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HADAMARD-TYPE FRACTIONAL CALCULUS

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ABSTRACT. The paper is devoted to the study of fractional integration and differentiation on a finite interval $[a, b]$ of the real axis in the frame of Hadamard setting. The constructions under consideration generalize the modified integration $\int_a^x (t/x)^\mu f(t)dt/t$ and the modified differentiation $\delta + \mu$ ($\delta = xD$, $D = d/dx$) with real μ, being taken n times. Conditions are given for such a Hadamard-type fractional integration operator to be bounded in the space $X_c^p(a, b)$ of Lebesgue measurable functions f on $\mathbf{R}_+ = (0, \infty)$ such that

$$\int_a^b |t^c f(t)|^p \frac{dt}{t} < \infty \quad (1 \leq p < \infty),$$
$$\text{ess sup}_{a \leq t \leq b}[u^c |f(t)|] < \infty \quad (p = \infty),$$

for $c \in \mathbf{R} = (-\infty, \infty)$, in particular in the space $L^p(0, \infty)$ $(1 \leq p \leq \infty)$. The existence almost everywhere is established for the corresponding Hadamard-type fractional derivative for a function $g(x)$ such that $x^\mu g(x)$ have δ derivatives up to order $n-1$ on $[a, b]$ and $\delta^{n-1}[x^\mu g(x)]$ is absolutely continuous on $[a, b]$. Semigroup and reciprocal properties for the above operators are proved.

1. Introduction

The purpose of this paper is to develop fractional integration and differentiation in the Hadamard setting. For natural $n \in \mathbf{N} = \{1, 2, \cdots\}$ and real μ and $a \geq 0$ such an aproach is based on the nth integral of the form

$$(1.1) \quad \begin{aligned}(\mathcal{J}_{\mu,a+}^n f)(x) &= x^{-\mu} \int_a^x \frac{dt_1}{t_1} \int_a^{t_1} \frac{dt_2}{t_2} \cdots \int_a^{t_{n-1}} t_n^\mu f(t_n) \frac{dt_n}{t_n} \\ &= \frac{1}{(n-1)!} \int_a^x \left(\frac{t}{x}\right)^\mu \left(\log \frac{x}{t}\right)^{n-1} f(t) \frac{dt}{t} \quad (x > a)\end{aligned}$$

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and the corresponding derivative

$$(1.2)\qquad \begin{aligned}(\mathcal{D}^1_{\mu,a+}g)(x) &= ((\delta+\mu)f)(x) = xf'(x)+\mu f(x), \quad \delta = x\frac{d}{dx},\\ \mathcal{D}^n_{\mu,a+}g &= \mathcal{D}^1_{\mu,a+}(\mathcal{D}^{n-1}_{\mu,a+}g)\ (n=2,3,\cdots)\ (x>a).\end{aligned}$$

The fractional versions of the integral (1.1) and the derivative (1.2) are given by

$$(1.3)\quad (\mathcal{J}^\alpha_{a+,\mu}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\frac{t}{x}\right)^\mu \left(\log\frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t}\ (\alpha>0;\ x>a)$$

and

$$(1.4)\qquad (\mathcal{D}^\alpha_{a+,\mu}g)(x) = x^{-\mu}\delta^n x^\mu \left(\mathcal{J}^{n-\alpha}_{0+,\mu}g\right)(x),\ \delta = x\frac{d}{dx}\quad (\alpha>0;\ n=[\alpha]+1,\ \mu\in\mathbf{R}),$$

respectively, $[\alpha]$ being integral part of α. When $\mu=0$, (1.3) and (1.4) take the forms

$$(1.5)\qquad (\mathcal{J}^\alpha_{a+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\log\frac{x}{u}\right)^{\alpha-1} f(u)\frac{du}{u}\ (\alpha>0;\ x>a)$$

and

$$(1.6)\qquad (\mathcal{D}^\alpha_{a+}y)(x) = \delta^n\left(\mathcal{J}^{n-\alpha}_{a+,\mu}g\right)(x),\ \delta = x\frac{d}{dx}\ (\alpha>0;\ n=[\alpha]+1).$$

The integral (1.5) was introduced by Hadamard [5] in the case $a=0$ and therefore $\mathcal{J}^\alpha_{a+}f$ and $\mathcal{D}^\alpha_{a+}f$ are often referred to as Hadamard fractional integral and derivative of order α [6, Section 18.3 and Section 23.1, notes to Section 18.3]. Therefore we may call the more general constructions in (1.3) and (1.4) Hadamard-type fractional integral and derivative of order α.

It is well developed an approach to fractional calculus by Riemann and Liouville based on the generalization of usual integration $\int_a^x f(t)dt$ and differentiation $D=d/dx$, see for example [6, Chapters 2 and 3]. Hadamard fractional calculus approach is studied less. Some facts for the Hadamard calculus operators (1.5) and (1.6) were presented in [6, Section 18.3]. The Mellin approach was suggested in [1] to study the properties of the operators $\mathcal{J}^\alpha_{0+,\mu}$ and $\mathcal{D}^\alpha_{0+,\mu}$ defined on the the half-axis $\mathbf{R}_+=(0,\infty)$. Some properties of the operator $\mathcal{J}^\alpha_{0+,\mu}$ and three of its modifications were invetsigated in [1]-[3].

