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INTEGRAL GRÜSS INEQUALITY FOR MAPPINGS WITH VALUES IN HILBERT SPACES AND APPLICATIONS

S. S. Dragomir

Abstract. In this paper we prove a version of Grüss' integral inequality for mappings with values in Hilbert spaces. Some applications for convex functions defined on Hilbert spaces are also given.

1. Introduction

In 1935, G. Grüss proved the following integral inequality [12]

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma), \tag{1.1}$$

provided that f and g are two integrable functions on $[a,b]$ and satisfy the condition

$$\phi\leq f(x)\leq\Phi \text{ and } \gamma\leq g(x)\leq\Gamma \text{ for a.e. } x\in[a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is the *best possible* and is achieved for

$$f(x)=g(x)=\operatorname{sgn}\left(x-\frac{a+b}{2}\right).$$

The discrete version of (1.1) states that:

If $a\leq a_i\leq A$, $b\leq b_i\leq B$ $(i=1,\dots,n)$ where a,A,a_i,b,B,b_i are real numbers, then

$$\left|\frac{1}{n}\sum_{i=1}^n a_ib_i-\frac{1}{n}\sum_{i=1}^n a_i\cdot\frac{1}{n}\sum_{i=1}^n b_i\right|\leq\frac{1}{4}(A-a)(B-b), \tag{1.3}$$

where the constant $\frac{1}{4}$ is the best possible.

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For an entire chapter devoted to the history of this inequality see the book [14] where further references are given.

New results in the domain can be found in the papers [1]-[9] and [13].

In the recent paper [2], the author proved the following generalization in inner product spaces.

THEOREM 1. *Let* $(X, \langle \cdot, \cdot \rangle)$ *be an inner product space over* $\mathbb{K}, \mathbb{K} = \mathbb{C}, \mathbb{R}$ *and* $e \in X, \|e\| = 1$. *If* $\phi, \Phi, \gamma, \Gamma \in \mathbb{K}$ *and* $x, y \in X$ *are such that*

$$\operatorname{Re} \langle \Phi e - x, x - \phi e \rangle \geq 0 \text{ and } \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0, \tag{1.4}$$

holds, then we have the inequality

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma|. \tag{1.5}$$

The constant $\frac{1}{4}$ *is the best possible.*

It has been shown in [1] that the above theorem, for real cases, contains the usual integral and discrete Grüss inequality and also some Grüss type inequalities for mappings defined on infinite intervals.

Namely, if $\rho : (-\infty, \infty) \to (-\infty, \infty)$ is a probabilistic density function, i.e., $\int_{-\infty}^{\infty} \rho(t)\, dt = 1$, then $\rho^{\frac{1}{2}} \in L^2(-\infty, \infty)$ and obviously $\left\| \rho^{\frac{1}{2}} \right\|_2 = 1$. Consequently, if we assume that $f, g \in L^2(-\infty, \infty)$ and

$$\alpha \cdot \rho^{\frac{1}{2}} \leq f \leq \psi \cdot \rho^{\frac{1}{2}},\ \beta \cdot \rho^{\frac{1}{2}} \leq g \leq \theta \cdot \rho^{\frac{1}{2}} \text{ a.e. on } (-\infty, \infty), \tag{1.6}$$

then we have the inequality

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} f(t) g(t)\, dt - \int_{-\infty}^{\infty} f(t) \rho^{\frac{1}{2}}(t)\, dt \cdot \int_{-\infty}^{\infty} g(t) \rho^{\frac{1}{2}}(t)\, dt \right| \\ &\leq \frac{1}{4} (\psi - \alpha)(\theta - \beta). \end{aligned} \tag{1.7}$$

Similarly, if $l = (l_i)_{i \in \mathbb{N}} \in l^2(\mathbb{R})$ with $\sum_{i \in \mathbb{N}} |l_i|^2 = 1$ and $x = (x_i)_{i \in \mathbb{N}}, y = (y_i)_{i \in \mathbb{N}} \in l^2(\mathbb{R})$ are such that

$$\alpha \cdot l_i \leq x_i \leq \psi \cdot l_i, \beta \cdot l_i \leq y_i \leq \theta \cdot l_i \tag{1.8}$$

for all $i \in \mathbb{N}$, then we have

$$\left| \sum_{i \in \mathbb{N}} x_i y_i - \sum_{i \in \mathbb{N}} x_i l_i \cdot \sum_{i \in \mathbb{N}} y_i l_i \right| \leq \frac{1}{4} (\psi - \alpha)(\theta - \beta). \tag{1.9}$$

In this paper we point out a Grüss type inequality for Bochner measurable functions with values in Hilbert spaces.

