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Projections of Extended Formulations with Precedence Variables for the Asymmetric Traveling Salesman Problem*

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ABSTRACT

Gouveia and Pires (European Journal of Operations Research 112 (1999) 134-146) have proposed four extended formulations having precedence variables as extra variables and characterized the projections of three of the four formulations into the natural variable space. In Gouveia and Pires (Discrete Applied Mathematics 112 (2001)), they also have introduced some other extended formulations with the same extra variables and conjectured that the projection of one of the proposed formulations is equivalent to the one proposed by Dantzig, Fulkerson, and Johnson (Operations Research 2 (1954) 393-410). In this paper, we provide a unifying framework based on which we give alternative proofs on the projections of three extended formulations and new proofs on those of two formulations appeared in Gouveia and Pires (1999, 2001).

1. INTRODUCTION

Given a loop-free directed graph $G = (V, A)$ where $V = \{1, \ldots, n\}$ and costs c_{ij} for each arc $(i, j) \in A$ the asymmetric traveling salesman problem (ATSP) is to find a least cost Hamiltonian cycle (tour) contained in the graph. A number of formulations for the ATSP have been proposed and they can be classified into the following two types: A natural formulation and an extended formulation. A natural formulation only contains arc variables that indicate whether or not to include arcs in a tour while an extended formulation have extra variables other than arc variables. Various different ATSP formulations, both natural and extended ones, can be found in Langevin, Soumis, and Desrosiers [7] and Gouveia and Pires

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[4].

Recently, Gouveia and Pires [4, 5] have proposed a class of extended formulations that contains precedence variables as extra variables. Precedence variables are defined on each pair of nodes to indicate which node in the pair precedes the other in the selected tour under the assumption that a tour starts from a predetermined node. Gouveia and Pires [4] developed four extended formulations and they characterized the projections of three of the four proposed formulations into the space of natural variables, that is, arc variables. In Gouveia and Pires [5], they also introduced several different extended formulations with precedence variables and provided a conjecture that the projection of one of the proposed formulations is equivalent to the formulation introduced by Dantzig, Fulkerson, and Johnson [2].

As is the case in integer and combinatorial optimization, the efficiency of an algorithm for solving the ATSP is dependent on the formulation that the algorithm is based on. Most ATSP algorithms need a lower bound on the value of the objective function and the efficiency of the algorithm depends on how to obtain the sharp lower bound in reasonable time. Therefore, an attempt to compare different formulations of the ATSP provides a valuable information to choose an appropriate formulation. A useful method of comparing various alternative formulations for the ATSP is to project out extra variables from extended formulations so that the resulting formulations have the same set of variables. Such method, called projection, has also been used in other researches to compare alternative formulations of combinatorial optimization problems [1, 3, 8].

In this paper, we develop a unifying framework that helps to characterize the projections of extended formulations with precedence variables into the natural variable space. Based on this framework we give alternative proofs on the projections of three extended formulations and newly characterize the projection of another formulation in Gouveia and Pires [4]. We also prove that the conjecture in Gouveia and Pires [5] is true. In Section 2, we provide the notation and introduce the formulations proposed by Gouveia and Pires [4, 5]. The main theorems and their proofs are given in Section 3. Finally, Section 4 provides the concluding remarks.

2. NOTATION AND FORMULATIONS

We assume that a given graph is complete. For a subset S of nodes, we let $\delta^+(S) = \{(i, j) \in A \mid i \in S \text{ and } j \in V\setminus S\}$, $\delta^-(S) = \{(i, j) \in A \mid i \in V\setminus S \text{ and } j \in S\}$, and $E(S) = \{(i, j) \in A \mid i, j \in S\}$. We also let $\delta(S) = \delta^+(S) \cup \delta^-(S)$. For simplicity, we let $\delta^+(i) = \delta^+(\{i\})$, $\delta^-(i) = \delta^-(\{i\})$, and $\delta(i) = \delta(\{i\})$ for every node i. If x is defined on an arc set A, then we denote $\sum_{(i,j)\in F} x_{ij}$ for $F \subseteq A$ by x (F). Let $V_1 = V\setminus\{1\}$ and $A_1 = A\setminus \delta(1)$. For a subset V' of V, let $G[V']$ denote the subgraph of G induced by the set of nodes V'. For each arc $(i, j) \in A$, let $x_{ij} = 1$ if arc (i, j) is included in a selected tour; $x_{ij} = 0$ otherwise. We assume that node 1 is the root node where any tour starts and finishes. Precedence variables v_{ij} for any pair of distinct nodes $i, j \in V_1$ are defined as follows: $v_{ij} = 1$ if node i is visited prior to node j in the optimal tour; $v_{ij} = 0$ otherwise.