The aim of this paper is to study the properties of the Hadamard-type fractional operators (1.3) and (1.4) on a finite interval $[a,b]$ of the real line $\mathbf{R}=(-\infty,\infty)$ for $a>0$. The paper is organized as follows. First in Section 2 we give conditions for the operator $\mathcal{J}^{\alpha}_{a+,\mu}f$ to be bounded in the space $X^p_c(a,b)$ ($c \in \mathbf{R}$, $1 \le p \le \infty$) of those complex-valued Lebesgue measurable functions f on $[a,b]$ for which $\|f\|_{X^p_c} < \infty$, where

$$\|f\|_{X^p_c} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{1/p} \quad (1 \le p < \infty,\ c \in \mathbf{R}) \tag{1.7}$$

and

$$\|f\|_{X^\infty_c} = \text{ess sup}_{a \le t \le b}[t^c |f(t)|] \quad (c \in \mathbf{R}). \tag{1.8}$$

In particular, when $c = 1/p$ ($1 \le p \le \infty$), the space $X^p_c(a,b)$ coincides with the classical $L^p(a,b)$-space: $L^p(a,b) \equiv X^p_{1/p}(a,b)$ with

$$\begin{aligned} \|f\|_p &= \left(\int_a^b |f(t)|^p dt\right)^{1/p} \quad (1 \le p < \infty), \\ \|f\|_\infty &= \text{ess sup}_{a \le t \le b}|f(t)|. \end{aligned} \tag{1.9}$$

Next in Section 3 we prove that the Hadamard-type fractional derivative $\mathcal{D}^{\alpha}_{a+,\mu}g$ exists almost everywhere for a function $g(x) \in AC^n_{\delta;\mu}[a,b]$ such that $x^\mu g(x)$ have $\delta = xD$ ($D = d/dx$) derivatives up to order $n-1$ on $[a,b]$ and $\delta^{n-1}[x^\mu g(x)]$ is absolutely continuous on $[a,b]$:

$$\begin{aligned} AC^n_{\delta;\mu}[a,b] = \{h :\ [a,b] \to \mathbf{C} : \delta^{n-1}[x^\mu h(x)] \in AC[a,b], \\ \mu \in \mathbf{R};\ \delta = x\frac{x}{dx}\}. \end{aligned} \tag{1.10}$$

Here $AC[a,b]$ is the set of absolutely continuous functions on $[a,b]$ which coincide with the space of primitives of Lebesgue measurable functions:

$$h(x) \in AC[a,b] \Leftrightarrow h(x) = c + \int_a^x \psi(t)dt,\ \psi(t) \in L_(a,b), \tag{1.11}$$

see, for example, [6, (1.4)].

In conclusion in Section 4 we establish semigroup and reciprocal properties for the operators $\mathcal{J}^{\alpha}_{a+,\mu}$ and $\mathcal{D}^{\alpha}_{a+,\mu}$.

We note that the corresponding results for the Hadamard fractional calculus operators (1.5) and (1.6) are also presented in Sections 2-4.

2. Hadamard-type fractional integration in the space $X^p_c(a,b)$

In this section we show that the Hadamard-type fractional integration operator $\mathcal{J}^{\alpha}_{a+,\mu}$ is defined on $X^p_c(a,b)$ for $\mu \ge c$. To formulate the result we need the incomplete gamma-function $\gamma(\nu,x)$ defined for $\nu > 0$ and $x > 0$ by [4, 6.9(2)]:

$$\gamma(\nu,x) = \int_0^x t^{\nu-1}e^{-t}dt. \tag{2.1}$$

THEOREM 2.1. *Let $\alpha > 0$, $1 \le p \le \infty$, $0 < a < b < \infty$ and let $\mu \in \mathbf{R}$ and $c \in \mathbf{R}$ be such that $\mu \ge c$. Then the operator $\mathcal{J}^{\alpha}_{a+,\mu}$ is bounded in $X^p_c(a,b)$ and*

$$\|\mathcal{J}^{\alpha}_{a+,\mu}f\|_{X^p_c} \le K\|f\|_{X^p_c}, \tag{2.2}$$

where

$$K = \frac{1}{\Gamma(\alpha+1)}\left(\log\frac{b}{a}\right)^{\alpha} \tag{2.3}$$

for $\mu = c$, while

$$K = \frac{1}{\Gamma(\alpha)}(\mu-c)^{-\alpha}\gamma\left[\alpha,(\mu-c)\log\left(\frac{b}{a}\right)\right] \tag{2.4}$$

for $\mu > c$.

