is a characterization, due to D. Voiculescu [13], of a version of approximate equivalence for representations of C*-algebras. For single operators we use [3] as a basic reference on approximate equivalence. Here is the most fundamental problem in approximate similarity.

Received January 10, 2001.
2000 Mathematics Subject Classification: 47A99.
Key words and phrases: approximate similarity, approximate equivalence, normal operator, subnormal operator, hyponormal operator, compact operator, GNS construction.

QUESTION 1. Is approximate similarity the equivalence relation generated by approximate equivalence and similarity?

What this question asks is whether, for each pair of approximately similar operators S and T there is a finite sequence

$$S = S_1, \dots, S_n = T$$

such that, for each k with $1 \leq k < n$, S_k is either similar or approximately equivalent to S_{k+1}. The following easy lemma is a tantalizing piece of positive evidence.

LEMMA 1. *Suppose $\{P_n\}$ is an invertibly bounded sequence of positive operators and $\|P_n^{-1} S P_n - T\| \to 0$. Then S is similar to T.*

Proof. There exist positive numbers r, R such that $r \leq P_n \leq R$ for each $n \geq 1$. Choose a subsequence $\{P_{n_k}\}$ such that $P_{n_k} \to P$ in the weak operator topology for some operator P. Since $r \leq P \leq R$, it follows that P is invertible. Since $SP_n - P_n T \to 0$ in norm and $SP_{n_k} - P_{n_k} T \to SP - PT$ weakly, it follows that $P^{-1} SP = T$. □

The preceding lemma makes it tempting to hope that the polar decomposition of an invertibly bounded sequence may lead to a result expressing approximate similarity as either similarity followed by approximate equivalence or approximate equivalence followed by similarity. In fact the following questions were posed in [4]:

1. (Question 3.8 in [4]) If $S \sim_{as} T$, must S be similar to an operator that is approximately equivalent to T?
2. (Question 3.9 in [4]) If S is similar to an operator that is approximately equivalent to T, must T be similar to an operator that is approximately equivalent to S?
3. (Question 3.10 in [4]) If a set of operators is closed under approximate equivalence and similarity, must it be closed under approximate similarity?

The third question is the same as the fundamental Question 1 at the beginning of the paper. We provide an answer to the first two questions in the following example.

EXAMPLE 1. Suppose T is an irreducible compact operator with zero kernel, and S is an irreducible operator that is similar to $T \oplus 0$. Clearly, $S \sim_{as} T$, since S is similar to $T \oplus 0$ and $T \oplus 0$ is approximately equivalent to T. On the other hand, every operator that is approximately equivalent to S has S as a direct summand [3], and thus has nonzero kernel. Hence, no operator that is approximately equivalent to S is similar to T. This shows that (2) above (i.e., Question 3.9 in [4]) has a negative answer.

If we are more careful in choosing T, we can include the property that T commutes with no idempotent other than 0 or 1 (e.g., T can be a compact weighted unilateral shift operator with positive weights). In this case any operator W that similar to T is an irreducible compact operator, and thus any operator approximately equivalent to W is unitarily equivalent to $W \oplus 0^{(m)}$ for some cardinal $m \geq 0$. No such operator can be equal to S, since S is irreducible and not similar to T. This negatively answers the first question above (Question 3.8 in [4]).

Here is another important bit of positive evidence. For an operator T, let $C^*(T)$ denote the unital C^*-algebra generated by T.

