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Reliability Models for Redundant Systems Using Phase-type Distributions

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ABSTRACT

This paper presents the reliability models for redundant systems composed of repairable components whose failure time and repair time distributions are phase-type. It is shown that the distribution of time to system failure is also phase-type. The dependency between components are considered and integrated into the model by the use of the rate adjustment factor. The phase-type representation is constructed for the system through algebraic operations on the parameters of components' failure time and repair time distributions and the corresponding rate adjustment factors. Types of system structures considered are parallel, k-out-of-N system with load sharing scheme and standby system with operation priority.

1. INTRODUCTION

Phase-type distribution (PH-distribution) is a type of distribution represented by Markov process with an absorbing state. The main feature of PH-distribution is the versatility that enables one to model any probabilistic behaviors. PH-distribution is expected to overcome the limitation of exponential distribution which is frequently assumed in reliability modeling. Several studies have been made on the estimation of PH-distribution from empirical data and approximation of popular distributions into PH-distribution [3, 9]. Another feature relates to the useful closure properties about mathematical operations, which can be applied to the reliability modeling from the knowledge of system structure. The only drawback is the computational complexity that highly depends on the order and sparsity of transition matrix. To enhance the analytical tractability, minimal representation is suggested as a Triangular PH-distribution (TPH-distribution) [3, 5, 11].

Although few studies on the application of PH-distribution to reliability modeling can be seen, [10] presents two basic theorems which show that the failure time distributions of series and parallel systems composed of 2 nonrepairable components are also represented by PH-distributions. This is a simple application that illustrates the modeling power of PH-distribution which makes it possible to integrate the structures into distributional representation of system failure time. But it is more realistic to consider repairable components. Furthermore, the dependencies between components need to be considered for redundant systems.

In this paper, the redundant systems composed of repairable components whose failure time and repair time distributions are given by different PH-distributions are considered. It can be shown that the distribution of time to failure of nonrepairable system is also represented by PH-distribution whose transition matrix is built by proper Kronecker operations on the transition matrices of individual components. Beginning with a simple case of state-independency, the results are readily extended to the case of state-dependency. Rate adjustment factor is introduced for the integration of state-dependency, which represents the change in failure or repair rate. Finally, it is possible to construct the system matrix for reliability from the knowledge of each component's failure time and repair time distributions, system structures and state-dependent factors. The coverage of system structures includes parallel, k-out-of-N system with load-sharing scheme and standby system with operation priority. Eigenvalue approach is adopted for the computation whose procedures are presented.

NOTATION

N	number of components
(i)	superscript, component index, $i = 1, \ldots, N$
$PH(\alpha^{(i)}, \mathbf{T}^{(i)})$	phase-type representation of failure time distribution of component i
$m^{(i)}$	order of $PH(\alpha^{(i)}, \mathbf{T}^{(i)})$
$PH(\beta^{(i)}, \mathbf{S}^{(i)})$	phase-type representation of repair time distribution of component i
$n^{(i)}$	order of $PH(\beta^{(i)}, \mathbf{S}^{(i)})$
$\mathbf{e}_p$	$(p \times 1)$ vector, all elements are 1
$\mathbf{v}_p$	$(1 \times p)$ vector which is equal to $(1, 2, \ldots, p)$

$\mathbf{E}_{pq}$	$(p \times q)$ matrix, all elements are 1
$\mathbf{I}_p$	$(p \times p)$ identity matrix
$\lVert\mathbf{v}\rVert$	Holder norm of vector equal to $\lVert\mathbf{v}\rVert_1$
$\mathbf{T}_0^{(i)}$	$-\mathbf{T}^{(i)}\mathbf{e}_{m^{(i)}}$
$\mathbf{S}_0^{(i)}$	$-\mathbf{S}^{(i)}\mathbf{e}_{n^{(i)}}$
$\overline{\mathbf{T}}_0^{(i)}$	$(m^{(i)} \times n^{(i)})$ matrix with identical columns $\mathbf{T}_0^{(i)}$
$\overline{\mathbf{S}}_0^{(i)}$	$(n^{(i)} \times m^{(i)})$ matrix with identical columns $\mathbf{S}_0^{(i)}$
$\mathbf{A}_0^{(i)}$	$(m^{(i)} \times m^{(i)})$ matrix equal to $diag(\alpha_1^{(i)}, \ldots, \alpha_{m^{(i)}}^{(i)})$
$\mathbf{B}_0^{(i)}$	$(n^{(i)} \times n^{(i)})$ matrix equal to $diag(\beta_1^{(i)}, \ldots, \beta_{n^{(i)}}^{(i)})$
$x^{(i)}$	state of component i, $x^{(i)} \in \{1, \ldots, m^{(i)}, \ldots, m^{(i)} + n^{(i)}\}$
$z^{(i)}$	super-state of component i, 1 if component i is operating and 0 if it is in failure
j	subscript, super-state index of system, $j = 1, \ldots, 2^N$
i^*	index of the component whose state has changed during transition from j to j', which is obtained by $< \mathbf{v}_N, \mathbf{z}_{j'} - \mathbf{z}_j >$
Ω	state space of system
Λ	state transition rate matrix of system
$\varphi_j(i_1, i_2)$	the number of states at super-state j when a subsystem comprising components $i_1, i_1{+}1, \ldots, i_2$ is considered, which is computed by $\prod_{i=i_1}^{i_2}\{m^{(i)}z_j^{(i)} + n^{(i)}(1 - z_j^{(i)})\}$, for $i_1 \le i_2$ and 1, otherwise
$H(t)$	distribution function of time to the first system failure
$R(t)$	reliability function of system

ASSUMPTION

(a) Initially each component is operative, which leads to setting $\alpha_{m^{(i)}+1}^{(i)} = 0$, for all i.