Proof. First consider the case $1 \le p < \infty$. Since $f(t) \in X^p_c(a,b)$, then $t^{c-1/p}f(t) \in L_p(a,b)$ and we can apply the generalized Minkowsky inequality (see, for example, [6, (1.33)]. In accordance with (1.3) and (1.7) we have

$$\begin{aligned}
\|\mathcal{J}^{\alpha}_{a+,\mu}f\|_{X^p_c} &= \left(\int_a^b x^{cp}\left|\frac{1}{\Gamma(\alpha)}\int_a^x\left(\frac{t}{x}\right)^{\mu}\left(\log\frac{x}{t}\right)^{\alpha-1}f(t)\frac{dt}{t}\right|^p\frac{dx}{x}\right)^{1/p}\\
&= \left(\int_a^b\left|\frac{1}{\Gamma(\alpha)}\int_1^{x/a}x^{c-1/p}u^{-\mu}(\log u)^{\alpha-1}f\left(\frac{x}{u}\right)\frac{du}{u}\right|^p dx\right)^{1/p}\\
&\le \int_1^{b/a}u^{-\mu-1}(\log u)^{\alpha-1}\left(\int_{at}^b x^{cp}\left|f\left(\frac{x}{u}\right)\right|^p\frac{dx}{x}\right)^{1/p}dt\\
&= \int_1^{b/a}u^{c-\mu-1}(\log u)^{\alpha-1}\left(\int_a^{b/u}|t^cf(t)|^p\frac{dt}{t}\right)^{1/p}du
\end{aligned}$$

and hence

$$\|\mathcal{J}^{\alpha}_{a+,\mu}f\|_{X^p_c} \le M\|f\|_{X^p_c},$$

where

$$M = \int_1^{b/a}u^{c-\mu-1}(\log u)^{\alpha-1}du.$$

Direct calculation show that M coincides with K given in (2.3) and (2.4), when $\mu = c$ and $\mu > c$, respectively. Thus (2.2) is proved for $1 \le p < \infty$.

Let now $p = \infty$. By (1.3) and (1.8) we have

$$|x^c(\mathcal{J}^\alpha_{a+,\mu}f)(x)| \le \frac{1}{\Gamma(\alpha)} \int_a^x \left(\frac{t}{x}\right)^{\mu-c} \left(\log\frac{x}{t}\right)^{\alpha-1} |t^c f(t)| \frac{dt}{t}$$

and thus

$$|x^c(\mathcal{J}^\alpha_{a+,\mu}f)(x)| \le K(x)\|f\|_{X^\infty_c}, \tag{2.5}$$

where

$$K(x) = \int_1^{x/a} u^{-(\mu-c)}(\log u)^{\alpha-1}\frac{du}{u}.$$

When $\mu = c$, then for any $a \le x \le b$

$$K(x) = \frac{1}{\Gamma(\alpha+1)}\left(\log\frac{x}{a}\right)^\alpha \le \frac{1}{\Gamma(\alpha+1)}\left(\log\frac{b}{a}\right)^\alpha. \tag{2.6}$$

If $\mu > c$, then making the change of variable $(\mu - c)u = y$ and taking (2.1) into account we find

$$K(x) = \frac{1}{\Gamma(\alpha)}(\mu - c)^{-\alpha}\gamma\left[\alpha, (\mu - c)\log\left(\frac{x}{a}\right)\right].$$

$\gamma(\nu, x)$ is increasing function and thus

$$K(x) \le \frac{1}{\Gamma(\alpha)}(\mu - c)^{-\alpha}\gamma\left[\alpha, (\mu - c)\log\left(\frac{b}{a}\right)\right] \tag{2.7}$$

for any $a \le x \le b$. It follows from (2.5)-(2.7) that for any $a \le x \le b$

$$|x^c(\mathcal{J}^\alpha_{a+,\mu}f)(x)| \le K\|f\|_{X^\infty_c}, \tag{2.8}$$

where K is given by (2.3) and (2.4) when $\mu = c$ and $\mu > c$, respectively. Hence, in accordance with (1.8), from (2.8) we obtain the result in (2.2) for $p = \infty$. This completes the proof of theorem. □

Putting $c = 1/p$ in Theorem 2.1 and taking (1.9) into account, we deduce the boundedness of the operator $\mathcal{J}^\alpha_{a+,\mu}$ in the space $L^p(a, b)$.

COROLLARY 2.2. *Let $\alpha > 0$, $1 \le p \le \infty$, $0 < a < b < \infty$ and let $\mu \in \mathbf{R}$ be such taht $\mu \ge 1/p$. Then the operator $\mathcal{J}^\alpha_{a+,\mu}$ is bounded in $L^p(a, b)$ and*

$$\|\mathcal{J}^\alpha_{a+,\mu}f\|_p \le K_1\|f\|_p, \tag{2.9}$$

where K_1 *is given by* (2.3) *for* $\mu = 1/p$, *while*

$$K_1 = \frac{1}{\Gamma(\alpha)}\left(\mu - \frac{1}{p}\right)^{-\alpha} \gamma\left[\alpha, \left(\mu - \frac{1}{p}\right)\log\left(\frac{b}{a}\right)\right] \tag{2.10}$$

for $\mu > 1/p$.