2. Preliminary results

The following lemma holds.

LEMMA 1. *Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over the real or complex number field $\mathbb{K}$, $f : \Omega \to H$ a Bochner measurable function on $\Omega \subset \mathbb{R}^n$ such that $f \in L_2(\Omega, H)$ and $\rho : \Omega \to [0, \infty)$ a Lebesgue integrable mapping on Ω so that $\int_\Omega \rho(t)\, d\mu(t) = 1$. If there exists the vectors $x, X \in H$ such that*

$$\operatorname{Re}\langle X - f(t), f(t) - x \rangle \geq 0 \text{ for a.e. } t \in \Omega, \tag{2.1}$$

then we have the inequality

$$\begin{aligned} 0 &\leq \int_\Omega \rho(t) \|f(t)\|^2 d\mu(t) - \left\| \int_\Omega \rho(t) f(t)\, d\mu(t) \right\|^2 \\ &\leq \frac{1}{4} \|X - x\|^2 . \end{aligned} \tag{2.2}$$

The constant $\frac{1}{4}$ is sharp.

Proof. Define

$$I_1 := \left\langle X - \int_\Omega \rho(t) f(t)\, d\mu(t), \int_\Omega \rho(t) f(t)\, d\mu(t) - x \right\rangle$$

and

$$I_2 := \int_\Omega \rho(t) \langle X - f(t), f(t) - x \rangle\, d\mu(t).$$

Then obviously

$$\begin{aligned} I_1 &= \int_\Omega \rho(t) \langle X, f(t) \rangle\, d\mu(t) - \langle X, x \rangle \\ &\quad - \left\| \int_\Omega \rho(t) f(t)\, d\mu(t) \right\|^2 + \int_\Omega \rho(t) \langle f(t), x \rangle\, d\mu(t) \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_\Omega \rho(t) \langle X, f(t) \rangle\, d\mu(t) - \langle X, x \rangle \\ &\quad - \int_\Omega \rho(t) \|f(t)\|^2 d\mu(t) + \int_\Omega \rho(t) \langle f(t), x \rangle\, d\mu(t). \end{aligned}$$

Consequently

$$I_1 - I_2 = \int_\Omega \rho(t) \|f(t)\|^2 d\mu(t) - \left\| \int_\Omega \rho(t) f(t)\, d\mu(t) \right\|^2 . \tag{2.3}$$

Taking the real value in (2.3), we can state

$$\begin{aligned}(2.4)\qquad &\int_{\Omega}\rho(t)\|f(t)\|^{2}d\mu(t)-\left\|\int_{\Omega}\rho(t)f(t)d\mu(t)\right\|^{2}\\ &= \operatorname{Re}\left\langle X-\int_{\Omega}\rho(t)f(t)d\mu(t),\int_{\Omega}\rho(t)f(t)d\mu(t)-x\right\rangle\\ &\quad-\int_{\Omega}\rho(t)\operatorname{Re}\langle X-f(t),f(t)-x\rangle d\mu(t)\end{aligned}$$

which is an identity of interest in itself.

Using the assumption (2.1), we may conclude, by (2.4), that

$$\begin{aligned}(2.5)\qquad &\int_{\Omega}\rho(t)\|f(t)\|^{2}d\mu(t)-\left\|\int_{\Omega}\rho(t)f(t)d\mu(t)\right\|^{2}\\ &\le \operatorname{Re}\left\langle X-\int_{\Omega}\rho(t)f(t)d\mu(t),\int_{\Omega}\rho(t)f(t)d\mu(t)-x\right\rangle.\end{aligned}$$

It is known that if $y,z\in H$, then

$$(2.6)\qquad 4\operatorname{Re}\langle z,y\rangle\le\|z+y\|^{2},$$

with equality iff $z=y$.

