ON SEMI-INVARIANT SUBMANIFOLDS OF LORENTZIAN ALMOST PARACONTACT MANIFOLDS

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ABSTRACT. Semi-invariant submanifolds of Lorentzian almost paracontact manifolds are studied. Integrability of certain distributions on the submanifold are investigated. It has been proved that a $LP$-Sasakian manifold does not admit a proper semi-invariant submanifold.

1. INTRODUCTION

Matsumoto [7] introduced the notion of a Lorentzian almost paracontact manifold. Submanifolds of a Lorentzian almost paracontact manifold have been studied in Prasad and Ojha [11]. In the present paper we study semi-invariant submanifolds of Lorentzian almost paracontact manifolds. The paper is organized as follows. Section 2 is devoted to preliminaries. In Section 3 some necessary and sufficient conditions for integrability of certain distributions on semi-invariant submanifolds are obtained. In the last section (Section 4), it has been shown that a $LP$-Sasakian manifold does not admit a proper semi-invariant submanifold.

2. PRELIMINARIES

Let $\bar{M}$ be a Lorentzian almost paracontact manifold (cf. [7], [8]) with a Lorentzian almost paracontact structure $ (\phi, \xi, \eta, g) $, that is, $\phi$ is a $(1, 1)$ tensor field, $\xi$ is a (timelike) vector field, $\eta$ is a 1-form and $g$ is a Lorentzian metric on $\bar{M}$ such that

\begin{align*}
\phi^2 &= I + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \\
g(\phi X, \phi Y) &= g(X, Y) + \eta(X)\eta(Y), \\
\Phi(X, Y) &\equiv g(\phi X, Y) = g(X, \phi Y) = \Phi(Y, X), \quad g(X, \xi) = \eta(X)
\end{align*}

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for all $X, Y \in T\bar{M}$.

A Lorentzian almost paracontact manifold is called (cf. Matsumoto [7]):

**Lorentzian paracontact manifold** if

$$\Phi(X, Y) = \frac{1}{2} \left( (\bar{\nabla}_X \eta)Y + (\bar{\nabla}_Y \eta)X \right),$$

(4)

**Lorentzian para-Sasakian** (in brief, LP-Sasakian) **manifold** if

$$(\bar{\nabla}_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X,$$

(5)

**Lorentzian special para-Sasakian** (in brief, LSP-Sasakian) **manifold** if

$$\Phi(X, Y) = eg(\phi X, \phi Y), \quad e^2 = 1.$$  

(6)

Here $\bar{\nabla}$ is the covariant differentiation with respect to $g$.

Let $M$ be a submanifold of a Lorentzian almost paracontact manifold $\bar{M}$ with Lorentzian almost paracontact structure $(\phi, \xi, \eta, g)$. Let the induced metric on $M$ also be denoted by $g$. Then Gauss and Weingarten formulae are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad X, Y \in TM,$$

(7)

$$\bar{\nabla}_X N = -A_N X + \nabla_X^1 N \quad N \in T^\perp M,$$

(8)

where $\nabla$ is the induced connection on $M$, $h$ is the second fundamental form of the immersion, and $-A_N X$ and $\nabla_X^1 N$ (resp.) are the tangential and normal (resp.) parts of $\bar{\nabla}_X N$. From (7) and (8) one gets

$$g(h(X, Y), N) = g(A_N X, Y).$$

(9)

Moreover, we have

$$(\bar{\nabla}_X \phi)Y$$

$$= ((\nabla_X P)Y - A_{FY} X - th(X, Y)) + ((\nabla_X F)Y + h(X, PY) - fh(X, Y)),$$