Gouveia and Pires [4] have introduced the following extended formulation with precedence variables for the ATSP.

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij}$$

$$\begin{array}{llll}
\text{s.t.} & x(\delta^+(i)) = 1 & i \in \mathrm{V} & (1)\\
& x(\delta^-(i)) = 1 & i \in \mathrm{V} & (2)\\
& x_{ij} + v_{pi} \le v_{pj} + 1 & p \in V_1;\ i,j \in V_1 \setminus \{p\} & (3a)\\
& x_{pi} \le v_{pi} & p \in V_1;\ i \in V_1 \setminus \{p\} & (4)\\
& x_{ip} + v_{pi} \le 1 & p \in V_1;\ i \in V_1 \setminus \{p\} & (5)\\
& 0 \le x_{ij} \le 1 & (i, j) \in A & (6)\\
& 0 \le v_{pi} \le 1 & p, i \in V_1 & (7)\\
& x_{ij}, v_{pi} \quad \text{integer} & (i, j) \in A;\ p, i \in V_1. & (8)
\end{array}$$

They also have shown that constraints (3a) can be lifted to each class of the following two constraints.

$$x_{ij} + x_{ji} + v_{pi} \le v_{pj} + 1 \qquad p \in V_1; i, j \in V_1 \setminus \{p\} \qquad (3b)$$

$$x_{ij} + x_{pj} + x_{ip} + v_{pi} \le v_{pj} + 1 \qquad p \in V_1; i, j \in V_1 \setminus \{p\} \qquad (3c)$$

In Gouveia and Pires [5], they have proposed generalizations of (3b) and (3c), one of which is the following generalized version of constraints (3b).

$$x(E(S)) + v_{pi} \le v_{pj} + |S| - 1 \quad p \in V_1; S \subseteq V_1 \setminus \{p\} \text{ such that } |S| \ge 2 \text{ and } i, j \in S \quad (3d)$$

Note that (3d) reduces to (3b) when $|S| = 2$.

Five different extended formulations for the ATSP can be constructed depending on which constraints are selected among (3a) - (3d). Consider the following five polytopes:

$$\text{PQ } 1 = \{ (x, v) \mid (x, v) \text{ satisfies (1), (2), (3a), (4) - (7)} \};$$

$$\text{PQ } 2 = \{ (x, v) \mid (x, v) \text{ satisfies (1), (2), (3b), (4) - (7)} \};$$

$$\text{PQ } 3 = \{ (x, v) \mid (x, v) \text{ satisfies (1), (2), (3c), (4) - (7)} \};$$

$$\text{PQ } 4 = \{ (x, v) \mid (x, v) \text{ satisfies (1), (2), (3d), (4) - (7)} \}.$$

$$\text{PQ } 5 = \{ (x, v) \mid (x, v) \text{ satisfies (1), (2), (3b), (3c), (4) - (7)} \}.$$

Let P1, P2, P3, P4, and P5 be the projections of PQ1, PQ2, PQ3, PQ4, and PQ5 into the x-space, respectively. To compare the above five extended formulations with the existing natural formulations, Gouveia and Pires [4, 5] have considered P1 - P5. To describe P1 - P5, we need the following constraints. For convenience, we will refer to a directed cycle as not only a set of arcs but a set of nodes that are included in the cycle.