(b) Repairing time cannot be zero, which leads to setting $\beta_{n^{(i)}+1}^{(i)} = 0$, for all i.

(c) Repair makes each component renewed.

(d) Repair facility is available always.

2. PH-DISTRIBUTION

2.1 PH-distribution

$PH(\alpha, \mathbf{T})$ of order m is defined as the time until absorption in the Markov process on the states $\{1,\ldots,m,m+1\}$ with infinitesimal generator

$$\mathbf{Q} = \begin{vmatrix} \mathbf{T} & \mathbf{T}_0 \\ \mathbf{0} & 0 \end{vmatrix} \tag{1}$$

where the $m \times m$ matrix $\mathbf{T}$ satisfies $T_{ii} < 0$ for $1 \le i \le m$, and $T_{ij} \ge 0$ for $i \ne j$. Also $\mathbf{T}\mathbf{e}_m + \mathbf{T}_0 = \mathbf{0}$ and the initial probability vector of $\mathbf{Q}$ is given by (α, α_{m+1}) with $\alpha\mathbf{e}_m + \alpha_{m+1} = 1$. When transition matrix $\mathbf{T}$ is a triangular matrix, it is called TPH-distribution. Distribution function and the i th moment are respectively given by

$$F(t) = 1 - \alpha \exp(\mathbf{T}t)\mathbf{e}_m , \tag{2a}$$

$$M_i = (-1)^i i! \alpha \mathbf{T}^{-i} \mathbf{e}_m . \tag{2b}$$

Consider component i, whose failure time and repair time distributions are represented by $PH(\alpha^{(i)}, \mathbf{T}^{(i)})$ and $PH(\beta^{(i)}, \mathbf{S}^{(i)})$, respectively. The state of component i alternates between operation and failure and the process is renewed after each repair, i.e. the behavior of individual component is described as an alternating renewal process. Transition matrix of each component is given by

$$\Lambda^{(i)} = \begin{vmatrix} \mathbf{T}^{(i)} & \overline{\mathbf{T}}_0^{(i)} \mathbf{B}_0^{(i)} \\ \overline{\mathbf{S}}_0^{(i)} \mathbf{A}_0^{(i)} & \mathbf{S}^{(i)} \end{vmatrix} . \tag{3}$$

Detailed explanation can be seen in [10].

2.2 Kronecker Algebra

Kronecker Algebra is frequently included in mathematical operations of PH-distribution. Let $\mathbf{A} \in R_{kl}$ and $\mathbf{B} \in R_{mn}$. The Kronecker Product of $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\mathbf{A} \otimes \mathbf{B} = \begin{vmatrix} a_{11}\mathbf{B} & \cdots & a_{1l}\mathbf{B} \\ \vdots & & \vdots \\ a_{k1}\mathbf{B} & \cdots & a_{kl}\mathbf{B} \end{vmatrix} . \tag{4}$$

Let $\mathbf{A} \in R_{mm}$ and $\mathbf{B} \in R_{nn}$. The Kronecker Sum of $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\mathbf{A} \oplus \mathbf{B} = (\mathbf{A} \otimes \mathbf{I}_n) + (\mathbf{I}_m \otimes \mathbf{B}) . \tag{5}$$

The following two properties are obtained in [4, 6].

Property 1

$$\text{(a)} \quad (\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C} . \tag{6a}$$

$$\text{(b)} \quad \mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C} . \tag{6b}$$

Property 2

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD} . \tag{7}$$

3. STATE-INDEPENDENT CASE

We start with state-independent case where failure and repair behaviors of each component are independent of the states of other components. System state is represented by $\mathbf{x} = (x^{(1)}, \ldots, x^{(N)})$. Considering all states, general transition matrix can be stated in terms of Kronecker Algebra [1], which is given by

$$\Lambda = \Lambda^{(1)} \oplus \Lambda^{(2)} \oplus \cdots \oplus \Lambda^{(N)} . \tag{8}$$

Since what we need to know to decide the system failure is only the information about whether the components are in operation or failure, we introduce the super-state vector $\mathbf{z} = (z^{(1)}, \ldots, z^{(N)})$. Systematic ordering of system's super-states is possible from the Kronecker Sum. Index j for a super-state is related to the component's super-state by the relation;

$$j = 2^N - \sum_{i=1}^{N} z_j^{(i)} 2^{N-i} . \tag{9}$$

System state space is divided into subspaces in terms of $\mathbf{z}$, namely,

$$\Omega = \bigcup_{j=1}^{2^N} \Omega_j . \tag{10}$$

In reliability model of redundant systems, not all states are available and there exists a set of states which indicate system failure, which is regarded as an absorbing state for PH-distribution modeling. For example, in case of parallel structure, the state subspace Ω_{2^N} includes all states indicating system failure. The transient super-states are defined only from the knowledge of system structure. The following base theorem shows that the PH-distribution representation of system failure time is constructed by the simple algebraic operations on the parameters of components' failure time and repair time distributions.