(10)

$$(\bar{\nabla}_X \phi)N$$

$$= ((\nabla_X t)N - A_{fN} X - PA_N X) + ((\nabla_X f)N + h(X, tN) - FA_N X),$$

(11)

where

$$\phi X \equiv PX + FX; \quad PX \in TM, \quad FX \in T^\perp M,$$

(12)

$$\phi N \equiv tN + fN; \quad tN \in TM, \quad fN \in T^\perp M,$$

(13)

$$(\nabla_X P)Y \equiv \nabla_X PY - P\nabla_X Y,$$

(14)

$$(\nabla_X F)Y \equiv \nabla_X^1 FY - F\nabla_X Y,$$

(15)
\[(\nabla_X t)N \equiv \nabla_X tN - t \nabla_X^1 N, \quad (16)\]
\[(\nabla_X f)N \equiv \nabla_X^1 fN - f \nabla_X N. \quad (17)\]

Let \(\xi \in TM\). We write \(TM = \{\xi\} \oplus \{\xi\}^\perp\), where \(\{\xi\}\) is the distribution spanned by \(\xi\) and \(\{\xi\}^\perp\) is the complementary orthogonal distribution of \(\{\xi\}\) in \(M\). Then we get
\[
P\xi = 0 = F\xi, \quad \eta \circ P = 0 = \eta \circ F, \quad (18)\]
\[
P^2 + tF = I + \eta \otimes \xi, \quad FP + fF = 0, \quad (19)\]
\[
f^2 + Ft = I, \quad tf + Pt = 0, \quad (20)\]
\[
\ker(P) = \ker(P^2) = \ker(tF - I - \eta \otimes \xi), \quad (21)\]
\[
\ker(F) = \ker(tF) = \ker(P^2 - I - \eta \otimes \xi), \quad (22)\]
\[
\ker(t) = \ker(Ft) = \ker(f^2 - I), \quad (23)\]
\[
\ker(f) = \ker(f^2) = \ker(Ft + I) \quad (24)\]
\[
\ker(P\rfloor_{\{\xi\}^\perp}) = \ker(P^2\rfloor_{\{\xi\}^\perp}) = \ker(tF\rfloor_{\{\xi\}^\perp} - I), \quad (25)\]
\[
\ker(F\rfloor_{\{\xi\}^\perp}) = \ker(tF\rfloor_{\{\xi\}^\perp}) = \ker(P^2\rfloor_{\{\xi\}^\perp} - I). \quad (26)\]

A submanifold \(M\) of a Lorentzian almost paracontact manifold \(\tilde{M}\) with \(\xi \in TM\) is called a semi-invariant submanifold of \(\tilde{M}\) if \(TM\) can be decomposed as a direct sum of mutually orthogonal differentiable distributions :
\[
TM = D^1 \oplus D^0 \oplus \{\xi\}, \quad (27)\]

where
\[
D^1 = \ker(F\rfloor_{\{\xi\}^\perp}) = \{X \in \{\xi\}^\perp : \|X\| = \|PX\|\} = TM \cap \phi(TM), \quad (28)\]
\[
D^0 = \ker(P\rfloor_{\{\xi\}^\perp}) = \{X \in \{\xi\}^\perp : \|X\| = \|FX\|\} = TM \cap \phi(T^\perp M). \quad (29)\]

Moreover, we have
\[
T^\perp M = \bar{D}^1 \oplus \bar{D}^0 \quad (30)\]

where
\[
\bar{D}^1 = \ker(t) = T^\perp M \cap \phi(T^\perp M), \quad \bar{D}^0 = \ker(f) = T^\perp M \cap \phi(TM), \quad F D^0 = \bar{D}^0, \quad (31)\]

and \(t\bar{D}^0 = D^0\). For \(X \in TM\) we can write
\[
X = U^1 X + U^0 X - \eta(X)\xi \quad (32)\]
where $U^1$ and $U^0$ are projection operators of $TM$ on $D^1$ and $D^0$ respectively.

A semi-invariant submanifold of a Lorentzian almost paracontact manifold is a \textit{\textit{invariant submanifold}} (resp. \textit{\textit{anti-invariant submanifold}}) if $D^0 = \{0\}$ (resp. $D^1 = \{0\}$). A semi-invariant submanifold is \textit{\textit{proper}} if $D^0 \neq \{0\} \neq D^1$.

