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ON NEARLY CONVERTIBLE (0, 1) MATRICES

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ABSTRACT. Let A be a nonnegative matrix of size $n \times n$. A is said to be *nearly convertible* if $A(i|j)$ is convertible for all integers $i, j \in \{1, 2, \cdots, n\}$ where $A(i|j)$ denote the submatrix obtained from A by deleting the i-th row and the j-th column. We investigate some properties of nearly convertible matrices and existence of (maximal) nearly convertible matrices of size n is proved for any integers $n(\geq 3)$.

1. INTRODUCTION

Let $A = [a_{ij}]$ be an $m \times n$ real matrix $(m \leq n)$. The *permanent* of A is defined by

$$\operatorname{per} A = \sum_{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{m\sigma(m)}$$

where the summation extend over all one-to-one functions from $\{1, \cdots, m\}$ to $\{1, \cdots, n\}$. A nonnegative $n \times n$ matrix A is called *convertible* if there exists a $(1, -1)$ matrix H such that $\operatorname{per} A = \det(H \circ A)$ where $H \circ A$ denotes the Hadamard product of H and A.

Let $T_n = [t_{ij}]$ denote the $(0, 1)$ matrix of size $n \times n$ with $t_{ij} = 0$ if and only if $j > i + 1$. For a matrix A, square or not, let $\pi(A)$ denote the number of positive entries of A. Gibson [4] has shown that for any $n \times n$ convertible $(0, 1)$ matrix A with $\operatorname{per} A > 0$, $\pi(A) \leq \pi(T_n) = (n^2 + 3n - 2)/2$ with equality if and only if $A \sim T_n$. Many authors (cf. Hwang and Kim [5]; Hwang, Kim and Song [6, 7]; and Kim [8]) investigated some properties of convertible matrices and constructed some maximal convertible matrices.

It is natural to ask the following question:

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For an $m \times n$ matrix A $(m \leq n)$, when can we compute permanent of A via using determinant of submatrices of A? i.e., when is each $m \times m$ submatrix of A convertible?

In association with this we can define a nearly convertible matrix. Let A be a nonnegative matrix of size $n \times n$. A is said to be *nearly convertible* if $A(i|j)$ is convertible for all integers $i, j \in \{1, 2, \cdots, n\}$. A nearly convertible matrix A is said to be *maximal* if replacing any zero entry with a 1 does not result in a nearly convertible matrix. In this paper, we investigate some properties of nearly convertible matrices and existence of (maximal) nearly convertible matrices of size n is proved for any integer $n \geq 3$.

For matrices A, B of the same size, A is said to be *permutation equivalent* to B, denoted by $A \sim B$, if there are permutation matrices P, Q such that $PAQ = B$. An $n \times n$ matrix is called *partly decomposable* if it contains a $t \times (n-t)$ zero submatrix for some $t > 0$. Square matrices which are not partly decomposable are called *fully indecomposable.*

For positive integers k and n with $k \leq n$, let $Q_{k,n}$ denote the set of all strictly increasing k-sequences from $\{1, \cdots, n\}$. For an $n \times n$ matrix A and for α, $\beta \in Q_{k,n}$, let $A(\alpha|\beta)$ denote the submatrix obtained from A by deleting rows α and columns β and let $A[\alpha|\beta]$ denote the matrix complementary to $A(\alpha|\beta)$ in A. Let E_{ij} denote the $n \times n$ matrix all of whose entries are 0 except for the (i, j) entry which is 1. For matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size, we write $A \leq B$ if $a_{ij} \leq b_{ij}$ for all (i, j) entries.

2. Nearly Convertible Matrices

Recall that an $n \times n$ nonnegative real matrix A is *nearly convertible* if $A(i|j)$ is convertible for any (i, j) and a nearly convertible matrix $A = [a_{ij}]$ is *maximal* if $A + cE_{ij}$ is not nearly convertible for real number $c > 0$ and any (i, j) with $a_{ij} = 0$. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then $A(i|j)$ is a 3×3 matrix having a zero entry and hence convertible matrix for any (i, j). Thus A is nearly convertible. Moreover, A is maximal nearly convertible matrix. For, replacing any zero entries with a 1 in A, we have

$$A(i|j) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

for some i, j. This matrix is not convertible.