Setting $\mu = 0$ in Theorem 2.1 we obtain the corresponding statement for the Hadamard fractional operator $\mathcal{J}^{\alpha}_{a+}$ in (1.5).

THEOREM 2.3. *Let* $\alpha > 0$, $1 \le p \le \infty$, $0 < a < b < \infty$ *and let* $c \le 0$. *Then the operator* $\mathcal{J}^{\alpha}_{a+}$ *is bounded in* $X^p_c(a,b)$ *and*

$$\|\mathcal{J}^{\alpha}_{a+} f\|_{X^p_c} \le K_2 \|f\|_{X^p_c}, \tag{2.11}$$

where

$$K_2 = \frac{1}{\Gamma(\alpha+1)}\left(\log\frac{b}{a}\right)^{\alpha} \tag{2.12}$$

for $c = 0$, *while*

$$K_2 = \frac{1}{\Gamma(\alpha)}(-c)^{-\alpha}\gamma\left[\alpha, -c\log\left(\frac{b}{a}\right)\right] \tag{2.13}$$

for $c < 0$.

REMARK 2.4. It follows from [1, Theorem 4(a)] that if $\mu > c$, then the Hadamard-type fractional operator $\mathcal{J}^{\alpha}_{0+,\mu}$ is bounded in $X^p_c(\mathbf{R}_+)$ and

$$\|\mathcal{J}^{\alpha}_{0+,\mu} f\|_{X^p_c} \le K_3 \|f\|_{X^p_c}.$$

Such a result formally follow from (2.2) and (2.4) if we put $a = 0$, $b = \infty$ and take into account the relation

$$\gamma(\nu, \infty) = \Gamma(\nu). \tag{2.14}$$

REMARK 2.5. It follows from Corollary 2.2 that the operator $\mathcal{J}^{\alpha}_{a+,\mu}$ is bounded in $L^p(a,b)$ for $\mu \ge 1/p$. Similar result can not be obtained from Theorem 2.3 for the Hadamard fractional operator $\mathcal{J}^{\alpha}_{a+}$. This fact leads to conjecture that the operator $\mathcal{J}^{\alpha}_{a+}$ is probably bounded from $L^p(a,b)$ into another space.

REMARK 2.6. The results in Theorem 2.1 and Theorem 2.3 for Hadamard-type and Hadamard fractional integration operators are analogues of those for the classsical Riemann-Liouville fractional integrals (see [6, Theorem 2.6]). We only note that the weighted space $X^p_c(a,b)$ is suitable for the former, while the space $L_p(a,b)$ for the latter.

3. Hadamard-type fractional differentiation in the space $AC^n_{\delta;\mu}[a,b]$

In this section we give sufficient conditions for the existence of the Hadamard-type fractional derivative $\mathcal{D}^{\alpha}_{a+,\mu}g$ in (1.4). Since the result will be state in terms of the space $AC^n_{\delta;\mu}[a,b]$ defined in (1.10), we first characterize this space.

THEOREM 3.1. *The space $AC^n_{\delta;\mu}[a,b]$ consists of those and only those functions $g(x)$, which are represented in the form*

$$g(x) = x^{-\mu}\left[\frac{1}{(n-1)!}\int_a^x \left(\log\frac{x}{t}\right)^{n-1}\varphi(t)dt + \sum_{k=0}^{n-1} c_k\left(\log\frac{x}{a}\right)^k\right], \tag{3.1}$$

where $\varphi(t) \in L^1(a,b)$ and c_k $(k=0,1,\cdots,n-1)$ are arbitrary constants.

Proof. First prove necessity. Let $g(x) \in AC^n_{\delta;\mu}[a,b]$, where $\delta = xD$ $(D = d/dx)$. Then by (1.10) $\delta^{n-1}[x^\mu g(x)] \in AC[a,b]$ and hence by (1.11)

$$\delta^{n-1}[x^\mu g(x)] = \int_a^x \varphi(t)dt + c_{n-1}, \tag{3.2}$$

where $\varphi(t) \in L^1(a,b)$ and c_{n-1} is an arbitrary constant. Rewrite (3.2) in the form

$$\frac{d}{dx}\delta^{n-2}[x^\mu g(x)] = \frac{1}{x}\int_a^x \varphi(t)dt + \frac{c_{n-1}}{x}.$$

Changing x to t and t to u and integration both sides of this relation we have

$$\delta^{n-2}[x^\mu g(x)] = \int_a^x \log\frac{x}{t}\varphi(t)dt + c_{n-2} + c_{n-1}\log\frac{x}{a},$$

where c_{n-2} and c_{n-1} are arbitrary constants. Repeating this procedure m $(1 \le m \le n-1)$ times we obtain

$$\begin{aligned}\delta^{n-m}[x^\mu g(x)] = &\int_a^x \left(\log\frac{x}{t}\right)^{m-1}\frac{\varphi(t)}{(m-1)!}dt \\ &+\sum_{k=0}^{m}\frac{c_{n-1+k-m}}{k!}\left(\log\frac{x}{a}\right)^k,\end{aligned} \tag{3.3}$$

where $c_{n-m-1},\cdots,c_{n-1}$ are arbitrary constants. Now (3.3) with $m=n$ yields (3.1), and necessity is proved.