3. \textbf{Integrability Conditions}

Let $M$ be a semi-invariant submanifold of a Lorentzian almost paracontact manifold $\tilde{M}$. The Nijenhuis tensor $[\phi, \phi]$ of $\phi$ is given by

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y],$$

(28)

for $X, Y \in \tilde{T}\tilde{M}$. Using (1), (27), (12) and $\ker(F) = D^1 \oplus \{\xi\}$, for $X, Y \in D^1 \oplus \{\xi\}$ we get

$$[\phi, \phi](X, Y) = [P, P](X, Y) + U^0[X, Y] - F([PX, Y] + [X, PY]).$$

(29)

Let superscripts $T$ and $\perp$ in a term denote its tangential and normal parts respectively. From (29) we can state the following.

**Proposition 3.1.** If $M$ is a semi-invariant submanifold of a Lorentzian almost paracontact manifold, then for $X, Y \in D^1 \oplus \{\xi\}$ we get

$$([\phi, \phi](X, Y))^T = [P, P](X, Y) + U^0[X, Y],$$

(30)

$$([\phi, \phi](X, Y))^\perp = -F([PX, Y] + [X, PY]).$$

(31)

Consequently, for $X \in D^1 \oplus \{\xi\}$ we have

$$([\phi, \phi](X, \xi))^T = [P, P](X, \xi) + U^0[X, \xi],$$

(32)

$$([\phi, \phi](X, \xi))^\perp = -F[PX, \xi].$$

(33)

In the following theorem, we find some necessary and sufficient condition for the integrability of the distribution $D^1 \oplus \{\xi\}$ on a semi-invariant submanifold of a Lorentzian almost paracontact manifold.

**Theorem 3.2.** Let $M$ be a semi-invariant submanifold of a Lorentzian almost paracontact manifold. Then the following three statements are equivalent:

(a) The distribution $D^1 \oplus \{\xi\}$ is integrable.

(b) $([\phi, \phi](X, Y))^T = [P, P](X, Y), \quad X, Y \in D^1 \oplus \{\xi\}$.

(c) $([\phi, \phi](X, Y))^\perp = 0, \quad U^0[P, P](X, Y) = 0, \quad X, Y \in D^1 \oplus \{\xi\}$. 
Proof. The distribution $\mathcal{D}^1 \oplus \{\xi\}$ is integrable if and only if

$$U^0[X, Y] = 0 \quad \text{for} \ X, Y \in \mathcal{D}^1 \oplus \{\xi\}.$$ 

In view of (30) it follows that (a) ⇔ (b).

Next, since

$$[P, P](X, Y) = U^1[X, Y] + [PX, PY] - P[PX, Y] - P[X, PY], \quad X, Y \in \mathcal{D}^1 \oplus \{\xi\},$$

operating by $U^0$ to this equation, in view of integrability of $\mathcal{D}^1 \oplus \{\xi\}$, we get

$$U^0[P, P](X, Y) = 0, \quad X, Y \in \mathcal{D}^1 \oplus \{\xi\}.$$ 

Taking account of (31), the integrability of $\mathcal{D}^1 \oplus \{\xi\}$ indicates

$$([\phi, \phi](X, Y))^\perp = 0, \quad X, Y \in \mathcal{D}^1 \oplus \{\xi\}.$$ 

Thus (a) ⇒ (c).

Conversely, let (c) be true. Then by (31) and $FX = \phi U^0 X$, we get

$$\phi U^0 ([PX, Y] + [X, PY]) = 0, \quad X, Y \in \mathcal{D}^1 \oplus \{\xi\}.$$ 

which operated by $\phi$ yields

$$U^0 ([PX, Y] + [X, PY]) = 0, \quad X, Y \in \mathcal{D}^1 \oplus \{\xi\}. \quad (34)$$

Next from $U^0[P, P](X, Y) = 0$, for $X, Y \in \mathcal{D}^1 \oplus \{\xi\}$, we get

$$U^0[PX, PY] = 0. \quad (35)$$

Now for $X, Y \in \mathcal{D}^1$ from (35) we have $U^0[X, Y] = 0$. In view of (34), we also have

$$0 = U^0 ([PX, P\xi] + [P^2 X, \xi]) = U^0[X, \xi], \quad X \in \mathcal{D}^1.$$