Theorem 1

Suppose the system fails when the number of failed components is greater than or equal to $\overline{N}(\leq N)$. Then $H(t)$ is a PH-distribution represented by $PH(\gamma, \mathbf{L})$ of order n and the parameters are given by

$$n = \sum_{\{j;\|\mathbf{z}_j\| > N-\overline{N}\}} \varphi_j(1,N) \quad , \tag{11a}$$

$$\gamma = (\alpha^{(1)} \otimes \cdots \otimes \alpha^{(N)}, \mathbf{0}), \quad \gamma_{n+1} = 0, \tag{11b}$$

$$\mathbf{L} = [\mathbf{L}_{jj'}]$$

where

$$\mathbf{L}_{jj'} = \begin{cases} \sum_{i=1}^{N} \mathbf{I}_{\varphi_j(1,i-1)} \otimes \left[\mathbf{T}^{(i)} z_j^{(i)} + \mathbf{S}^{(i)}(1 - z_j^{(i)})\right] \otimes \mathbf{I}_{\varphi_j(i+1,N)} & , if \left\| \mathbf{z}_{j'} - \mathbf{z}_j \right\| = 0 \\ \mathbf{I}_{\varphi_j(1,i^*-1)} \otimes \left[\overline{\mathbf{T}}_0^{(i^*)} \mathbf{B}_0^{(i^*)} z_j^{(i^*)} + \overline{\mathbf{S}}_0^{(i^*)} \mathbf{A}_0^{(i^*)} (1 - z_j^{(i^*)})\right] \otimes \mathbf{I}_{\varphi_j(i^*+1,N)} & , if \left\| \mathbf{z}_{j'} - \mathbf{z}_j \right\| = 1 \\ \mathbf{0} & , o/w \end{cases},$$

$$\tag{11c}$$

$$\mathbf{L}_0 = [\mathbf{L}_j^0]$$

where

$$\mathbf{L}_j^0 = \begin{cases} \sum_{\{i; z_j^{(i)}=1\}} \mathbf{e}_{\varphi_j(1,i-1)} \otimes \mathbf{T}_0^{(i)} \otimes \mathbf{e}_{\varphi_j(i+1,N)} & , if \left\|\mathbf{z}_j\right\| = N - \overline{N} + 1 \\ \mathbf{0} & , o/w \end{cases}. \tag{11d}$$

(Proof) The proof is given in the Appendix.

From Theorem 1, we can derive the reliability function and mean time to failure(MTTF) of system, which are respectively given by

$$R(t) = 1 - H(t) = \gamma \exp(\mathbf{L}t)\mathbf{e}_n, \tag{12a}$$

$$MTTF = -\gamma \mathbf{L}^{-1}\mathbf{e}_n . \tag{12b}$$

Theorem 1 is directly applicable to the reliability model of parallel and k-out-of-N system. For simple illustration, consider a parallel system of $N = 2$. Transient super-states of system are ordered as follows;

$$\mathbf{z}_1 = (1,1),\ \mathbf{z}_2 = (1,0),\ \mathbf{z}_3 = (0,1) \tag{13}$$

and the representation $PH(\gamma, \mathbf{L})$ of $H(t)$ is given by

$$\gamma = (\alpha^{(1)} \otimes \alpha^{(2)},\ \mathbf{0}), \tag{14a}$$

$$\mathbf{L} = \begin{vmatrix} \mathbf{T}^{(1)} \otimes \mathbf{I}_{m^{(2)}} + \mathbf{I}_{m^{(1)}} \otimes \mathbf{T}^{(2)} & \mathbf{I}_{m^{(1)}} \otimes \overline{\mathbf{T}}_0^{(2)}\mathbf{B}_0^{(2)} & \overline{\mathbf{T}}_0^{(1)}\mathbf{B}_0^{(1)} \otimes \mathbf{I}_{m^{(2)}} \\ \mathbf{I}_{m^{(1)}} \otimes \overline{\mathbf{S}}_0^{(2)}\mathbf{A}_0^{(2)} & \mathbf{T}^{(1)} \otimes \mathbf{I}_{n^{(2)}} + \mathbf{I}_{m^{(1)}} \otimes \mathbf{S}^{(2)} & \mathbf{0} \\ \overline{\mathbf{S}}_0^{(1)}\mathbf{A}_0^{(1)} \otimes \mathbf{I}_{m^{(2)}}. & \mathbf{0} & \mathbf{S}^{(1)} \otimes \mathbf{I}_{m^{(2)}} + \mathbf{I}_{n^{(1)}} \otimes \mathbf{T}^{(2)} \end{vmatrix}, \tag{14b}$$

$$\mathbf{L}_0 = \begin{vmatrix} \mathbf{0} \\ \mathbf{T}_0^{(1)} \otimes \mathbf{e}_{n^{(2)}} \\ \mathbf{e}_{n^{(1)}} \otimes \mathbf{T}_0^{(2)} \end{vmatrix}. \tag{14c}$$

4. STATE-DEPENDENT CASE

In this case, the failure rate of each component changes with the system state. It is a practical consideration in redundant system [2, 12]. We introduce the rate adjustment factor that is determined by the system state. First we need to inspect the distributional change incurred by the addition of constant value to failure rate

Theorem 2
Let $\lambda(t)$ denote the failure rate function of failure time distribution F represented by $PH(\alpha, \mathbf{T})$ of order m. Assume $\alpha_{m+1} = 0$. Then for, $\delta \geq 0$, $\tilde{\lambda}(t) = \lambda(t) + \delta$ is the failure rate function of $\tilde{F}$ represented by $PH(\alpha, \mathbf{T} - \delta \mathbf{I}_m)$.
(Proof) The proof is given in the Appendix.

Remark 1
In view of transition rate matrix of PH-distribution, adding constant value to the failure rate function is equivalent to adding the constant value to each transition rate to the absorbing state while other transition rates are remained unchanged.

The following property shows that MTTF increases by the addition of rate adjustment factor.

Property 3
For $\delta \geq 0$, the mean of $PH(\boldsymbol{\alpha}, \mathbf{T} - \delta\mathbf{I})$ is greater than or equal to that of $PH(\boldsymbol{\alpha}, \mathbf{T})$.
(Proof) The proof is given in the Appendix.