$$\sum_{(i,j)\in C} x_{ij} \le |C| - 1 \quad \text{for any directed cycle } C \text{ in } G[V_1]; \tag{9a}$$

$$\sum_{(i,j)\in C} x_{ij} + \sum_{(i,j)\in C;\, i\ne p;\, j\ne p} x_{ji} \le |C| - 1 \quad \text{for any directed cycle } C \text{ in } G[V_1] \text{ and } p \in C; \tag{9b}$$

$$\sum_{(i,j)\in C} x_{ij} + \sum_{i\in C} (x_{ip} + x_{pi}) \le |C| \quad \text{for any } p \in V_1 \text{ and directed cycle } C \text{ in } G[V_1 \setminus \{p\}]; \tag{9c}$$

$$x_{ij} + x_{ji} \le 1 \quad \text{for any } (i, j) \in A_1; \tag{9d}$$

$$x(E(S)) \le |S| - 1 \quad \text{for any subset } S \subseteq V_1 \text{ such that } 2 \le |S| \le n - 2; \tag{9e}$$

$$\sum_{(i,j)\in C} f_{ij}(x) \le |C| \quad \text{for any } p \in V_1 \text{ and directed cycle } C \text{ in } G[V_1 \setminus \{p\}]. \tag{9f}$$

where $f_{ij}(x)$ is either $x_{ij} + x_{ji}$ or $x_{ij} + x_{ip} + x_{pj}$.

It is not difficult to know that any feasible solution to the ATSP satisfies all the above inequalities. Notice that constraints (9e) dominate (9a) - (9d). Strictly speaking, to make the statement correct we have to include (9e) for $|S| = n - 1$. However, such constraint can be derived from (1) and (2). Notice that (9f) includes (9c) as its specific case, since if $f_{ij}(x)$ is $x_{ij} + x_{ip} + x_{pj}$ for all $(i, j) \in C$, the resulting constraint (9f) is nothing but (9c). On the other hand, if $f_{ij}(x)$ is $x_{ij} + x_{ji}$ for all $(i, j) \in C$, the resulting constraint (9f) is dominated by constraints (9d) since this specific version of (9f) can be obtained by adding (9d) one for each arc in a cycle.

Interestingly, constraints (9f) include well-known $D3$ inequalities. Consider a directed cycle C in $G[V_1 \setminus \{p\}]$ for some $p \in V_1$ with $|C| = 2$, say $C = \{(i, j), (j, i)\}$, then the corresponding inequality (9f) has the following form:

$$(x_{ij} + x_{ip} + x_{pj}) + (x_{ji} + x_{ij}) \leq 2 .$$

The above inequality is a $D3$ inequality and can be obtained by lifting a 3-cycle inequality (9a) corresponding the cycle $\{(i, p), (p, j), (j, i)\}$. Grötchel and Padberg [6] have proposed $D3$ inequalities as a specific case of more general $\bar{D}_k$ (or $\bar{D}_k$) inequalities.

Consider the following five polytopes:

Q1 = $\{ x \mid x$ satisfies (1), (2), (6), (9a) $\}$;

Q2 = $\{ x \mid x$ satisfies (1), (2), (6), (9b), (9d) $\}$;

Q3 = $\{ x \mid x$ satisfies (1), (2), (6), (9c), (9d) $\}$;

Q4 = $\{ x \mid x$ satisfies (1), (2), (6), (9e) $\}$;

Q5 = $\{ x \mid x$ satisfies (1), (2), (6), (9b), (9d), (9f) $\}$.

Adding the integrality constraints for the x variables to each of Q1 - Q5, we can construct five natural formulations for the ATSP. Gouveia and Pires [4] presented the first three formulations and Dantzig, Fulkerson, and Johnson [2] did Q4.

In the next section, we will prove the following theorem.

Theorem 1. $P1 = Q1$, $P2 = Q2$, $P3 = Q3$, $P4 = Q4$, and $P5 = Q5$.

Gouveia and Pires [4, 5] have proved the first three relations and conjectured that $P4 = Q4$. They have also shown that any feasible $x \in P5$ satisfies $D3$ inequalities and neither of $P4$ and $P5$ contains the other. This observation is consistent with our characterization of $P5$, i.e., $Q5$. Moreover, from $Q5$, we can know that $P5$ is contained in $P4$ with additional $D3$ inequalities. Note that (9f) is weaker than (9e) except when $|C| = 2$. Also note that (9f) with $|C| = 2$ other than $D3$ inequalities are included in (9e).