Thus taking account of

$$U^0[X, Y] = 0 = U^0[X, \xi], \quad X \in \mathcal{D}^1,$$

we get

$$U^0[X, Y] = 0 \quad X, Y \in \mathcal{D}^1 \oplus \{\xi\},$$

which makes (c) ⇒ (a). \qed

Theorem 3.3. The distribution $\mathcal{D}^0 \oplus \{\xi\}$ on a semi-invariant submanifold $M$ of a Lorentzian almost paracontact manifold $\tilde{M}$ is integrable if and only if

$$[P, P](X, Y) = 0, \quad X, Y \in \mathcal{D}^0 \oplus \{\xi\}.$$
\textbf{Proof.} In view of $\ker(P) = D^0 \oplus \{\xi\}$ and
\[ [P,P](X,Y) = [PX, PY] + P^2[X,Y] - P[PX,Y] - P[X, PY], \quad X, Y \in TM, \]
the proof follows immediately. \hfill \square

4. NONEXISTENCE OF PROPER SEMI-INVARIANT SUBMANIFOLDS

From the definition of $LP$-Sasakian manifold, we get
\[ \phi X = \nabla_X \xi, \quad X, Y \in TM. \tag{36} \]
We call a Lorentzian almost paracontact manifold $\tilde{M}$, a \textit{Lorentzian special paracontact manifold} if it satisfies (36). Obviously, a $LP$-Sasakian manifold is a Lorentzian special paracontact manifold. Now, we prove the following theorem.

\textbf{Theorem 4.1.} On a Lorentzian special paracontact manifold $\tilde{M}$ the distribution $\mathcal{T}$ determined by $\eta$ is integrable.

\textbf{Proof.} Let $X, Y \in \mathcal{T}$. Then $\eta(X) = 0 = \eta(Y)$ and consequently, in view of (2), from (36) and (3) it follows that $\eta[X,Y] = 0$, for $X, Y \in \mathcal{T}$. \hfill \square

This theorem implies the following theorem.

\textbf{Theorem 4.2.} Let $M$ be a semi-invariant submanifold of a Lorentzian special paracontact manifold. Then the distribution $D^1 \oplus D^0$ is integrable.

Let $M$ be a submanifold of a Lorentzian special paracontact manifold $\tilde{M}$ with $\xi \in TM$. Then, in view of (36), from (7) and (12) we get
\[ PX = \nabla_X \xi, \quad FX = h(X, \xi) \quad (\leftrightarrow tN = A_N \xi). \]
Consequently, we get
\[ \eta(A_N X) = g(FX, N). \tag{37} \]
Moreover, if $\tilde{M}$ is $LP$-Sasakian, then in view of (5) and (10) we get
\[ (\nabla_X P)Y - A_{FY} X - th(X,Y) = g(\phi X, \phi Y) \xi + \eta(Y) \phi^2 X. \tag{38} \]

Finally, we prove the following theorem.

\textbf{Theorem 4.3.} A $LP$-Sasakian manifold does not admit a proper semi-invariant submanifold.
Proof. We shall prove that $D^0 = \{0\}$. Let $X \in D^0$ and $Y \in TM$. We get

$$
g(A_X X, Y) = g(h(Y, X), F X) = g(th(Y, X), X)$$

$$
= g(\nabla_Y P X - P \nabla_Y X - A_F X Y - g(\phi Y, \phi X) \xi - \eta(X) \phi^2 Y, X)
$$

$$
= -g(\nabla_Y X, P X) - g(A_F X Y, X) = -g(A_F X X, Y),
$$

which implies that

$$A_F X X = 0, \quad X \in D^0$$

and consequently

$$0 = \eta(A_F X X) = g(F X, F X) = g(\phi X, \phi X) = g(X, X),$$

that is, $D^0 = \{0\}$. \qed

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