Let now $g(x)$ is represented by (3.1), or

$$x^\mu g(x) = \int_a^x \left(\log\frac{x}{t}\right)^{n-1} \frac{\varphi(t)}{(n-1)!}dt + \sum_{k=0}^{n-1} c_k \left(\log\frac{x}{a}\right)^k .$$

Taking δ-derivative m $(1 \leq m \leq n-1)$ times, we have

$$\delta^m[x^\mu g(x)] = \int_a^x \left(\log\frac{x}{t}\right)^{n-m-1} \frac{\varphi(t)}{(n-m-1)!}dt + \sum_{k=m}^{n-1} \frac{k!c_k}{(k-m)!} \left(\log\frac{x}{a}\right)^{k-m} .$$

From here for $m = n-1$ we obtain (3.2) (with $c = (n-1)!c_{n-1}$) and hence, in accordance with (1.10) and (1.11), $g(x) \in AC^n_{\delta;\mu}[a,b]$. This completes the proof of theorem. □

Note that it follows from our proof that $\varphi(t)$ and c_k are given by

$$\varphi(t) = g'_{n-1}(t),\ c_k = \frac{g_k(a)}{k!}\ (k = 0, 1, \cdots, n-1), \tag{3.4}$$

where

$$g_k(x) = \delta^k[x^\mu g(x)]\ (k = 0, 1, \cdots, n-1),\ g_0(x) = x^\mu g(x). \tag{3.5}$$

Hence (3.1) can be rewritten in the form

$$g(x) = x^{-\mu}\left[\int_a^x \left(\log\frac{x}{t}\right)^{n-1} \frac{g'_{n-1}(t)}{(n-1)!}dt + \sum_{k=0}^{n-1} \frac{g_k(a)}{k!}\left(\log\frac{x}{a}\right)^k\right], \tag{3.6}$$

Now we ready to prove the result giving sufficient conditions for the existence of the Hadamard-type fractional derivative (1.4).

THEOREM 3.2. *Let $\alpha > 0$, $n = [\alpha]+1$, $\mu \in \mathbf{R}$ and $g(x) \in AC^n_{\delta;\mu}[a,b]$. Then the Hadamard-type fractional derivative $\mathcal{D}^\alpha_{a+,\mu}g$ exists almost everywhere on $[a,b]$ and may be represented in the form*

$$(\mathcal{D}^\alpha_{a+,\mu}g)(x) = x^{-\mu}\Big[\frac{1}{\Gamma(n-\alpha)}\int_a^x \left(\log\frac{x}{t}\right)^{n-\alpha-1} g'_{n-1}(t)dt + \sum_{k=0}^{n-1} \frac{g_k(a)}{\Gamma(k-\alpha+1)}\left(\log\frac{x}{a}\right)^{k-\alpha}\Big], \tag{3.7}$$

where $g_k(a)$ $(k = 0, 1, \cdots, n-1)$ are given by (3.5).

Proof. Since $g(x) \in AC^n_{\delta;\mu}[a,b]$, we have representation (3.6). Substituting this relation into (1.4) we have

$$(\mathcal{D}^\alpha_{a+,\mu}g)(x) = x^{-\mu}\delta^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\log\frac{x}{t}\right)^{n-\alpha-1} \times \Bigg[\int_a^t \left(\log\frac{t}{u}\right)^{n-1} \frac{g'_{n-1}(u)}{(n-1)!}du + \sum_{k=0}^{n-1} \frac{g_k(a)}{k!}\left(\log\frac{t}{a}\right)^{k-\alpha} \Bigg]\frac{dt}{t}. \tag{3.8}$$

Interchanging the order of integration and applying the Dirichlet formula (see, for example, [6, (1.33)]) we have

$$\int_a^x \left(\log\frac{x}{t}\right)^{n-\alpha-1}\frac{dt}{t}\int_a^t\left(\log\frac{t}{u}\right)^{n-1} g'_{n-1}(u)du = \int_a^x g'_{n-1}(u)du\int_u^x \left(\log\frac{x}{t}\right)^{n-\alpha-1}\left(\log\frac{t}{u}\right)^{n-1}\frac{dt}{t}.$$