Let $\lambda_j^{(i)}$ and $\delta_j^{(i)}$ denote the failure rate and the rate adjustment factor of component i when system's super-state is j, respectively. Furthermore, let $\lambda_0{}^{(i)}$ denote the reference failure rate of component i. Then the state-dependency for failure behaviors is represented by

$$\lambda_j^{(i)}(t) = \lambda_0^{(i)}(t) + \delta_j^{(i)}. \tag{15}$$

Similar consideration is possible for the repair rate. Let $\mu_j^{(i)}$ and $\zeta_j^{(i)}$ denote the repair rate and the rate adjustment factor of component i when system's super-state is j, respectively. Furthermore, let $\mu_0{}^{(i)}$ denote the reference repair rate of component i. Then the state-dependency for repair behaviors is represented by

$$\mu_j^{(i)}(t) = \mu_0^{(i)}(t) + \zeta_j^{(i)}. \tag{16}$$

The following cases exemplify this model.

CASE 1 Load-sharing parallel and k-out-of-N system
Consider the case when failure rate of each component is affected by the number of failed components. This effect is modeled as follows: Let $\lambda_0^{(i)}$ represent the failure rate of component i when all the components are operating. Then

$$\delta_j^{(i)} = \delta^{(i)} \left\| \overline{\mathbf{z}}_j \right\| \text{ where } \overline{\mathbf{z}}_j = (1 - z_j^{(1)}, \ldots, 1 - z_j^{(N)}). \tag{17}$$

CASE 2 Stanby system with operation priority
Each component has different failure rates in operation and in standby state. All components are ordered by the operation priority. The component of highest priority is operating if available and the other components are in standby state. If operating component fails, the component of next highest priority among available components starts operation and it will be switched to the standby state as soon as the component of the highest priority becomes available. Assuming the

component index is ordered by its priority, this effect is modeled as follows: Let i^1 denote the only component that is currently operating. i^1 is the first index in $\mathbf{z}_j$ such that $z_j^{(i)} = 1$. Let $\lambda_0^{(i)}$ represent the failure rate of component i when it is in the standby state. Then

$$\delta_j^{(i)} = \begin{cases} \delta^{(i)} & , \text{ if } i = i^1 \\ 0 & , \text{ if } i > i^1 \end{cases}. \tag{18}$$

CASE 3 Multiple Repair facilities

When there are multiple repair facilities and they are always available, repairing speed increases as the number of operating components increases. This effect is modeled as follows: Let $\mu_0^{(i)}$ represent the repair rate of component i when all the components are failed. Then

$$\zeta_j^{(i)} = \zeta^{(i)} \left\| \mathbf{z}_j \right\|. \tag{19}$$

Now, the modified PH-distribution representation of system failure time is constructed by the algebraic operations on the parameters of components' failure time and repair time distributions and the associated rate adjustment factors. The result is given by theorem 3.

Theorem 3

Let $\delta_j^{(i)}$ and $\zeta_j^{(i)}$ denote the positive rate adjustment factors for failure rate and repair rate, respectively. Then, $H(t)$ is a PH-distribution represented by $PH(\gamma, \mathbf{L})$ of order n and the modified parameters are given by

$$\gamma = (\alpha^{(1)} \otimes \cdots \otimes \alpha^{(N)}, \mathbf{0}), \ \gamma_{n+1} = 0, \tag{20a}$$

$$\mathbf{L} = [\mathbf{L}_{jj'}]$$

where

$$\mathbf{L}_{jj'} = \begin{cases} \sum_{i=1}^{N} \mathbf{I}_{\varphi,(1,i-1)} \otimes \left[\left(\mathbf{T}^{(i)} - \delta_j^{(i)} \mathbf{I}_{m^{(i)}} \right) z_j^{(i)} + \left(\mathbf{S}^{(i)} - \zeta_j^{(i)} \mathbf{I}_{n^{(i)}} \right) (1 - z_j^{(i)}) \right] \otimes \mathbf{I}_{\varphi,(i+1,N)} & , \text{if } \left\| \mathbf{z}_{j'} - \mathbf{z}_j \right\| = 0 \\ \mathbf{I}_{\varphi,(1,i^*-1)} \otimes \left[\left(\overline{\mathbf{T}}_0^{(i^*)} + \delta_j^{(i^*)} \mathbf{E}_{m^{(i^*)}, n^{(i^*)}} \right) \mathbf{B}_0^{(i^*)} z_j^{(i^*)} + \left(\overline{\mathbf{S}}_0^{(i^*)} + \zeta_j^{(i^*)} \mathbf{E}_{n^{(i^*)}, m^{(i^*)}} \right) \mathbf{A}_0^{(i^*)} (1 - z_j^{(i^*)}) \right] \otimes \mathbf{I}_{\varphi,(i^*+1,N)} & , \text{if } \left\| \mathbf{z}_{j'} - \mathbf{z}_j \right\| = 1 \\ \mathbf{0} & , o/w \end{cases}$$

(20b)

$\mathbf{L}_0 = [\mathbf{L}_j^0]$

where

$$\mathbf{L}_j^0 = \begin{cases} \sum_{\{i;z_j^{(i)}=1\}} \mathbf{e}_{\varphi,(1,i-1)} \otimes \left(\mathbf{T}_0^{(i)} + \delta_j^{(i)} \mathbf{e}_{m^{(i)}}\right) \otimes \mathbf{e}_{\varphi,(i+1,N)} & , if \left\| \mathbf{z}_j \right\| = N - \overline{N} + 1 \\ \mathbf{0} & , o/w \end{cases} . \tag{20c}$$

(Proof) The proof is given in the Appendix.