Notice that A is not convertible (cf. [8]). Hence nearly convertibility does not imply convertibility. Also the converse does not hold as shown in the following example.

Example. Let

$$T_n = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & \cdots & \cdots & 1 \end{pmatrix}$$

be the n-square lower Hessenberg matrix. Then T_n is (maximal) convertible (cf. Gibson [4]). However, $T_n(1|n)$ is not convertible by Gibson [4] and hence T_n is not nearly convertible.

However we may have a convertible and nearly convertible matrix. For example,

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

is a maximal convertible matrix (cf. Hwang and Kim [5]) and a maximal nearly convertible matrix.

We rewrite some well-known properties of convertible matrices before we mention our first result.

Lemma A (cf. [2]). *Let $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ be a (maximal) convertible $(0,1)$ matrix of order n. Then for $k \in \{1, 2, \cdots, n\}$, let*

$$B = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \mathbf{a}_k & \mathbf{a}_1 & \cdots & \mathbf{a}_{k-1} & \mathbf{a}_k & \mathbf{a}_{k+1} & \cdots & \mathbf{a}_n \end{pmatrix}$$

is a (maximal) convertible matrix of order $n+1$.

A convertible matrix C is called a *column expansion* of convertible matrix A if $C \sim B$ for A, B in Lemma A. A *row expansion* of a convertible matrix is similarly defined. A matrix is called an *expansion* of convertible matrix B if it is a row expansion or a column expansion of B.

Lemma B (cf. [2]). *Let A be a maximal convertible matrix of order n whose first column is equal to $[1, 1, 0, \cdots, 0]^T$. Then the first two rows of A are identical and the matrix obtained from A by deleting row 1 and column 1 is a maximal convertible matrix.*

Lemma C (cf. [5]). *Let A be an $n \times n$ nonnegative convertible matrix with a converter H and let k be a positive integer with $k \leq n$. If*

$$\text{per}\,(A[\alpha|\beta])\text{per}\,(A(\alpha|\beta)) \neq 0 \quad \text{for} \quad \alpha, \beta \in Q_{k,n},$$

then $|\det(H \circ A)[\alpha|\beta]| = \text{per}\,(A[\alpha|\beta])$. *Especially $(H \circ A)[\alpha|\beta]$ is nonsingular.*

Theorm 2.1. *Let $A = [a_{ij}]$ be a nonnegative nearly convertible matrix. Then $A(i|j)$ is nearly convertible if $a_{ij} > 0$.*

Proof. Let $B = A(i|j)$ for any (i, j) with $a_{ij} > 0$. We show that $B(s|t)$ is convertible for all s, t. Since A is nearly convertible, $A(s|t)$ is a convertible matrix. Since $a_{ij} > 0$, $B(s|t) = A(i, s|j, t)$ is convertible (cf. Gibson [4]). □

For an $n \times n$ matrix $A = [a_{ij}]$, the $n \times n$ $(0, 1)$ matrix $\text{supp}\, A = [a^*_{ij}]$ defined by

$$a^*_{ij} = \begin{cases} 1, & \text{if } a_{ij} \neq 0, \\ 0, & \text{if } a_{ij} = 0 \end{cases}$$

is called the *support* of A.

It comes from Hwang and Kim [5] that a nonnegative square matrix is nearly convertible if and only if its support is nearly convertible. Moreover If A be an $n \times n$ nonnegative nearly convertible matrix, then any matrix B, not necessarily nonnegative, such that $\text{supp}\, B \leq \text{supp}\, A$ is nearly convertible. From now on, nearly convertible matrices we consider here are assumed to be $(0, 1)$ matrix.

We can make a new nearly convertible matrix from an old nearly convertible matrix. Let $\mathbf{e}$ be the column vector whose entries are all 1 of appropriate size.