The inner integral is evaluated by the change of variable $y = \log(t/u)/\log(x/u)$ and using the formulas [4, 1.5(1) and 1.5(5)] for the beta function:

$$\int_u^x \left(\log\frac{x}{t}\right)^{n-\alpha-1}\left(\log\frac{t}{u}\right)^{n-1}\frac{dt}{t} = \frac{\Gamma(n-\alpha)\Gamma(n)}{\Gamma(2n-\alpha)}\left(\log\frac{x}{u}\right)^{n-\alpha-1}$$

and hence

$$\int_a^x \left(\log\frac{x}{t}\right)^{n-\alpha-1}\frac{dt}{t}\int_a^t\left(\log\frac{t}{u}\right)^{n-1} g'_{n-1}(u)du = \frac{\Gamma(n-\alpha)\Gamma(n)}{\Gamma(2n-\alpha)}\int_a^x\left(\log\frac{x}{u}\right)^{2n-\alpha-1} g'_{n-1}(u)du.$$

Substituting this relation into (3.8) and taking δ^n-differentiation, we obtain (3.7). Thus theorem is proved. □

COROLLARY 3.3. *If* $0 < \alpha < 1$, $\mu \in \mathbf{R}$ *and* $g(x) \in AC^1_{\delta;\mu}[a,b]$, *then* $\mathcal{D}^\alpha_{a+,\mu}g$ *exists almost everywhere on* $[a,b]$ *and*

$$(\mathcal{D}^\alpha_{a+,\mu}g)(x) = \frac{x^{-\mu}}{\Gamma(1-\alpha)}\Bigg[\int_a^x\left(\log\frac{x}{t}\right)^{-\alpha}[t^\mu g(t)]'\frac{dt}{t} + \lim_{t\to a+}[t^\mu g(t)]\left(\log\frac{x}{a}\right)^{-\alpha}\Bigg]. \tag{3.9}$$

When $\mu = 0$, from Theorem 3.2 we deduce sufficient conditions for the existence of the Hadamard fractional derivative (1.6).

THEOREM 3.4. *Let $\alpha > 0$, $n = [\alpha] + 1$ and $g(x) \in AC^n_{\delta;0}[a,b]$. Then the Hadamard fractional derivative $\mathcal{D}^\alpha_{a+}g$ exists almost everywhere on $[a,b]$ and may be represented in the form*

$$(3.10)\qquad \begin{aligned}(\mathcal{D}^\alpha_{a+}g)(x) &= \frac{1}{\Gamma(n-\alpha)}\int_a^x \left(\log\frac{x}{t}\right)^{n-\alpha-1}(\delta^n g)(t)dt\\ &+\sum_{k=0}^{n-1}\frac{(\delta^k g)(a)}{\Gamma(k-\alpha+1)}\left(\log\frac{x}{a}\right)^{k-\alpha}.\end{aligned}$$

COROLLARY 3.5. *If $0 < \alpha < 1$ and $g(x) \in AC^1_{\delta;0}[a,b]$, then $\mathcal{D}^\alpha_{a+}g$ exists almost everywhere on $[a,b]$ and*

$$(3.11)\qquad \begin{aligned}(\mathcal{D}^\alpha_{a+}g)(x) &= \frac{1}{\Gamma(1-\alpha)}\int_a^x \left(\log\frac{x}{t}\right)^{-\alpha} g'(t)\frac{dt}{t}\\ &+\frac{g(a)}{\Gamma(1-\alpha)}\left(\log\frac{x}{a}\right)^{-\alpha}.\end{aligned}$$

REMARK 3.6. The results in Theorem 3.2 and Theorem 3.4 for Hadamard-type and Hadamard fractional differentiation operators are analogues of those for the classsical Riemann-Liouville fractional derivatives (see [6, Theorem 2.2]). We only note that the weighted space $AC^n_{\delta;\mu}[a,b]$ is suitable for the former, while the space $AC^n[a,b]$ for the latter.

4. Semigroup and reciprocal properties of Hadamard-type fractional calculus operators

First we prove the semigroup property for the Hadamard-type fractional integration operator $\mathcal{J}^\alpha_{a+,\mu}$ in (1.3).

THEOREM 4.1. *Let $\alpha > 0$, $\beta > 0$, $1 \leq p \leq \infty$, $0 < a < b < \infty$ and let $\mu \in \mathbf{R}$ and $c \in \mathbf{R}$ be such that $\mu \geq c$. Then for $f \in X^p_c(a,b)$ the semigroup property holds*

$$(4.1)\qquad \mathcal{J}^\alpha_{a+,\mu}\mathcal{J}^\beta_{a+,\mu}f = \mathcal{J}^{\alpha+\beta}_{a+,\mu}f.$$

Proof. First we prove (4.1) for "sufficiently good" functions f. Applying Fubini's theorem we find

$$
\begin{aligned}
&(\mathcal{J}^{\alpha}_{a+,\mu}\mathcal{J}^{\beta}_{a+,\mu}f)(x)\\
&= \frac{1}{\Gamma(\alpha)}\int_a^x \left(\frac{u}{x}\right)^{\mu}\left(\log\frac{x}{u}\right)^{\alpha-1}\frac{du}{u}\\
&\times \frac{1}{\Gamma(\beta)}\int_a^u \left(\frac{t}{u}\right)^{\mu}\left(\log\frac{u}{t}\right)^{\beta-1} f(t)\frac{dt}{t} \qquad (4.2)\\
&= \frac{x^{-\mu}}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x t^{\mu-1}f(t)dt\int_t^x \left(\log\frac{x}{u}\right)^{\alpha-1}\left(\log\frac{u}{t}\right)^{\beta-1}\frac{du}{u}.
\end{aligned}
$$