5. COMPUTATION METHOD

Main complexity in reliability computation is caused from matrix-exponential solution. Typically a numerical solution by Runge-Kutta method can be used. In many practical reliability computations, however, this method requires large computation time. In this paper, eigenvalue approach is adopted, where the reliability is computed by the use of Vandermonde system, after obtaining eigenvalues. The procedures are summarized as follows:

5.1 Eigenvalue Solution

Most efficient method of finding eigenvalues of large matrix is known to be Iterative QR-method based on Schur decomposition theorem [6,8]. To reduce the computations, the matrix is first transformed into a Hessenberg matrix by Householder method or Givens method. Since the target matrix $\mathbf{L}$ that is given by theorem 1 has many zero entries, the computational efforts can be reduced. After the iterations, good approximations for the eigenvalues can be obtained. The procedures can be summarized as

(a) $\mathbf{U}^T\mathbf{L}\mathbf{U} = \mathbf{H} = \mathbf{H}_1$ (Hessenberg reduction) (21)

where $\mathbf{U}$ is a product of several transformation matrices.

(b) $\mathbf{H}_k = \mathbf{Q}_k\mathbf{R}_k$ (QR-factorization)

$$\mathbf{H}_{k+1} = \mathbf{R}_k\mathbf{Q}_k = \mathbf{Q}_k^T\mathbf{H}_k\mathbf{Q}_k \quad , \; k = 1, 2, \ldots \tag{22}$$

where $\mathbf{Q}_k$ is a unitary matrix and $\mathbf{R}_k$ is a upper triangular matrix.

5.2 Matrix-exponential Solution

Details for the exact solution of the matrix-exponential is available in [6]. In many reliability models the eigenvalues are distinct [7]. Hence, we summarize the main results when matrix $\mathbf{L}$ is semi-simple, that is, when $\mathbf{L}$ has either distinct eigenvalues only, or multiple eigenvalues but with corresponding linearly independent eigenvectors. The proof of theorem 4 is omitted since it can be found in many texts including [6].

Theorem 4
Suppose $\mathbf{A} \in R_{nn}$ is a semi-simple matrix with distinct eigenvalues $\rho_1, \rho_2, \dots, \rho_s$, where $s \le n$. Then

$$e^{\mathbf{A}t} = \sum_{i=1}^{s} \prod_{\substack{j=1 \\ j \ne i}}^{s} \left(\frac{\mathbf{A} - \rho_j \mathbf{I}}{\rho_i - \rho_j} \right) e^{\rho_i t}. \tag{23}$$

6. NUMERICAL EXAMPLES

The following examples present the accuracy of computation and the effect of rate adjustment factors for some simple structures. CASE I and CASE 2 are considered for exemplifying state-dependent cases. Each system is composed of 3 components and each component has PH-type failure time and repair time distributions. Specific descriptions of input distributions are given by Table 1.

Table 1. Descriptions of input distributions

Component (i)	Failure time distribution			Repair time distribution		
	$\alpha^{(i)}$	$T^{(i)}$	Mean	$\beta^{(i)}$	$S^{(i)}$	Mean
1	(0.4, 0.6)	$\begin{pmatrix} -0.5 & 0.3 \\ 0 & -0.3 \end{pmatrix}$	3.60	(0.2, 0.8)	$\begin{pmatrix} -1.2 & 0.4 \\ 0 & -2.1 \end{pmatrix}$	0.58
2	(0.4, 0.6)	$\begin{pmatrix} -0.5 & 0.2 \\ 0 & -0.4 \end{pmatrix}$	2.70	(0.2, 0.8)	$\begin{pmatrix} -1.2 & 0.4 \\ 0 & -2.1 \end{pmatrix}$	0.58
3	(0.4, 0.6)	$\begin{pmatrix} -0.5 & 0.1 \\ 0 & -0.6 \end{pmatrix}$	1.93	(0.2, 0.8)	$\begin{pmatrix} -1.2 & 0.4 \\ 0 & -2.1 \end{pmatrix}$	0.58

In case of load-sharing system, equivalent loads are assumed for all components, i.e. rate adjustment factors are set to be $\delta_j^{(i)} = \delta \left\| \overline{\mathbf{z}}_j \right\|$, for all i. Figure 1 and Figure 2 present the reliability functions for parallel and 2-out-of-3 system respectively, where the dots represent the simulation results. In these figures, curve (a) represents an independent case with no load-sharing scheme. For the state-dependent case, reliability function of respective systems are computed for various values of δ . As can be seen in the figures, computation results are well matched with the simulation results.