3. PROOF OF THEOREM 1

In this section, we prove Theorem 1. It is not difficult to know that we can obtain each of (9a) - (9f) if we select an appropriate set of inequalities among (3a) - (3d), (4), and (5), and aggregate them to eliminate v-variables. Gouveia and Pires [4, 5] have shown how to select inequalities among (3a) - (3d), (4), and (5) to derive (9a) - (9d). As for (9f), for any $p \in V_1$ and directed cycle C in $G[V_1 \setminus \{p\}]$, if we select either (3b) or (3c) corresponding to each arc in the cycle and add all the selected constraints, then we can have (9f) for p and C.

However, it is not trivial to show the opposite direction, that is, to show that the constraints in $Q1 - Q5$ are sufficient to describe $P1 - P5$, respectively. To prove that direction, we have to show that for any feasible $x \in Q$t, there always exist variables v_{pi} for each $i \in V_1 \setminus \{p\}$ and $p \in V_1$ such that (x, v) satisfies the system of inequalities in PQt, for $t = 1, \ldots, 5$. When proving that $Q1 \subseteq P1$ and $Q2 \subseteq P2$, Gouveia and Pires [4] used Farkas' Lemma in a general way to check whether there exists such vector v. Here we develop a more simple condition for the existence of a vector v such that $(x, v) \in PQ$t, for $t = 1, \ldots, 5$. This condition is based on the structural properties of (3a) - (3d), (4), and (5) such that the projected inequalities (9a) - (9f) are obtained by combining an appropriate set of inequali-ties (3a) - (3d), (4), and (5) in a cycle fashion. The following lemma provides a unifying framework that produces the proofs for Qt $\subseteq P$t for $t = 1, \ldots, 5$. Given a graph $G = (V, A)$ and arc lengths d_{ij} for each $(i, j) \in A$, we define the length of a directed path (cycle) as the sum of the lengths of arcs in the path (cycle). Recall that $V_1 = V \setminus \{1\}$ and $A_1 = A \setminus \delta(1)$.

Lemma 2. For a subgraph $G_1 = (V_1, A_1)$ of G with an arc length d_{ij} for each $(i, j) \in A_1$, suppose that for some $p \in V_1$, the following two conditions hold:

(C1) No directed cycle in $G[V_1 \setminus \{p\}]$ has positive length;

(C2) No directed cycle in $G[V_1]$ has length greater than 1.

Then there exist variables v_{pi} for each $i \in V_1 \setminus \{p\}$ that satisfy the following system of inequalities.

$$v_{pj} - v_{pi} \geq d_{ij} \qquad i, j \in V_1 \setminus \{p\} \tag{10}$$

$$v_{pi} \geq d_{pi} \qquad i \in V_1 \setminus \{p\} \tag{11}$$

$$v_{pi} \le 1 - d_{ip} \qquad\qquad i \in V_1 \setminus \{p\}. \tag{12}$$

Proof. Suppose that arc lengths d_{ij} for each $(i, j) \in A_1$ satisfy the conditions (C1) and (C2). For each $i \in V_1 \setminus \{p\}$ set v_{pi} as the length of the longest path from p to i. Obviously, the resulting variables v_{pi} satisfy (11) by definition and (12) by (C2). Now we show that the variables v_{pi} also satisfy (10). Suppose that $v_{pj} - v_{pi} < d_{ij}$ for some pair of nodes $i, j \in V_1 \setminus \{p\}$. Consider a longest path from p to i whose length equals to v_{pi}. If this path does not pass node j, we can construct a path from p to j by connecting the longest path from p to i and arc (i, j) whose length is greater than v_{pj}, thereby contradicting the assumption on v_{pj}. On the other hand, if the longest path from p to i passes node j, we can construct a positive length cycle by connecting arc (i, j) and a subpath from j to i of the longest path from p to i. It contradicts (C1). □

Notice that constraints (3a) - (3d) have the form of $v_{pj} - v_{pi} \ge d_{ij}$ assuming that d_{ij} represents a formula of the x variables and that (4) and (5) take the form of (11) and (12), respectively. So, Lemma 2 can be used to characterize the projections of extended formulations having constraints that can be described as (10) - (12). Although we don't show it, the reverse of the statement in Lemma 2 also holds. Using the lemma, we complete the proof of Theorem 1 by showing that Qt $\subseteq$ Pt for $t = 1, \ldots, 5$.