Now, by (2.6), we can state that

$$\begin{aligned}&\operatorname{Re}\left\langle X-\int_{\Omega}\rho(t)f(t)d\mu(t),\int_{\Omega}\rho(t)f(t)d\mu(t)-x\right\rangle\\ &\le\frac{1}{4}\left\|X-\int_{\Omega}\rho(t)f(t)d\mu(t)+\int_{\Omega}\rho(t)f(t)d\mu(t)-x\right\|^{2}\\ &=\frac{1}{4}\|X-x\|^{2}.\end{aligned}$$

Using (2.5), we deduce (2.2).

To prove the sharpness of the constant $\frac{1}{4}$, let us assume that the inequality (2.2) holds with a constant $c>0$. That is,

$$\begin{aligned}&\int_{\Omega}\rho(t)\|f(t)\|^{2}d\mu(t)-\left\|\int_{\Omega}\rho(t)f(t)d\mu(t)\right\|^{2}\\ &\le c\|X-x\|^{2}\end{aligned}$$

for any Ω, μ, ρ, f and x, X as above.

Choose $\Omega = \{1, 2\}$, $\rho(1) = \rho(2) = \frac{1}{2}$, μ is the discrete measure on Ω and $f(1) = x$, $f(2) = X$ $(x \neq X)$. Then obviously

$$\begin{aligned} \int_\Omega \rho(t) \|f(t)\|^2 d\mu(t) &= \frac{\|x\|^2 + \|X\|^2}{2}, \\ \left\| \int_\Omega \rho(t) f(t) d\mu(t) \right\|^2 &= \left\| \frac{x + X}{2} \right\|^2 \end{aligned}$$

and by (2.1), we deduce

$$\frac{\|x\|^2 + \|X\|^2}{2} - \left\| \frac{x + X}{2} \right\|^2 \leq c \|X - x\|^2$$

which is clearly equivalent to

$$\frac{1}{4} \|X - x\|^2 \leq c \|X - x\|^2$$

implying that $c \geq \frac{1}{4}$. □

Remark 1. The assumption (2.1) can be replaced by the more general condition

$$\int_\Omega \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle d\mu(t) \geq 0 \tag{2.7}$$

and the conclusion (2.2) will still be valid.

The following corollary is natural.

Corollary 1. *Let $g : \Omega \to \mathbb{K}$ be Lebesgue integrable on Ω and ρ be as above. If $a, A \in \mathbb{K}$ are such that*

$$\operatorname{Re} \left[(A - g(t)) \left(\overline{g(t)} - \bar{a} \right) \right] \geq 0 \text{ a.e on } \Omega, \tag{2.8}$$

then we have the inequality

$$\begin{aligned} 0 &\leq \int_\Omega \rho(t) |g(t)|^2 d\mu(t) - \left| \int_\Omega \rho(t) g(t) d\mu(t) \right|^2 \\ &\leq \frac{1}{4} |A - a|^2. \end{aligned} \tag{2.9}$$

The constant $\frac{1}{4}$ is sharp.

Remark 2. The condition (2.8) can be replaced by the more general assumption

$$\int_\Omega \rho(t) \operatorname{Re} \left[(A - g(t)) \left(\overline{g(t)} - \bar{a} \right) \right] d\mu(t) \geq 0. \tag{2.10}$$

REMARK 3. If we assume that $\mathbb{K} = \mathbb{R}$, then (2.8) is equivalent with

$$a \le g(t) \le A \text{ a.e. on } \Omega \tag{2.11}$$

and then, with the assumption (2.11), we get the Grüss type inequality

$$\begin{aligned} 0 &\le \int_\Omega \rho(t) g^2(t) \, d\mu(t) - \left(\int_\Omega \rho(t) g(t) \, d\mu(t) \right)^2 \\ &\le \frac{1}{4} (A - a)^2 . \end{aligned} \tag{2.12}$$

3. An inequality of Grüss type

The following Grüss type inequality holds.

THEOREM 2. *Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{K}$, $\mathbb{K} = \mathbb{C}$, $\mathbb{R}$, $f, g : \Omega \to \mathbb{K}$ be Bochner measurable and $f, g \in L_2(\Omega, H)$ and $\rho : \Omega \to [0, \infty)$ Lebesgue integrable so that $\int_\Omega \rho(t) \, d\mu(t) = 1$. If there exists the vectors $x, X, y, Y \in H$ such that*

$$\operatorname{Re} \langle X - f(t), f(t) - x \rangle \ge 0 \text{ and } \operatorname{Re} \langle Y - g(t), g(t) - y \rangle \ge 0 \text{ for a.e. } t \text{ on } \Omega, \tag{3.1}$$

then we have the inequality

$$\begin{aligned} &\left| \int_\Omega \rho(t) \langle f(t), g(t) \rangle \, d\mu(t) \right. \\ &\left. \quad - \left\langle \int_\Omega \rho(t) f(t) \, d\mu(t), \int_\Omega \rho(t) g(t) \, d\mu(t) \right\rangle \right| \\ &\le \frac{1}{4} \|X - x\| \, \|Y - y\| . \end{aligned} \tag{3.2}$$

The constant $\frac{1}{4}$ is sharp.