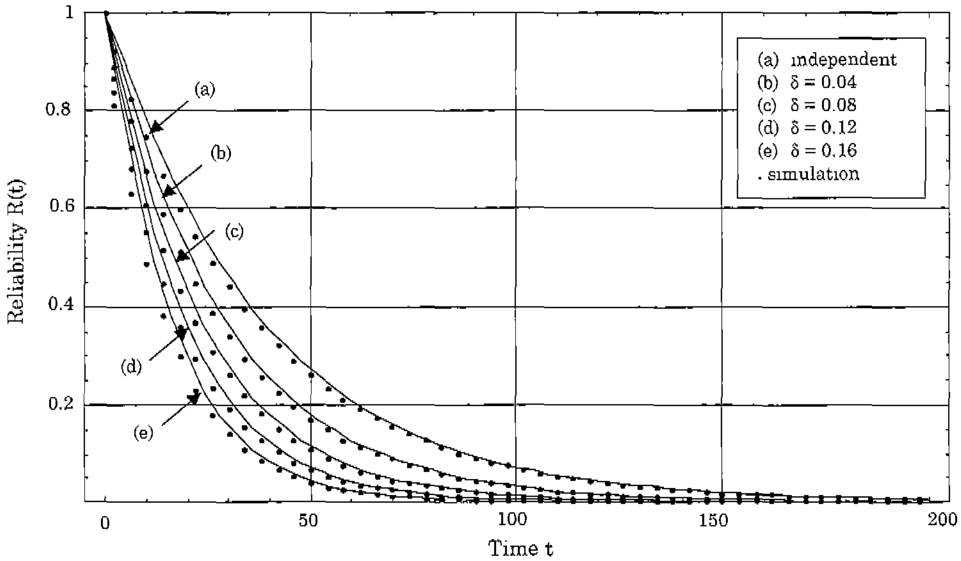

Figure 1. Load-sharing, 3-components parallel system

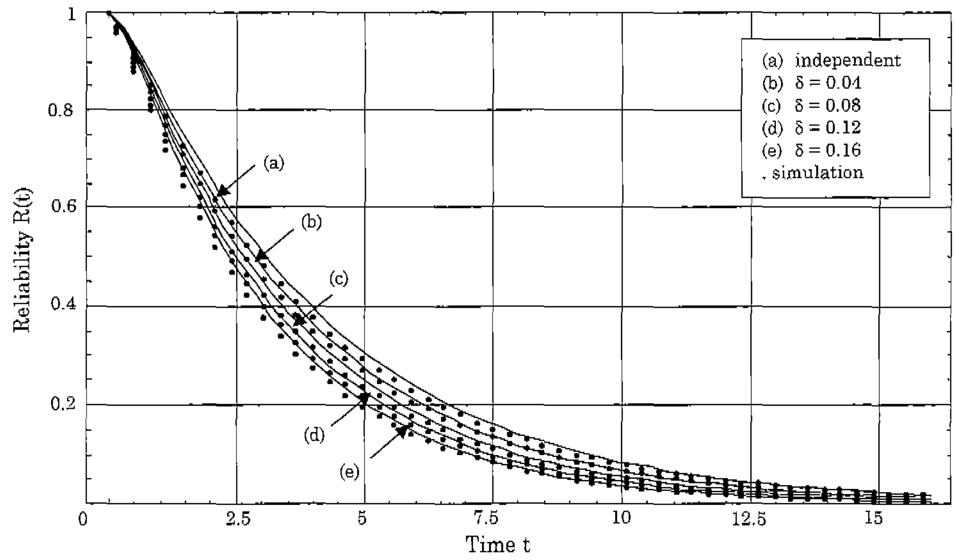

Figure 2. Load-sharing 2-out-of-3 system

In case of standby system with operation priority, the increments of failure rate by switching from standby state to operation state are assumed to be the same for all components, i.e. if component i is operating, then rate adjustment factor for component i is set to be $\delta_j^{(i)} = \delta$. Figure 3 also shows that the results from computation and simulation are well matched in this case.

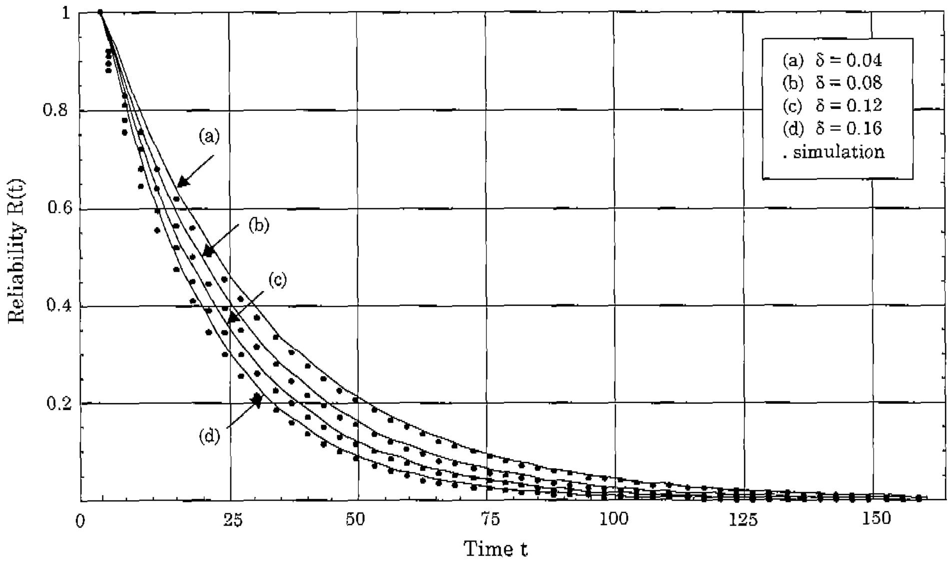

Figure 3. 3-components standby system with operation priority

7. CONCLUSIONS

The main result of this paper is the reliability modeling of redundant systems from the knowledge of parameters of PH-distributions representing each component's failure and repair behaviors, system structure and state dependency. It is shown that the structural property of a system is integrated into the phase-type reliability modeling.

This study is expected to serve as a starting point to construct unified phase-type reliability models for systems composed of many subsystems of different configurations. For example, the reliability of a system composed of 2 subsystems in series each of which is composed of components in parallel can be represented by the phase-type model. Reliability modeling for other types of standby structures using PH-distribution is left as a further study. Finally, unified phase-type reliability modeling for general systems need to be further investigated.