Theorm 2.2. *Let*

$$A = \begin{pmatrix} 1 & \mathbf{e}^T \\ \mathbf{e} & * \end{pmatrix}$$

be an n-square $(0,1)$ convertible and nearly convertible matrix. Then the $(n+1)$-square matrix

$$B = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & & & & \\ 0 & & & & \\ \vdots & & & A & \\ 0 & & & & \end{pmatrix}$$

is also convertible and nearly convertible.

Proof. By Lemma A, matrix B is convertible. We will show that $B(i|j)$ is convertible for any pair (i,j) of integers with $1 \leq i,j \leq n+1$. For $(i,j) = (1,1)$, $B(1|1) = A$ is convertible by hypothesis. For $(i,j) = (1,j)$ with $2 \leq j \leq n+1$ or $(i,j) = (i,1)$ with $2 \leq i \leq n+1$, $B(i|j)P \leq A$ or $PB(i|j) \leq A$ where P is a permutation matrix and hence $B(i|j)$ is convertible. For $(i,j) = (2,2)$, $B(2|2) = 1 \oplus A(1|1)$. Since $A(1|1)$ is convertible, $B(2|2)$ is convertible. For (i,j) with $2 \leq i,j \leq n+1$ except $(i,j) = (2,2)$, $B(i|j)P \leq C$ where P is a permutation matrix and C an expansion of $A(i-1|j-1)$, which is convertible. Hence B is nearly convertible. □

Notice that we can find such a matrix A in Theorem 2.2 using T_n.

Now we can construct an $n \times n$ maximal nearly convertible matrix. Let P_n be the fully cyclic permutation on $\{1, 2, \cdots, n\}$.

Lemma 2.3 (cf. [9]). *Let*

$$A = \begin{pmatrix} \mathbf{e} & P_{n-1} + I_{n-1} \\ 0 & \mathbf{e}^T \end{pmatrix}.$$

Then A is a (doubly indecomposable) maximal convertible matrix.

Lemma 2.4 (cf. [1]). *Let A be a $(0,1)$ matrix of order n. Then* $\operatorname{per}(A) = 1$ *if and only if the lines of A may be permuted to yield a triangular matrix with 1's in the n main diagonal positions and with 0's above the main diagonal.*

Lemma 2.5. $\operatorname{per}(P_n + I_n)(i|j)$ *is permutation equivalent to a triangular matrix for any (i,j).*

Proof. Let $P_n + I_n = [h_{ij}]$. It is sufficient to show that $\operatorname{per}(P_n + I_n)(i|j) \leq 1$ by Lemma 2.4. For (i,j) with $h_{ij} = 1$, it is easy to show that $\operatorname{per}(P_n + I_n)(i|j) = \operatorname{per}(P_n + I_n)(1|1) = 1$. Also we have $\operatorname{per}(P_n + I_n)(i|j) = \operatorname{per}(P_n + I_n)(1|n) = 1$ for (i,j) with $h_{ij} = 0$. Hence we have the result. □

Theorm 2.6. *Let*

$$A = \begin{pmatrix} \mathbf{e} & P_{n-1} + I_{n-1} \\ 1 & \mathbf{e}^T \end{pmatrix}.$$

Then A is nearly convertible.

Proof. Let

$$A = \begin{pmatrix} \mathbf{e} & P_{n-1} + I_{n-1} \\ 1 & \mathbf{e}^T \end{pmatrix}, \quad B = \begin{pmatrix} \mathbf{e} & P_{n-1} + I_{n-1} \\ 0 & \mathbf{e}^T \end{pmatrix}.$$