Proof. A simple calculation shows that

$$\begin{aligned} &\int_\Omega \rho(t) \langle f(t), g(t) \rangle \, d\mu(t) \\ &\quad - \left\langle \int_\Omega \rho(t) f(t) \, d\mu(t), \int_\Omega \rho(t) g(t) \, d\mu(t) \right\rangle \\ &= \frac{1}{2} \int_\Omega \int_\Omega \rho(t) \rho(s) \langle f(t) - f(s), g(t) - g(s) \rangle \, d\mu(t) \, d\mu(s) . \end{aligned} \tag{3.3}$$

Taking the modulus in both parts of (3.3), and using the generalised triangle inequality, we obtain

$$\begin{aligned}
&\left|\int_{\Omega}\rho(t)\langle f(t),g(t)\rangle d\mu(t)\right.\\
(3.4)\qquad&\left.-\left\langle\int_{\Omega}\rho(t)f(t)\,d\mu(t),\int_{\Omega}\rho(t)g(t)\,d\mu(t)\right\rangle\right|\\
&\leq\frac{1}{2}\int_{\Omega}\int_{\Omega}\rho(t)\rho(s)\left|\langle f(t)-f(s),g(t)-g(s)\rangle\right|d\mu(t)\,d\mu(s).
\end{aligned}$$

By Schwartz's inequality in inner product spaces, we have

$$|\langle f(t)-f(s),g(t)-g(s)\rangle|\leq\|f(t)-f(s)\|\,\|g(t)-g(s)\| \tag{3.5}$$

for all $t,s\in\Omega$, and therefore

$$\begin{aligned}
&\left|\int_{\Omega}\rho(t)\langle f(t),g(t)\rangle d\mu(t)\right.\\
(3.6)\qquad&\left.-\left\langle\int_{\Omega}\rho(t)f(t)\,d\mu(t),\int_{\Omega}\rho(t)g(t)\,d\mu(t)\right\rangle\right|\\
&\leq\frac{1}{2}\int_{\Omega}\int_{\Omega}\rho(t)\rho(s)\|f(t)-f(s)\|\,\|g(t)-g(s)\|\,d\mu(t)\,d\mu(s).
\end{aligned}$$

Using the Cauchy-Buniakowski-Schwartz inequality for double integrals, we can state that

$$\begin{aligned}
(3.7)\qquad&\frac{1}{2}\int_{\Omega}\int_{\Omega}\rho(t)\rho(s)\|f(t)-f(s)\|\,\|g(t)-g(s)\|\,d\mu(t)\,d\mu(s)\\
\leq\;&\left(\frac{1}{2}\int_{\Omega}\int_{\Omega}\rho(t)\rho(s)\|f(t)-f(s)\|^{2}d\mu(t)\,d\mu(s)\right)^{\frac{1}{2}}\\
&\times\left(\frac{1}{2}\int_{\Omega}\int_{\Omega}\rho(t)\rho(s)\|g(t)-g(s)\|^{2}d\mu(t)\,d\mu(s)\right)^{\frac{1}{2}}.
\end{aligned}$$

As a simple calculation shows that

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega}\int_{\Omega}\rho(t)\rho(s)\|f(t)-f(s)\|^{2}d\mu(t)\,d\mu(s)\\
=\;&\int_{\Omega}\rho(t)\|f(t)\|^{2}d\mu(t)-\left\|\int_{\Omega}\rho(t)f(t)\,d\mu(t)\right\|^{2}
\end{aligned}$$

and

$$\frac{1}{2}\int_{\Omega}\int_{\Omega}\rho(t)\rho(s)\|g(t)-g(s)\|^2 d\mu(t)\,d\mu(s) = \int_{\Omega}\rho(t)\|g(t)\|^2 d\mu(t) - \left\|\int_{\Omega}\rho(t)g(t)\,d\mu(t)\right\|^2,$$