The inner integral is evaluated by the change of variable $\tau = \log\,(u/t)/\log(x/t)$:

$$\int_t^x \left(\log\frac{x}{u}\right)^{\alpha-1}\left(\log\frac{u}{t}\right)^{\beta-1}\frac{du}{u} = \left(\log\frac{x}{t}\right)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Substituting this relation into (4.2) and taking (1.3) into account we have

$$
\begin{aligned}
(\mathcal{J}^{\alpha}_{a+,\mu}\mathcal{J}^{\beta}_{a+,\mu}f)(x) &= \frac{1}{\Gamma(\alpha+\beta)}\int_a^x \left(\frac{u}{x}\right)^{\mu}\left(\log\frac{x}{u}\right)^{\alpha+\beta-1}\frac{du}{u}\\
&= (\mathcal{J}^{\alpha+\beta}_{a+,\mu}f)(x),
\end{aligned}
$$

and thus (4.1) is proved for "sufficiently good" functions f.

If $\mu \geq c$, then by Theorem 2.1 the operators $\mathcal{J}^{\alpha}_{a+,\mu}$, $\mathcal{J}^{\beta}_{a+,\mu}$ and $\mathcal{J}^{\alpha+\beta}_{a+,\mu}$ are bounded in $X^p_c(a,b)$, hence the relation (4.1) is true for $f \in X^p_c(a,b)$. This completes the proof of theorem. □

COROLLARY 4.2. *Let $\alpha > 0$, $\beta > 0$, $1 \leq p \leq \infty$, $0 < a < b < \infty$ and let $\mu \in \mathbf{R}$ be such that $\mu \leq 1/p$. Then for $f \in L^p(a,b)$ the semigroup property* (4.1) *holds.*

When $m = 0$, from Theorem 4.1 and Theorem 2.3 we obtain the semigroup property for the Hadamard fractional integration operators (1.5).

THEOREM 4.3. *Let $\alpha > 0$, $\beta > 0$, $1 \leq p \leq \infty$, $0 < a < b < \infty$ and $c \leq 0$. Then for $f \in X^p_c(a,b)$ the semigroup property holds*

$$\mathcal{J}^{\alpha}_{a+}\mathcal{J}^{\beta}_{a+}f = \mathcal{J}^{\alpha+\beta}_{a+}f. \qquad (4.3)$$

Next we consider the composition between the operators of Hadamard-type fractional differentiation (1.4) and fractional integration (1.3).

THEOREM 4.4. *Let $\alpha > \beta > 0$, $1 \leq p \leq \infty$, $0 < a < b < \infty$ and let $\mu \in \mathbf{R}$ and $c \in \mathbf{R}$ be such that $\mu \geq c$. Then for $f \in X_c^p(a,b)$ there holds*

$$\mathcal{D}_{a+,\mu}^{\beta}\mathcal{J}_{a+,\mu}^{\alpha}f = \mathcal{J}_{a+,\mu}^{\alpha-\beta}f. \tag{4.4}$$

In particular, if $\beta = m \in \mathbf{N}$, then

$$\mathcal{D}_{a+,\mu}^{m}\mathcal{J}_{a+,\mu}^{\alpha}f = \mathcal{J}_{a+,\mu}^{\alpha-m}f. \tag{4.5}$$

Proof. Let $m-1 < \beta \leq m$ $(m \in \mathbf{N})$. If $\beta = m$, then by (1.4)

$$(\mathcal{D}_{a+,\mu}^{m}y)(x) = x^{-\mu}\left(x\frac{d}{dx}\right)^{m} x^{\mu}y(x) \tag{4.6}$$

and hence

$$(\mathcal{D}_{a+,\mu}^{m}\mathcal{J}_{a+,\mu}^{\alpha}f)(x) = x^{-\mu}\left(x\frac{d}{dx}\right)^{m-1} x\frac{d}{dx}\frac{1}{\Gamma(\alpha)}\int_a^x u^{\mu}\left(\log\frac{x}{u}\right)^{\alpha-1} f(u)\frac{du}{u}.$$

Applying the formula of differentaition under the integral sign and using the relation for the gamma-function [4, 1.2(1)] and (1.3) we obtain

$$\begin{aligned}
&(\mathcal{D}_{a+,\mu}^{m}\mathcal{J}_{a+,\mu}^{\alpha}f)(x)\\
&= x^{-\mu}\left(x\frac{d}{dx}\right)^{m-1} \frac{x}{\Gamma(\alpha)}\int_a^x u^{\mu}\frac{\partial}{\partial x}\left(\log\frac{x}{u}\right)^{\alpha-1} f(u)\frac{du}{u}\\
&= x^{-\mu}\left(x\frac{d}{dx}\right)^{m-1} \frac{1}{\Gamma(\alpha-1)}\int_a^x u^{\mu}\frac{\partial}{\partial x}\left(\log\frac{x}{u}\right)^{\alpha-2} f(u)\frac{du}{u}\\
&= x^{-\mu}\left(x\frac{d}{dx}\right)^{m-1} x^{\mu}\mathcal{J}_{a+,\mu}^{\alpha-1}f)(x).
\end{aligned}$$