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Appendix

(Proof of Theorem 1)

First, general transition matrix Λ is rewritten in terms of the parameters of failure time and repair time distributions of components. Then, by aggregating into an absorbing state, the phase-type representation of time to system failure is obtained.

Let $\varphi(i_1, i_2)$ denote the number of all states when a subsystem comprising component $i_1, i_1+1, \dots, i_2$ is considered, which is computed by $\prod_{i=i_1}^{i_2}(m^{(i)}+n^{(i)})$, for $i_1 \le i_2$ and 1, otherwise. Let Λ_k denote the transition rate matrix of the subsystem comprising the first k components for $k \ge 2$. Then from (8),

$$\begin{aligned}\Lambda_k &= \Lambda_{k-1} \otimes \mathbf{I}_{\varphi(k,k)} + \mathbf{I}_{\varphi(1,k-1)} \otimes \Lambda^{(k)} \\ &= \Lambda_{k-2} \otimes \mathbf{I}_{\varphi(k-1,k-1)} \otimes \mathbf{I}_{\varphi(k,k)} + \mathbf{I}_{\varphi(1,k-2)} \otimes \Lambda^{(k-1)} \otimes \mathbf{I}_{\varphi(k,k)} + \mathbf{I}_{\varphi(1,k-1)} \otimes \Lambda^{(k)} \\ &= \Lambda^{(1)} \otimes \mathbf{I}_{\varphi(2,2)} \otimes \cdots \otimes \mathbf{I}_{\varphi(k,k)} + \mathbf{I}_{\varphi(1,1)} \otimes \Lambda^{(2)} \otimes \mathbf{I}_{\varphi(3,3)} \otimes \cdots \otimes \mathbf{I}_{\varphi(k,k)} + \cdots \\ &\quad + \mathbf{I}_{\varphi(1,k-2)} \otimes \Lambda^{(k-1)} \otimes \mathbf{I}_{\varphi(k,k)} + \mathbf{I}_{\varphi(1,k-1)} \otimes \Lambda^{(k)}.\end{aligned}$$

Hence, Λ_k can be expressed by

$$\Lambda_k = \sum_{i=1}^{k} \mathbf{D}_k^{(i)} \quad , \textit{ where } \mathbf{D}_k^{(i)} = \mathbf{I}_{\varphi(1,i-1)} \otimes \Lambda^{(i)} \otimes \mathbf{I}_{\varphi(i+1,k)}.$$

Consequently,

$$\Lambda = \Lambda_N = \sum_{i=1}^{N} \mathbf{D}_N^{(i)} .$$

The matrix $\mathbf{D}_N^{(i)}$ is pertinent to component i and can be expressed as Kronecker Product form;

$$\mathbf{D}_N^{(i)} = \mathbf{I}_{\varphi(1,i-1)} \otimes \begin{vmatrix} \mathbf{T}^{(i)} & \overline{\mathbf{T}}_0^{(i)}\mathbf{B}_0^{(i)} \\ \overline{\mathbf{S}}_0^{(i)}\mathbf{A}_0^{(i)} & \mathbf{S}^{(i)} \end{vmatrix} \otimes \mathbf{I}_{\varphi(i+1,N)} .$$

The role of identity matrices $\mathbf{I}_{\varphi(1,i-1)}$ and $\mathbf{I}_{\varphi(i+1,N)}$ is to construct the structure of Λ. The structure of $\mathbf{I}_{\varphi(1,i-1)}$ represents all the possible states of $(x^{(1)}, x^{(2)}, \ldots, x^{(i-1)})$ and it is composed of 2^{i-1} identity matrices of which each represents the structure corresponding to the partial super-state $(z^{(1)}, z^{(2)}, \ldots, z^{(i-1)})$. Namely,

$$\mathbf{I}_{\varphi(1,i-1)} = \begin{vmatrix} \mathbf{I}_{(1,1,\ldots,1)} & & & \mathbf{0} \\ & \mathbf{I}_{(1,1,\ldots,0)} & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{I}_{(0,0,\ldots,0)} \end{vmatrix} .$$

Similar interpretation is possible for $\mathbf{I}_{\varphi(i+1,N)}$. Thus if we let $\mathbf{I}_{i-}$ and $\mathbf{I}_{i+}$ represent identity matrices whose structures correspond to the partial super-state $(z^{(1)}, \ldots, z^{(i-1)})$ and $(z^{(i+1)}, \ldots, z^{(N)})$ respectively, then

$$\mathbf{D}_N^{(i)} = \begin{vmatrix} \ddots & & & \\ & \mathbf{I}_{i-} \otimes \mathbf{T}^{(i)} \otimes \mathbf{I}_{i+} & \mathbf{I}_{i-} \otimes \overline{\mathbf{T}}_0^{(i)}\mathbf{B}_0^{(i)} \otimes \mathbf{I}_{i+} & \\ & \mathbf{I}_{i-} \otimes \overline{\mathbf{S}}_0^{(i)}\mathbf{B}_0^{(i)} \otimes \mathbf{I}_{i+} & \mathbf{I}_{i-} \otimes \mathbf{S}^{(i)} \otimes \mathbf{I}_{i+} & \\ & & & \ddots \end{vmatrix} .$$

Now considering the meaning of Kronecker Product, the following facts are obtained;

(a) $\mathbf{D}_N^{(i)}$ adds $\mathbf{I}_{i-} \otimes \mathbf{T}^{(i)} \otimes \mathbf{I}_{i+}$ in all diagonal entries which corresponds to the system's super-state $\mathbf{z}$ where $z^{(i)} = 1$, and such an entry is added by all i.