(Proof of Theorem 1)

The outline of our proof is as follows. For each $t = 1, 2, 3, 4, 5$, we select $x \in$ Qt and set $d_{ij}(p)$ for each $p \in V_1$ and $(i, j) \in A_1$ as a formula of the x variables such that for all $t = 1, \ldots, 5$, $d_{pi}(p) = x_{pi}$ for each $i \in V_1 \setminus \{p\}$ and $d_{ip}(p) = x_{ip}$ for each $i \in V_1 \setminus \{p\}$ and that $d_{ij}(p)$ for each $(i, j) \in A_1 \setminus \delta(p)$ is equal to $-1 + x_{ij}$, $-1 + x_{ij} + x_{ji}$, $-1 + x_{ij} + x_{pj} + x_{ip}$, $\max\{-|S| + 1 + x(E(S)) \mid S \subseteq V_1 \setminus \{p\}$ such that $|S| \ge 2$ and $i, j \in S\}$, $\max\{-1 + x_{ij} + x_{ji}, -1 + x_{ij} + x_{pj} + x_{ip}\}$ for $t = 1, 2, 3, 4, 5$, respectively. Notice that for all t, (11) and (12) are nothing but (4) and (5), respectively and that (10) for $t = 1$ (2, 3, 4, 5, respectively) corresponds to (3a) ((3b), (3c), (3d), both (3b) and (3c), respectively). Therefore, if we show that for all $p \in V_1$, the resulting lengths $d_{ij}(p)$ satisfy the conditions (C1) and (C2), we would prove that $(x, v) \in$ PQt, for each $t = 1, 2, 3, 4, 5$.

We first show that Q1 $\subseteq$ P1. Assume that $x \in$ Q1 and $d_{ij}(p)$ is set as we defined for $t = 1$. Since $d_{ij}(p) \le 0$ for each $(i, j) \in A_1 \setminus \delta(p)$, the condition $(C1)$ holds. Consider any directed cycle C in $G[V_1]$ which passes node p. Then

$$\sum_{(i,j)\in C} d_{ij}(p) = \sum_{(i,j)\in C} x_{ij} - (|C| - 2) \le 1.$$

By (9a), the inequality holds, so the length of the directed cycle C is less than or equal to 1.

To show that Q2 $\subseteq$ P2, select x $\in$ Q2 and set $d_{ij}(p)$ as we defined for $t = 2$. By (9d), $d_{ij}(p) \le 0$ for each $(i, j) \in A_1 \setminus \delta(p)$, so the condition (C1) holds. Each directed cycle C with $|C| = 2$ in $G[V_1]$ that passes node p, satisfies the condition (C2) by (9d). Consider any directed cycle C with $|C| \ge 3$ in $G[V_1]$ which passes node p. Then

$$\sum_{(i,j)\in C} d_{ij}(p) = \sum_{(i,j)\in C} x_{ij} + \sum_{(i,j)\in C; i\ne p, j\ne p} x_{ji} - (|C| - 2) \le 1.$$

The inequality holds by (9b).