THEOREM 1. *Assume H, M are infinite-dimensional Hilbert spaces and H is separable. Suppose $S, T \in B(H)$ and $C^*(S) \cap \mathcal{K}(H) = C^*(T) \cap \mathcal{K}(H) = \{0\}$. Suppose also $\pi : C^*(S) \to B(M)$ and $\rho : C^*(T) \to B(M)$ are faithful unital $*$-homomorphisms such that $\pi(S)$ is similar to $\rho(T)$. Then there exist similar operators S' and T' such that $S \sim_a S'$ and $T \sim_a T'$.*

Proof. Suppose $W \in B(M)$ and $W^{-1}\pi(S)W = \rho(T)$. Since $C^*(\{W, \pi(S), \rho(T)\})$ is separable, there is a faithful unital $*$-homomorphism $\tau : C^*(\{W, \pi(S), \rho(T)\}) \to B(H)$ such that τ is unitarily equivalent to $\tau^{(\infty)} = \tau \oplus \tau \oplus \cdots$. It follows from Voiculescu's theorem [13] that $S \sim_a (\tau \circ \pi)(S)$ and $T \sim_a (\tau \circ \rho)(T)$ and that $\tau(W)$ makes $(\tau \circ \pi)(S)$ similar to $(\tau \circ \rho)(T)$. □

COROLLARY 1. *Assume H, M are infinite-dimensional Hilbert spaces and H is separable. Suppose $S, T \in B(H)$ and $C^*(S) \cap \mathcal{K}(H) = C^*(T) \cap \mathcal{K}(H) = \{0\}$. If $S \sim_{as} T$, then there exist similar operators S' and T' such that $S \sim_a S'$ and $T \sim_a T'$.*

Proof. Let $\mathcal{A}$ be the C^*-direct product (i.e., ℓ^∞-product) of countably many copies of $B(H)$ and let $\mathcal{J}$ be the C^*-direct sum (i.e., the c_0-product) of countably many copies of $B(H)$ and define $\delta : B(H) \to \mathcal{A}/\mathcal{J}$ by $\delta(T) = [(T, T, T, \ldots)]$. Then $S, T \in B(H)$ are approximately similar if and only if $\delta(S)$ and $\delta(T)$ are similar in $\mathcal{A}/\mathcal{J}$. Since the GNS construction says that $\mathcal{A}/\mathcal{J}$ is $*$-isometric to an algebra of operators on a Hilbert space M, the corollary is implied by the theorem. □

COROLLARY 2. *If $S, T \in B(H)$ and $S \sim_{as} T$, then there are operators $S', T' \in B(H)$ such that $S \sim_a S'$, $T \sim_a T'$, and $(S')^{(\infty)}$ is similar to $(T')^{(\infty)}$.*

Proof. Suppose $S \sim_{as} T$. Then $S^{(\infty)} \sim_{as} T^{(\infty)}$, and it follows from the Corollary 1 that there are operators A, B such that $S^{(\infty)} \sim_a A, T^{(\infty)} \sim_a B$ and A is similar to B. It follows that there are operators $S' \sim_a S$ and $T' \sim_a T$ such that $(S')^{(\infty)}$ is unitarily equivalent to $A^{(\infty)}$ and $(T')^{(\infty)}$ is unitarily equivalent to $B^{(\infty)}$. Hence $(S')^{(\infty)}$ is similar to $(T')^{(\infty)}$. □

It follows that if $S, T \in B(H)$ and S is similar to S_1 and T is similar to T_1 with $C^*(S_1) \cap \mathcal{K}(H) = C^*(T_1) \cap \mathcal{K}(H) = \{0\}$ then $S \sim_{as} T$ implies $S_1 \sim_{as} T_1$, and Corollary 1 implies there are similar operators $S' \sim_a S_1$ and $T' \sim_a T_1$. Hence such pairs S, T confirm the assertion in Question 1. Call an operator T *approximately infinite* if T is similar to an operator T_1 with $C^*(T_1) \cap \mathcal{K}(H) = \{0\}$?

QUESTION 2. Is there a nice characterization of approximately infinite operators? Is the set of approximately infinite operators the set of all operators T such that $T \sim_{as} T^{(\infty)}$?

Although we don't know the answer to Question 2, we can say something about the norm closure. Here $\mathrm{Ind}(T)$ denotes the Fredholm index of the operator T.