Repeating this procedure k $(1 \leq k \leq m)$ times we have

$$(\mathcal{D}_{a+,\mu}^{m}\mathcal{J}_{a+,\mu}^{\alpha}f)(x) = x^{-\mu}\left(x\frac{d}{dx}\right)^{m-k} x^{\mu}(\mathcal{J}_{a+,\mu}^{\alpha-k}f)(x),$$

and (4.5) follows for $k = m$.

If $m-1 < \beta < m$, then (4.4) follows from (4.1) and (4.5):

$$\mathcal{D}_{a+,\mu}^{\beta}\mathcal{J}_{a+,\mu}^{\alpha}f = \mathcal{D}_{a+,\mu}^{m}\mathcal{J}_{a+,\mu}^{m-\beta}\mathcal{J}_{a+,\mu}^{\alpha}f = \mathcal{D}_{a+,\mu}^{m}\mathcal{J}_{a+,\mu}^{m+\alpha-\beta}f = \mathcal{J}_{a+,\mu}^{\alpha-\beta}f.$$

Thus theorem is proved. □

COROLLARY 4.5. *Let $\alpha > \beta > 0$, $1 \le p \le \infty$, $0 < a < b < \infty$ and let $\mu \in \mathbf{R}$ be such that $\mu \ge 1/p$. Then for $f \in L^p(a,b)$ the relation (4.4) holds. In particular, (4.5) is valid for $\beta = m \in \mathbf{N}$.*

THEOREM 4.6. *Let $\alpha > \beta > 0$, $1 \le p \le \infty$, $0 < a < b < \infty$ and $c \le 0$. Then for $f \in X_c^p(a,b)$ the relation holds*

$$\mathcal{D}_{a+}^{\beta}\mathcal{J}_{a+}^{\alpha}f = \mathcal{J}_{a+}^{\alpha-\beta}f. \tag{4.7}$$

In particular, if $\beta = m \in \mathbf{N}$, then

$$\mathcal{D}_{a+}^{m}\mathcal{J}_{a+}^{\alpha}f = \mathcal{J}_{a+}^{\alpha-m}f. \tag{4.8}$$

Theorem 4.4 is also true for $\alpha = \beta$ which means that the Hadamard-type fractional differentiation (1.6) and integration (1.5) are reciprocal operations if the former is applied first. The result below is proved similarly to the proof of Theorem 4.4.

THEOREM 4.7. *Let $\alpha > 0$, $1 \le p \le \infty$, $0 < a < b < \infty$ and let $\mu \in \mathbf{R}$ and $c \in \mathbf{R}$ be such that $\mu \ge c$. Then for $f \in X_c^p(a,b)$ there holds*

$$\mathcal{D}_{a+,\mu}^{\alpha}\mathcal{J}_{a+,\mu}^{\alpha}f = f. \tag{4.9}$$

In particular, if $\mu \ge 1/p$, then (4.10) is valid for $f \in L^p(a,b)$.

THEOREM 4.8. *Let $\alpha > 0$, $1 \le p \le \infty$, $0 < a < b < \infty$ and let $c \le 0$. Then for $f \in X_c^p(a,b)$ there holds*

$$\mathcal{D}_{a+}^{\alpha}\mathcal{J}_{a+}^{\alpha}f = f. \tag{4.10}$$

REMARK 4.9. It follows from [2, Theorem 1(a)] that if $\alpha > 0$, $\beta > 0$, $1 \le p \le \infty$ and $\mu > c$, then for $f \in X_c^p(\mathbf{R}_+)$ the semigroup property

$$\mathcal{J}_{0+,\mu}^{\alpha}\mathcal{J}_{0+,\mu}^{\beta}f = \mathcal{J}_{0+,\mu}^{\alpha+\beta}f \tag{4.11}$$

holds. Such a result follows from (4.1) if we put $a = 0$ and $b = \infty$ and take into account that the operators $\mathcal{J}_{0+,\mu}^{\alpha}$, $\mathcal{J}_{0+,\mu}^{\beta}$ and $\mathcal{J}_{0+,\mu}^{\alpha+\beta}$ are bounded in $X_c^p(\mathbf{R}_+)$ when $\mu > c$.

REMARK 4.10. The results presented in Theorems 4.7 and 4.8 show that the Hadamard-type and Hadamard fractional differentiation (1.4) and (1.6) and integration (1.3) and (1.5) are reciprocal operations if the formers are applied first. It is the problem when the latters can be applied first. Such a problem is solved for the Riemann-Liouville fractional calculus operators (see [6, Theorem 2.4]).

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