then we obtain

$$\begin{aligned}&\left|\int_{\Omega}\rho(t)\langle f(t),g(t)\rangle d\mu(t) - \left\langle\int_{\Omega}\rho(t)f(t)\,d\mu(t),\int_{\Omega}\rho(t)g(t)\,d\mu(t)\right\rangle\right| \\ &\le \left(\int_{\Omega}\rho(t)\|f(t)\|^2 d\mu(t) - \left\|\int_{\Omega}\rho(t)f(t)\,d\mu(t)\right\|^2\right)^{\frac{1}{2}} \\ &\times \left(\int_{\Omega}\rho(t)\|g(t)\|^2 d\mu(t) - \left\|\int_{\Omega}\rho(t)g(t)\,d\mu(t)\right\|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{3.8}$$

Using Lemma 1, we know that

$$\left(\int_{\Omega}\rho(t)\|f(t)\|^2 d\mu(t) - \left\|\int_{\Omega}\rho(t)f(t)\,d\mu(t)\right\|^2\right)^{\frac{1}{2}} \le \frac{1}{2}\|X-x\|$$

and

$$\left(\int_{\Omega}\rho(t)\|g(t)\|^2 d\mu(t) - \left\|\int_{\Omega}\rho(t)g(t)\,d\mu(t)\right\|^2\right)^{\frac{1}{2}} \le \frac{1}{2}\|Y-y\|$$

and then, by (3.8), we deduce the desired inequality (3.2).

The sharpness of the constant is obvious by Lemma 1 and we omit the details. □

REMARK 4. The condition (3.1) can be replaced by the more general assumption:

$$\begin{aligned}\int_{\Omega}\rho(t)\operatorname{Re}\langle X-f(t),f(t)-x\rangle d\mu(t) &\ge 0, \\ \int_{\Omega}\rho(t)\operatorname{Re}\langle Y-g(t),g(t)-y\rangle d\mu(t) &\ge 0\end{aligned} \tag{3.9}$$

and the conclusion (3.2) still remains valid.

The following corollary for real or complex functions holds.

COROLLARY 2. *Let $f, g : \Omega \to \mathbb{K}$ be in $L_2(\Omega, \mathbb{K})$ and $\rho : \Omega \to [0, \infty)$ be as above. If $a, A, b, B \in \mathbb{K}$ are such that*

$$\begin{aligned} \operatorname{Re}\left[(A - f(t))\left(\overline{f(t)} - \bar{a}\right)\right] &\geq 0, \\ \operatorname{Re}\left[(B - g(t))\left(\overline{g(t)} - \bar{b}\right)\right] &\geq 0 \text{ for a.e. } t \in \Omega, \end{aligned} \tag{3.10}$$

then we have the inequality

$$\begin{aligned} &\left| \int_\Omega \rho(t) f(t) \bar{g}(t) \, d\mu(t) - \int_\Omega \rho(t) f(t) \, d\mu(t) \cdot \int_\Omega \rho(t) \bar{g}(t) \, d\mu(t) \right| \\ \leq\ & \frac{1}{4} |A - a| \, |B - b| \end{aligned} \tag{3.11}$$

and the constant $\frac{1}{4}$ is sharp.

The proof is obvious by Theorem 2 applied for the Hilbert space $(\mathbb{K}, \langle \cdot, \cdot \rangle)$, $\langle x, y \rangle = x \cdot \bar{y}$. We omit the details.

REMARK 5. The condition (3.10) can be replaced by the more general condition

$$\begin{aligned} \int_\Omega \rho(t) \operatorname{Re}\left[(A - f(t))\left(\overline{f(t)} - \bar{a}\right)\right] d\mu(t) &\geq 0, \\ \int_\Omega \rho(t) \operatorname{Re}\left[(B - g(t))\left(\overline{g(t)} - \bar{b}\right)\right] d\mu(t) &\geq 0 \end{aligned} \tag{3.12}$$

and the conclusion of the above corollary will still remain valid.