PROPOSITION 1. *The norm closure of the set of approximately infinite operators is the set of all operators T such that $\sigma(T) = \sigma_e(T)$ and, for every $\lambda \in \mathbb{C} \backslash \sigma_e(T)$, Ind$(T - \lambda)$ is in the set $\{-\infty, 0, \infty\}$.*

Proof. This follows from the general similarity theorem [7]. □

In [4] the author proved that an operator that is approximately similar to a normal operator is similar to a normal operator, and that the result remains true when "normal" is replaced with "subnormal". It was also proved in [4] that if T is similar to a normal (respectively, subnormal) operator and $\pi : C^*(T) \to B(M)$ is a unital $*$-homomorphism, then $\pi(T)$ is similar to a normal (respectively, subnormal) operator. A class $\mathcal{C}$ of operators is a *part class* if it is closed under unitary equivalence, and a direct sum of operators is in $\mathcal{C}$ if and only if each summand is in $\mathcal{C}$.

THEOREM 2. *Suppose $\mathcal{C}$ is a part class, and let $\mathcal{S}(\mathcal{C})$ denote the class of operators that are similar to an operator in $\mathcal{C}$. Assume that whenever $A \oplus B \in \mathcal{S}(\mathcal{C})$ we have $A \in \mathcal{S}(\mathcal{C})$. The following are equivalent:*

1. *$\mathcal{S}(\mathcal{C})$ is closed under approximate similarity.*
2. *$\mathcal{S}(\mathcal{C})$ is closed under approximate equivalence.*

3. *If* $T \in \mathcal{S}(\mathcal{C})$, *$M$ is a separable Hilbert space and* $\pi : C^*(T) \to B(M)$ *is a unital $*$-homomorphism, then* $\pi(T) \in \mathcal{S}(\mathcal{C})$.
4. *If* $\{A_n\}$ *is an invertibly bounded sequence,* $\{T_n\}$ *is a bounded sequence in* $\mathcal{C}$, *and* $A_n^{-1}T_nA_n \to S$ *in the $*$-strong operator topology, then* $S \in \mathcal{S}(\mathcal{C})$.

Proof. It is clear that (4)⇒(1)⇒(2).The implication (2)⇒(3) follows from the fact that $\pi(T) \oplus T^{(\infty)} \sim_a T^{(\infty)}$ and $T^{(\infty)} \in \mathcal{S}(\mathcal{C})$. To prove the implication (3)⇒(4) let $T = T_1 \oplus T_2 \oplus \cdots$ and $A = A_1 \oplus A_2 \oplus \cdots$. Then $T \in \mathcal{C}$. Also the convergence $A_n^{-1}T_nA_n \to S$ in the $*$-strong operator topology implies that there is a representation $\pi : C^*(A^{-1}TA) \to B(H)$ such that $\pi(A^{-1}TA) = S$. It now follows from (3) that $S \in \mathcal{S}(\mathcal{C})$. □

REMARK 1.

1. If the part-class $\mathcal{C}$ is closed under limits in the strong operator topology, then $\mathcal{C}$ is closed under restrictions to invariant subspaces [5]. In this case it is clear that the condition
$$A \oplus B \in \mathcal{S}(\mathcal{C}) \Rightarrow A \in \mathcal{S}(\mathcal{C})$$
is satisfied.
2. Without the assumption $A \oplus B \in \mathcal{S}(\mathcal{C}) \Rightarrow A \in \mathcal{S}(\mathcal{C})$, the implications (3)⇒(4)⇒(1)⇒(2) remain true.

The following example, which derives from Example 1, shows that not every part-class $\mathcal{C}$ that is closed under $*$-strong limits has the property that $\mathcal{S}(\mathcal{C})$ is closed under approximate equivalence.