We now show that Q3 $\subseteq$ P3. Consider $x \in$ Q3 and $d_{ij}(p)$ as we defined for t = 3. Consider any directed cycle C in $G[V_1 \setminus \{p\}]$. Then

$$\sum_{(i,j)\in C} d_{ij}(p) = \sum_{(i,j)\in C} x_{ij} + \sum_{i\in C} (x_{ip} + x_{pi}) - |C| \le 0.$$

The inequality holds by (9c), thereby the condition (C1) is satisfied. Next consider a directed cycle C in $G[V_1]$ which passes node p. Any directed cycle C with $|C| = 2$ satisfies the condition (C2) by (9d), so we assume that $|C| \ge 3$. Let $C = \{(p, i_1), (i_1, i_2), \ldots, (i_{r-1}, i_r), (i_r, p)\}$ and $C' = \{(i_1, i_2), \ldots, (i_{r-1}, i_r), (i_r, i_1)\}$. Note that $|C'| = |C| - 1$ and $p \notin C'$. Then

$$\sum_{(i,j)\in C} d_{ij}(p) = \sum_{(i,j)\in C'} x_{ij} + \sum_{i\in C'} (x_{ip} + x_{pi}) - x_{i_r i_1} - (|C| - 2) \le 1.$$

The inequality holds since the summation of the first two terms next to equal sign is less than or equal to $|C'| (= |C| - 1)$ by (9c).

To show that Q4 $\subseteq$ P4, we assume that $x \in$ Q4 and $d_{ij}(p)$ is set as we defined for $t = 4$. By (9e), $d_{ij}(p) \le 0$ for each $(i, j) \in A_1 \setminus \delta(p)$, so the condition (C1) holds. In order to prove that the condition (C2) holds, we first show that for all $i, j, k \in V_1 \setminus \{p\}$, $d_{ij}(p) + d_{jk}(p) \le d_{ik}(p)$. For any $i, j, k \in V_1 \setminus \{p\}$, let $d_{ij}(p) = -|S_1| + 1 + x(E(S_1))$ and $d_{jk}(p) = -|S_2| + 1 + x(E(S_2))$. Since $i, k \in S_1 \cup S_2$,

$$\begin{aligned}
d_{ik}(p) &\ge -|S_1 \cup S_2| + 1 + x(E(S_1 \cup S_2)) \\
&\ge -|S_1| - |S_2| + |S_1 \cap S_2| + 1 + x(E(S_1)) + x(E(S_2)) - x(E(S_1 \cap S_2)) \\
&\ge -|S_1| + 1 + x(E(S_1)) - |S_2| + 1 + x(E(S_2)) - 1 + |S_1 \cap S_2| - x(E(S_1 \cap S_2)) \\
&\ge d_{ij}(p) + d_{jk}(p).
\end{aligned}$$

The second inequality is due to the fact that $x(E(S_1 \cup S_2)) \ge x(E(S_1)) + x(E(S_2))$ -

$x(E(S_1 \cap S_2))$ and the last inequality holds by (9e) for $S = S_1 \cap S_2$. Consider a directed cycle C in $G[V_1]$ which passes node p. Any directed cycle C with $|C| = 2$ satisfies the condition (C2) by (9e) for $|S| = 2$. Suppose that $|C| \geq 3$ and assume that $C = \{(p, i), \ldots, (j, p)\}$ and $d_{ij}(p) = -|S_1| + 1 + x(E(S_1))$. Then

$$\begin{aligned}\sum_{(q,r)\in C} d_{qr}(p) &\leq d_{pi}(p) + d_{ij}(p) + d_{jp}(p) \\ &\leq -|S_1| + 1 + x(E(S_1)) + x_{pi} + x_{jp} \\ &\leq -|S_1| + 1 + x(E(S_1 \cup \{p\})) \\ &\leq -|S_1| + 1 + |S_1| + 1 - 1 \\ &\leq 1.\end{aligned}$$

Recall that $d_{qr}(p) + d_{rs}(p) \leq d_{qs}(p)$ for all $q, r, s \in V_1 \backslash \{p\}$. Therefore, $d_{ij}(p)$ is greater than or equal to the length of any path from i to j. It follows that the first inequality holds.