REMARK 6. If we assume that f, g, a, b, A, B are real, then (3.10) is equivalent to

$$a \leq f(t) \leq A, \quad b \leq g(t) \leq B \text{ for a.e. } t \text{ on } \Omega, \tag{3.13}$$

and (3.11) becomes

$$\begin{aligned} &\left| \int_\Omega \rho(t) f(t) g(t) \, d\mu(t) - \int_\Omega \rho(t) f(t) \, d\mu(t) \cdot \int_\Omega \rho(t) g(t) \, d\mu(t) \right| \\ \leq\ & \frac{1}{4} (A - a)(B - b) \end{aligned} \tag{3.14}$$

which is the classical Grüss inequality for real valued functions.

4. Applications for convex functions

Let $(H;\langle\cdot,\cdot\rangle)$ be a real Hilbert space and $F:H\to\mathbb{R}$ a Fréchet differentiable convex mapping on H. Then we have the "gradient inequality", that is,

$$F(x)-F(y)\geq\langle\nabla F(y),x-y\rangle \tag{4.1}$$

for all $x,y\in H$, where $\nabla F:H\to H$ is the gradient operator associated to the convex function F.

The following theorem containing a reverse inequality for Jensen's inequality for the measurable space Ω with $\mu(\Omega)<\infty$, holds.

THEOREM 3. *Let $F:H\to\mathbb{R}$ be as above, $f:\Omega\to H$ a Bochner measurable function such that there exists the vectors n, $N\in H$ with the property*

$$\langle f(t)-n,N-f(t)\rangle\geq 0\quad \text{a.e. on } \Omega, \tag{4.2}$$

and the vectors $m,M\in H$ such that

$$\langle\nabla F(f(t))-m,M-\nabla F(f(t))\rangle\geq 0\ \text{ a.e. on } \Omega. \tag{4.3}$$

If $\rho:\Omega\to[0,\infty)$ is Lebesgue measurable on Ω such that $\int_\Omega\rho(t)\,d\mu(t)>0$ and $\rho F(f(\cdot))\in L(\Omega)$, $\rho\cdot f\in L(\Omega,H)$, then we have the inequality

$$\begin{aligned}0 &\leq \frac{1}{\int_\Omega\rho(t)\,d\mu(t)}\int_\Omega\rho(t)F(f(t))\,d\mu(t)\\ &\quad -F\left(\frac{1}{\int_\Omega\rho(t)\,d\mu(t)}\int_\Omega\rho(t)f(t)\,d\mu(t)\right)\\ &\leq \frac{1}{4}\|N-n\|\,\|M-m\|.\end{aligned} \tag{4.4}$$

Proof. Choose in (4.1)

$$x=\frac{1}{\int_\Omega\rho(t)\,d\mu(t)}\int_\Omega\rho(t)f(t)\,d\mu(t)\quad\text{and}\quad y=f(s)$$

to obtain

$$\begin{aligned}&F\left(\frac{1}{\int_\Omega\rho(t)\,d\mu(t)}\int_\Omega\rho(t)f(t)\,d\mu(t)\right)-F(f(s))\\ &\geq\left\langle\nabla F(f(s)),\frac{1}{\int_\Omega\rho(t)\,d\mu(t)}\int_\Omega\rho(t)f(t)\,d\mu(t)-f(s)\right\rangle\end{aligned} \tag{4.5}$$

for a.e. $s\in\Omega$.

If we multiply (4.5) by $\rho(s) \geq 0$ and integrate on Ω, we have

$$\begin{aligned}
&\int_{\Omega} \rho(s)\, d\mu(s)\, F\left(\frac{1}{\int_{\Omega} \rho(t)\, d\mu(t)} \int_{\Omega} \rho(t)\, f(t)\, d\mu(t)\right) \\
&- \int_{\Omega} \rho(s)\, (F \circ f)(s)\, d\mu(s) \\
\geq\ & \frac{1}{\int_{\Omega} \rho(t)\, d\mu(t)} \left\langle \int_{\Omega} \rho(t) \left((\nabla F) \circ (f)\right)(s)\, d\mu(t), \int_{\Omega} \rho(t)\, f(t)\, d\mu(t) \right\rangle \\
&- \int_{\Omega} \rho(s) \left\langle \nabla F(f(s)), f(s) \right\rangle d\mu(s).
\end{aligned}$$