EXAMPLE 2. Suppose $K \in B(H)$ is a compact operator with zero kernel and commuting with no idempotents. Choose an irreducible operator T that is similar to $K \oplus 0$. Let $\mathcal{C}$ denote the class of all operators unitarily equivalent to $T^{(m)} \oplus 0^{(k)}$ for cardinals $k, m \geq 0$ with $k+m > 0$. Since $C^*(T) = \mathcal{K}(H)$, it follows that $\mathcal{C}$ is the class of all operators of the form $\pi(T)$, where π is a representation of $C^*(T)$. It follows that each element of $\mathcal{C}$, and thus each element of $\mathcal{S}(\mathcal{C})$, has nonzero kernel. However, $K \sim_a K \oplus 0 \in \mathcal{S}(\mathcal{C})$, but $K \notin \mathcal{S}(\mathcal{C})$ since $\ker K = \{0\}$. Thus $\mathcal{S}(\mathcal{C})$ is not closed under approximate equivalence. Note, however, that $K \oplus 0 \in \mathcal{S}(\mathcal{C})$, but $K \notin \mathcal{S}(\mathcal{C})$. Hence $\mathcal{C}$ does not satisfy the condition $A \oplus B \in \mathcal{S}(\mathcal{C}) \Rightarrow A \in \mathcal{S}(\mathcal{C})$.

A natural test case is the class $\mathcal{C}$ of *hyponormal* operators ($T^*T - TT^* \geq 0$). This class is closed under strong limits, so we have $A \oplus B \in \mathcal{S}(\mathcal{C}) \Rightarrow A \in \mathcal{S}(\mathcal{C})$. Moreover, this class contains the classes of normal and subnormal operators.

QUESTION 3. If T is similar to a hyponormal operator and $S \sim_{as} T$, must S be similar to a hyponormal operator?

Voiculescu [13] proved that if two operators are approximately equivalent, then one is unitarily equivalent to arbitrarily small *compact* perturbations of the other. If Question 1 has an affirmative answer, then it follows that approximately similar operators must be similar modulo the compact operators.

QUESTION 4. Is every pair of approximately similar operators similar in the Calkin algebra?

A natural class of operators for which unitary equivalence in the Calkin algebra is understood is the class of essentially normal operators [1]. This class is not closed under direct sums, so it is not a part-class. However, this class is a natural one for investigating Question 4. Since approximate similarity preserves spectrum, essential spectrum and Fredholm index, it follows from [1] that approximately similar essentially normal operators are unitarily equivalent modulo the compact operators.

Here is a specific example. Suppose U is the unilateral shift operator with multiplicity 1, and let $T = U \oplus U^*$. What are the operators that are approximately similar to T? If $S \sim_{as} T$, then $\sigma(S)$ is the closed unit disk, the essential spectrum $\sigma_e(S)$ is the unit circle, and if $\nu : B(H) \to B(H)/\mathcal{K}(H)$ is the Calkin map, then

$$\sup_{n \in \mathbb{Z}} \|\nu(S)^n\| < \infty.$$

It follows from a theorem of Sz.-Nagy [10] that if $\rho : B(H)/\mathcal{K}(H) \to B(M)$ is a unital $*$-homomorphism, then $\rho(\nu(S))$ is similar to a unitary operator. Is $\nu(S)$ similar to a unitary element in the Calkin algebra? Since T is a compact perturbation of the bilateral shift operator, $\mathrm{Ind}(S) = 0$. Hence if $\nu(S)$ were similar to a unitary element, it would follow from [1] that $\nu(S)$ is similar to $\nu(T)$. Note that, since $\ker T^2 \neq \ker T$, we know that T is not similar to a normal operator, and hence T is not approximately similar to a normal operator.

QUESTION 5. What are the operators that are approximately similar to $U \oplus U^*$?

Another natural place to look for evidence is among the class of weighted shift operators. Both similarity [12] and approximate equivalence [11] are well understood among the class of weighted shift operators. Moreover, $(\mathcal{U} + \mathcal{K})$-equivalence is understood among these operators [8].

QUESTION 6. When are two weighted shifts approximately similar?