Finally, we now show that Q5 $\subseteq$ P5. Consider $x \in$ Q5 and set $d_{ij}(p)$ as we defined for $t = 5$. Consider any directed cycle C in $G[V_1 \backslash \{p\}]$. Then

$$\begin{aligned}\sum_{(i,j)\in C} d_{ij}(p) &= \sum_{(i,j)\in C} \max\{-1 + x_{ij} + x_{ji}, -1 + x_{ij} + x_{pj} + x_{ip}\} \\ &= \sum_{(i,j)\in C} \max\{x_{ij} + x_{ji}, x_{ij} + x_{pj} + x_{ip}\} - |C| \\ &\leq 0\end{aligned}$$

The inequality holds by (9f), thereby the condition (C1) is satisfied.

Consider a directed cycle C in $G[V_1]$ which passes node p. Any directed cycle C with $|C| = 2$ satisfies the condition (C2) by (9d), so we assume that $|C| \geq 3$. Let $C = \{(p, i_1), (i_1, i_2), \ldots, (i_{r-1}, i_r), (i_r, p)\}$ and $C' = \{(i_1, i_2), \ldots, (i_{r-1}, i_r), (i_r, i_1)\}$. Note that $|C'| = |C| - 1$ and $p \notin C'$. Then

$$\begin{aligned}\sum_{(q,r)\in C} d_{qr}(p) &= \sum_{(q,r)\in C'} d_{qr}(p) + d_{pi_1}(p) + d_{i_r p}(p) - d_{i_r i_1}(p) \\ &\leq \sum_{(q,r)\in C'} d_{qr}(p) + x_{pi_1} + x_{i_r p} - (-1 + x_{i_r i_1} + x_{pi_1} + x_{i_r p}) \\ &= \sum_{(q,r)\in C'} d_{qr}(p) + 1 - x_{i_r i_1} \\ &\leq 1\end{aligned}$$

The last inequality holds because the length of the cycle C' is less than or equal to 0. □

4. CONCLUSIONS

In this paper, we have produced proofs to characterize the projections of five extended formulations with precedence variables for the ATSP, developed by Gouveia and Pires [4, 5], into the natural variable space. By exploiting the longest distance property, we have developed a unifying framework that enabled to give alternative proofs for the projections of three formulations and new proofs for the remaining two. Gouveia and Pires [5] have also introduced some other generalizations of (9c) that result in extended formulations with precedence variables other than those introduced in this paper. When analyzing the projections of these formulations into the natural variable space, we could use the same framework developed in this paper.

To prove that new formulations with precedence variables are any help to solve the ATSP, we need to have an efficient algorithm to solve the relaxations of the proposed formulations. Therefore, finding such an algorithm would be an interesting research issue. Another future research is to find a formulation with precedence variables stronger than the ones developed by Gouveia and Pires [4, 5] by identifying valid inequalities.

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REFERENCES

[1] Balas, E, and W. R. Pulleyblank, "The perfectly matchable subgraph polytope of a bipartite graph," *Networks* 13 (1983), 495-516.

[2] Dantzig, G. B., D. R. Fulkerson, and S. M. Johnson, "Solutions of large scale traveling salesman problem," *Operations Research* 2 (1954), 393-410.

[3] Goemans, M. X. and Y. S. Myung, "A catalog of Steiner tree formulations," *Networks* 23 (1993), 19-28.

[4] Gouveia, L. and J. M. Pires, "The asymmetric travelling salesman problem and a reformulation of the Miller-Tucker-Zemlin constraints," *European Journal of Operational Research* 112 (1999), 134-146.

[5] Gouveia, L. and J. M. Pires, "The asymmetric travelling salesman problem: on generalizations of disaggregated Miller-Tucker-Zemlin constraints," *Discrete Applied Mathematics* 112 (2001), 129-145.

[6] Grötchel, M. and M. Padberg, "Polyhedral Theory," in E. L. Lawler, J. K. Lenstra, A. H. G. Rinnooy Kan, and D. B. Shmoys (Eds.), *The Travelng Salesman Problem*, John Wiley and Sons, New York, 1985.

[7] Langevin, A., F. Soumis, and J. Desrosiers, "Classifications of travelling salesman problem formulation," *Operations Research Letters* 9 (1990), 127-132.

[8] Padberg, M. and T. Y. Sung, "An analytical comparison of different formulations of the travelling salesman problem," *Mathematical Programming* 52 (1991), 315-357.