Then $A(i|j) = B(i|j)$ for all $(i,j) = (i,1)$ or $(i,j) = (n,j)$ with $i,j = 1,2,\cdots,n$. Since $B(i|j)$ is convertible by Lemma C and Lemma 2.3, $A(i|j)$ is convertible for $(i,j) = (i,1)$ or $(i,j) = (n,j)$ with $i = 1,2,\cdots,n-1$; $j = 2,3,\cdots,n$. $A(n|1) = P_{n-1} + I_{n-1} \leq T_{n-1}$, which is convertible. Hence $A(n|1)$ is convertible. Next we will show that $A(i|j)$ is convertible for $i = 1,2,\cdots,n-1$; $j = 2,3,\cdots,n$. Since $A(i,n|1,j) = (P_{n-1} + I_{n-1})(i|j-1)$, $A(i,n|1,j)$ is permutation equivalent to a triangular matrix by Lemma 2.5. Hence $A(i|j)$ is permutation equivalent to $B \leq T_n$, which is convertible. Thus $A(i|j)$ is convertible for all (i,j) and hence A is nearly convertible. □

In fact, the matrix in the Theorem 2.6 is maximal nearly convertible.

Theorm 2.7. *The matrix in Theorem 2.6 is maximal nearly convertible.*

Proof. We will show that $A + E_{ij}$ is not nearly convertible for any (i,j) with $a_{ij} = 0$. That is, we will show that $(A + E_{ij})(k|l)$ is not convertible for some (k,l). Without loss of generality, we may assume that $(i,j) = (1,n)$. Then $(A + E_{1n})(n-1|2)$ is not convertible. For, we have $(A + E_{1n})[1,n-2,n|1,n] = J_{3\times 2}$ and it is easy to show that $\operatorname{per}(A + E_{1n})(p,q,n-1|1,2,n) > 0$ for $(p,q) = (1,n-2),(1,n)$ or $(n-2,n)$. This is impossible by Lemma C if $(A + E_{1n})(n-1|2)$ is convertible. Thus we have the result. □

Theorm 2.8. *Let A be an n-square fully indecomposable, maximal nearly convertible matrix containing T_{n-1} as a submatrix. Then*

$$A \sim \begin{pmatrix} \mathbf{a} & 0 \\ T_{n-1} & \mathbf{a}^T \end{pmatrix}$$

where $\mathbf{a} = (1,0,1,0,\cdots,0)$.

Proof. Without loss of generality, we may assume that A is of the form

$$A = [a_{ij}] = \begin{pmatrix} A_{11} & A_{12} \\ T_{n-1} & A_{22} \end{pmatrix}.$$

Since A is nearly convertible, $A(2|n-1)$ is convertible. If $a_{1n} = 1$, then $A(1,2|n-1,n)$ is convertible. This is impossible because $A(1,2|n-1,n)$ contains the maximal convertible matrix T_{n-2} properly. Hence $a_{1n} = 0$. If $a_{1,n-1}$ or a_{2n} is 1, then $A(1,2|n-1,n)$ is convertible, which is impossible. Hence $a_{1,n-1} = a_{2n} = 0$. Since $A(1|n-1)$ is convertible containing $A(1,n|n-1,n) \sim T_{n-2}$ as a submatrix, $PA(1|n-1) \leq T_{n-1}$ for some permutation P by Gibson's Theorem [4]. Since A is fully indecomposable, $A(1|n-1) \sim T_{n-1}$. Hence

$$A \sim \begin{pmatrix} * & 0 \\ T_{n-1} & \mathbf{a}^T \end{pmatrix}$$

where $\mathbf{a} = (1,0,1,0,\cdots,0)$. Similarly, $A(2|n) \sim T_{n-1}$. Thus we have

$$A \sim \begin{pmatrix} \mathbf{a} & 0 \\ T_{n-1} & \mathbf{a}^T \end{pmatrix}. \qquad \square$$

From the well-known Little Theorem (cf. [1], [10]), we can derive a characterization of nearly convertible matrices:

> Let A be a (0,1) matrix of order n. Then A is nearly convertible if and only if there does not exist permutation matrices P and Q of order n and a (0,1) matrix B with $B \leq A$ such that for some integer k with $1 \leq k < n-3$, $PBQ = I_k + B'$ or $PBQ = I_{k-1} + 0 + B'$ where the all 1's matrix J_3 of order 3 can be obtained by a sequence of contractions starting with the matrix B'.

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