The smallest part class containing all weighted shifts that is closed under approximate equivalence is the class of centered operators [9]. An operator T is a *centered operator* if

$$\{(T^*)^n T^n : n \geq 0\} \cup \{T^n (T^*)^n : n > 0\}$$

is a commuting family.

QUESTION 7. If $A \oplus B$ is similar to a centered operator, must A be similar to a centered operator? If T is approximately similar to a centered operator, must T be similar to some centered operator?

The fact that an operator that is similar to a normal operator must be similar to another normal operator combined with the fact that approximately similar normal operators are approximately equivalent [4] gives a complete characterization of the class of operators that are approximately similar to a given normal operator.

PROPOSITION 2. *Suppose $T \in B(H)$ is normal. An operator S is approximately similar to T if and only if S is a scalar-type spectral operator, $\sigma(S) = \sigma(T)$, and the isolated eigenvalues of S and T have the same multiplicities.*

It was shown in [4] that an operator approximately similar to a subnormal operator must be similar to some subnormal operator. However, approximately similar subnormal operators need not be approximately equivalent [6]. Thus the question of the operators that are approximately similar to a given subnormal operators reduces to the following.

QUESTION 8. When are two subnormal operators approximately similar? When are they similar?

We conclude with a natural concept related to Question 1 above. Suppose S and T are approximately similar operators. We define $\ell(S, T)$ to be the minimal length of a chain $S = W_1, \ldots, W_n = T$ where each

W_k is either similar or approximately equivalent to W_{k+1} for $1 \le k < n$. If there is no such chain, we define $\ell(S,T) = \infty$. Question 1 asks if $\ell(S,T) < \infty$ for every pair S,T of approximately similar operators. We define $\ell(S,T) = 0$ when S is not approximately similar to T. Here is a stronger version of Question 1.

QUESTION 9. Is $\sup\{\ell(S,T) : S,T \in B(H)\} < \infty$?

Clearly, an affirmative answer to Question 9 implies an affirmative answer to Question 1. There is a certain reverse implication, but it involves the norms of the similarities and their inverses that implement the approximate similarity being uniformly bounded. Suppose $M \ge 1$ and $S,T \in B(H)$. We say that $S \sim_{as,M} T$ if there is a sequence $\{A_k\}$ of invertible operators such that $\left\|A_k^{-1} S A_k - T\right\| \to 0$ and $\sup \|A_k\| \left\|A_k^{-1}\right\| \le M$. The relation $\sim_{as,M}$ is symmetric and reflexive, but generally not transitive. However, $\sim_{as,1}$ is the same as $\sim_a$. If $\{A_n\}$ is an invertibly bounded sequence, then the set

$$\{S : \{A_n^{-1} S A_n\} \text{ is norm convergent}\}$$

is a norm-closed inverse-closed unital algebra and the map $\rho(S) = \lim_{n\to\infty} A_n^{-1} S A_n$ is a bounded unital homomorphism on this algebra.

LEMMA 2. *Suppose $\{U_n\}$ is a sequence of unitary operators in $B(H)$, and $S,T,P,Q \in B(H)$ are such that*

1. *P,Q are idempotents, and*
2. *$U_n^* S U_n \to T$ and $U_n^* P U_n \to Q$.*

Then $S|\operatorname{ran} P \sim_a T|\operatorname{ran} Q$.

Proof. Define $S_1 = S|\operatorname{ran} P$ and $T_1 = T|\operatorname{ran} Q$. Let $Q_n = U_n^* P U_n$, let $A_n = Q_n Q + (1-Q_n)(1-Q)$, and let $W_n = U_n A_n$ for $n = 1, 2, \ldots$. Since $A_n \to 1$, we can assume that each A_n is invertible and $W_n^{-1} S W_n \to T$. Moreover, since $Q_n A_n = A_n Q$, it follows that