(b) $\mathbf{D}_N^{(i)}$ adds $\mathbf{I}_{i-}\otimes\overline{\mathbf{T}}_0^{(i)}\mathbf{B}_0^{(i)}\otimes\mathbf{I}_{i+}$ in all entries which represents the transition of system's super-state from $\mathbf{z}$ to $\mathbf{z}'$ where $z^{(i)}=1$ and $z'^{(i)}=0$, and this is the only addition to such an entry.

(c) $\mathbf{D}_N^{(i)}$ adds $\mathbf{I}_{i-}\otimes\mathbf{S}^{(i)}\otimes\mathbf{I}_{i+}$ in all diagonal entries which corresponds to the system's super-state $\mathbf{z}$ where $z^{(i)}=0$, and such an entry is added by all i.

(d) $\mathbf{D}_N^{(i)}$ adds $\mathbf{I}_{i-}\otimes\overline{\mathbf{S}}_0^{(i)}\mathbf{B}_0^{(i)}\otimes\mathbf{I}_{i+}$ in all entries which represents the transition of system's super-state from $\mathbf{z}$ to $\mathbf{z}'$ where $z^{(i)}=0$ and $z'^{(i)}=1$, and this is the only addition to such an entry.

The dimension of identity matrices $\mathbf{I}_{i-}$ and $\mathbf{I}_{i+}$ is determined by the current super-state of all components except component i. From these facts, complete transition matrix Λ is built as seen in the theorem. But in reliability model, not all states are available. All the states indicating the system failure are aggregated to an absorbing state. Consequently the transition rate to the absorbing state must be recalculated, which is done by summation of all the row elements in the transition rate matrix from $\mathbf{z}_j$ to $\mathbf{z}_{j'}$ where $\|\mathbf{z}_j\|=N-\overline{N}+1$ and $\|\mathbf{z}_{j'}\|=N-\overline{N}$. Thus the summation is given by

$$\begin{aligned}
&\sum_{\{i;z_j^{(i)}=1\}}\left(\mathbf{I}_{\varphi_j(1,i-1)}\otimes\overline{\mathbf{T}}_0^{(i)}\mathbf{B}_0^{(i)}\otimes\mathbf{I}_{\varphi_j(i+1,N)}\right)\mathbf{e}_{\varphi_j(1,N)}\\
&\quad=\sum_{\{i;z_j^{(i)}=1\}}\left(\mathbf{I}_{\varphi_j(1,i-1)}\mathbf{e}_{\varphi_j(1,i-1)}\right)\otimes\left(\overline{\mathbf{T}}_0^{(i)}\mathbf{B}_0^{(i)}\mathbf{e}_{\varphi_j(i,i)}\right)\otimes\left(\mathbf{I}_{\varphi_j(i+1,N)}\mathbf{e}_{\varphi_j(i+1,N)}\right)\\
&\quad=\sum_{\{i;z_j^{(i)}=1\}}\mathbf{e}_{\varphi_j(1,i-1)}\otimes\mathbf{T}_0^{(i)}\otimes\mathbf{e}_{\varphi_j(i+1,N)}.
\end{aligned}$$

Finally the phase-type representation of $H(t)$ is obtained.

Q.E.D.

(Proof of Theorem 2)

In [2], failure rate is given by

$$\lambda(t) = \frac{F'(t)}{R(t)}.$$

By definition of PH-distribution, the failure rate of PH-distribution becomes

$$\begin{aligned}
\tilde{\lambda}(t) &= \frac{\alpha \exp(\mathbf{T}t)\mathbf{T}_0}{\alpha \exp(\mathbf{T}t)\mathbf{e}_m} + \delta \\
&= \frac{\alpha \exp[(\mathbf{T}-\delta\mathbf{I})t](\mathbf{T}_0 + \delta\mathbf{e}_m)}{\alpha \exp[(\mathbf{T}-\delta\mathbf{I})t]\mathbf{e}_m} \\
&= \frac{\alpha \exp(\tilde{\mathbf{T}}t)\tilde{\mathbf{T}}_0}{\alpha \exp(\tilde{\mathbf{T}}t)\mathbf{e}_m} = \frac{\tilde{F}'(t)}{\tilde{R}(t)}.
\end{aligned}$$

Thus, $\tilde{\lambda}(t)$ is the failure rate function of new PH-distribution represented by $PH(\alpha, \tilde{\mathbf{T}})$.

Q.E.D.

(Proof of Property 3)

The difference of means between two PH-distributions is given by

$$\begin{aligned}
(-\alpha\mathbf{T}^{-1}\mathbf{e}_m) - \{-\alpha(\mathbf{T}-\delta\mathbf{I})^{-1}\mathbf{e}_m\} \\
&= \alpha\{(\mathbf{T}-\delta\mathbf{I})^{-1} - \mathbf{T}^{-1}\}\mathbf{e}_m \\
&= \delta\alpha(\mathbf{T}-\delta\mathbf{I})^{-1}\mathbf{T}^{-1}\mathbf{e}_m \\
&= \delta\alpha\{-(\mathbf{T}-\delta\mathbf{I})^{-1}(-\mathbf{T}^{-1})\}\mathbf{e}_m.
\end{aligned}$$

Since $-(\mathbf{T}-\delta\mathbf{I})^{-1}$ and $-\mathbf{T}^{-1}$ are both positive matrices from the property of transition rate matrix, the difference is greater than or equal to zero.

Q.E.D.

(Proof of Theorem 3)

The structure of Λ depends on the transition diagram, not on the state-dependency From theorem 2, it can be said that the state-dependency only appears in the parameters of PH-distribution. Hence, theorem 1 is readily extended to theorem 3 by replacing the parameters of failure time and repair time distributions.

Q.E.D.

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