Dividing by $\int_{\Omega} \rho(t)\, d\mu(t) > 0$, we obtain the inequality

$$\begin{aligned}
(4.6) \qquad 0 \leq\ & \frac{1}{\int_{\Omega} \rho(t)\, d\mu(t)} \int_{\Omega} \rho(s)\, (F \circ f)(s)\, d\mu(s) \\
& - F\left(\frac{1}{\int_{\Omega} \rho(t)\, d\mu(t)} \int_{\Omega} \rho(t)\, f(t)\, d\mu(t)\right) \\
\leq\ & \frac{1}{\int_{\Omega} \rho(t)\, d\mu(t)} \int_{\Omega} \rho(t) \left\langle ((\nabla F) \circ f)(t), f(t) d\mu(t) \right\rangle \\
& - \left\langle \frac{1}{\int_{\Omega} \rho(t)\, d\mu(t)} \int_{\Omega} \rho(t)\, (\nabla F \circ f)(t)\, d\mu(t), \right. \\
& \qquad \left. \frac{1}{\int_{\Omega} \rho(t)\, d\mu(t)} \int_{\Omega} \rho(t)\, f(t)\, d\mu(t) \right\rangle
\end{aligned}$$

which is a generalisation for the case of inner product spaces of the result by Dragomir-Goh established in 1996 for the case of differentiable mappings defined on $\mathbb{R}^n$ [10].

Applying Theorem 1 for the case of real inner product spaces, $X = N$, $x = n$, $g(t) = (\nabla F \circ \rho)(t)$, $y = m$, $Y = M$, we easily deduce

$$\begin{aligned}
(4.7) \qquad & \frac{1}{\int_{\Omega} \rho(t)\, d\mu(t)} \int_{\Omega} \rho(t) \left\langle (\nabla F \circ f)(t), f(t) \right\rangle d\mu(t) \\
& - \left\langle \frac{1}{\int_{\Omega} \rho(t)\, d\mu(t)} \int_{\Omega} \rho(t)\, (\nabla F \circ f)(t)\, d\mu(t), \right. \\
& \qquad \left. \frac{1}{\int_{\Omega} \rho(t)\, d\mu(t)} \int_{\Omega} \rho(t)\, f(t)\, d\mu(t) \right\rangle \\
\leq\ & \frac{1}{4} \|N - n\|\, \|M - m\|
\end{aligned}$$

and then, by (4.6) and (4.7), we can conclude that the desired inequality (4.4) holds. □

REMARK 7. The conditions (4.2) and (4.3) can be replaced by the more general assumptions

$$\int_{\Omega} \rho(t) \langle f(t) - n, N - f(t) \rangle \, d\mu(t) \geq 0 \tag{4.8}$$

and

$$\int_{\Omega} \rho(t) \langle (\nabla F \circ f)(t) - m, M - (\nabla F \circ f)(t) \rangle \, d\mu(t) \geq 0 \tag{4.9}$$

and the conclusion (4.4) will still be valid.

REMARK 8. Even if the inequality (4.4) is not as sharp as (4.6), it can be more useful in practice when only some bounds of the gradient operator ∇F and the mapping f are known. On other words, it provides the opportunity to estimate the difference

$$\begin{aligned} \Delta(F, f, \rho) \quad := \quad & \frac{1}{\int_{\Omega} \rho(t) \, d\mu(t)} \int_{\Omega} \rho(t) (F \circ f)(t) \, d\mu(t) \\ & -F\left(\frac{1}{\int_{\Omega} \rho(t) \, d\mu(t)} \int_{\Omega} \rho(t) f(t) \, d\mu(t) \right), \end{aligned}$$

when the differences $\|N - n\|$ and $\|M - m\|$ are known.

For example, if we know that

$$\begin{aligned} \langle (\nabla F \circ f)(t) - m, M - (\nabla F \circ f)(t) \rangle \quad & \geq \quad 0 \quad \text{for a.e. } t \in \Omega, \\ \langle f(t) - n, N - f(t) \rangle \quad & \geq \quad 0 \quad \text{for a.e. } t \in \Omega, \end{aligned}$$

and

$$\|N - n\| \leq \frac{4\varepsilon}{\|M - m\|} \quad (\varepsilon > 0),$$

then by (4.4) we can conclude that

$$0 \leq \Delta(F, f, \rho) \leq \varepsilon.$$

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School of Communications and Informatics
Victoria University of Technology
Melbourne City MC
Victoria 8001, Australia
E-mail: sever@matilda.vu.edu.au
URL: `http://rgmia.vu.edu.au/SSDragomirWeb.html`