$$W_n^{-1} P W_n = A_n^{-1} U_n^* P U_n A_n = A_n^{-1} Q_n A_n = Q,$$

and

$$W_n^* W_n \to 1, W_n W_n^* \to 1.$$

Thus $W_n(\operatorname{ran} Q) = \operatorname{ran} P$ for every n, and $B_n = W_n|\operatorname{ran} Q : \operatorname{ran} Q \to \operatorname{ran} P$ is invertible, $B_n^{-1} = W_n^{-1}|\operatorname{ran} P$, and $B_n^* = W_n^*|\operatorname{ran} P$. Hence $B_n^{-1} S_1 B_n \to T_1$ and $B_n B_n^* = W_n W_n^*|\operatorname{ran} P \to 1|\operatorname{ran} P$. It follows that $V_n = (B_n^* B_n)^{-\frac{1}{2}} B_n$ is the unitary part of the polar decomposition of B_n and $V_n^{-1} S_1 V_n \to T_1$. Therefore $S_1 \sim_a T_1$. □

COROLLARY 3. *Suppose $S \sim_{as} T$ and Ω_1, Ω_2 are disjoint open sets that cover $\sigma(S)$ and such that each intersects $\sigma(S)$. Let f be the function on $\Omega = \Omega_1 \cup \Omega_2$ that is 1 on Ω_1 and 0 on Ω_2. If $S_1 = S|\mathrm{ran} f(S)$ and $T_1 = T|\mathrm{ran} f(T)$, then $\ell(S_1, T_1) \leq \ell(S, T)$.*

Proof. If A and B are operators whose spectrum is $\sigma(S)$, then any bounded unital homomorphism on the inverse-closed norm closed unital algebra generated by A that sends A to B must also send $f(A)$ to $f(B)$. Hence if W is invertible and $W^{-1}AW = B$, then $W^{-1}f(A)W = f(B)$, and if $X = W|\mathrm{ran} f(B)$, then

$$X^{-1}\left(A|\mathrm{ran} f(A)\right) X = B|\mathrm{ran} f(B).$$

It follows from the preceding lemma that if $A \sim_a B$, then $A|\mathrm{ran} f(A) \sim_a B|\mathrm{ran} f(B)$. The corollary is now obvious. □

PROPOSITION 3. *Suppose $M \geq 1$, $\{S_n\}, \{T_n\}$ are sequences in $B(H)$ such that $\ell(S_n, T_n) \to \infty$ and such that $S_n \sim_{as,M} T_n$ for $n = 1, 2, \dots$. Then there are operators $S, T \in B(H)$ such that $\ell(S, T) = \infty$.*

Proof. Since, for each n, we can replace S_n and T_n with $\alpha S_n + \beta_n$ and $\alpha T_n + \beta_n$, respectively, without changing $\ell(S_n, T_n)$, we can assume that there is a strictly decreasing sequence $\{d_n\}$ of positive numbers such that $d_n \to 0$, and if $r_n = (d_n - d_{n+1})/3$, then $\sigma(S_n) \subset D(d_n, r_n)$ and $\|S_n\| \leq d_n + r_n$.

Let $S = \sum^{\oplus}_{1 \leq n < \infty} S_n$ and $T = \sum^{\oplus}_{1 \leq n < \infty} T_n$. Then $S \sim_{as,M} T$. However, if $n \geq 1$ and $\Omega_{n,1} = D(d_n, r_n)$ and $\Omega_{n,2} = \mathbb{C} \backslash \overline{D(d_n, r_n)}$, and if $f_n(z) = 1$ on $\Omega_{n,1}$ and $f_n(z) = 0$ on $\Omega_{n,2}$, then $S|\mathrm{ran} f_n(S) = S_n$ and $T|\mathrm{ran} f_n(T) = T_n$. It follows from the preceding corollary that $\ell(S, T) \geq \ell(S_n, T_n)$ for each n. Hence, $\ell(S, T) = \infty$. □

ACKNOWLEDGEMENT. The author would like to thank Professor Jo Ward for many valuable suggestions concerning this paper.

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