TOPOLOGICAL CHARACTERIZATIONS
OF CERTAIN LIMIT POINTS FOR MÖBIUS GROUPS

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ABSTRACT. A limit point $p$ of a Möbius group acting on $B^m$ is
called a concentration point if for every sufficiently small connected
open neighborhood of $p$, the set of translates contains a local basis
for the topology of $p$. For the case of two generator Schottky groups
acting on $B^2$, we give characterizations for several different kinds
of limit points.

1. Introduction

Let $\Gamma$ be a discrete subgroup of hyperbolic isometries acting on the
Poincaré disc $B^m$, $m \geq 2$. The discrete group $\Gamma$ acts properly discontinuously in $B^m$, and acts on $\partial B^m$ as a group of conformal homeomorphisms, but need not act properly discontinuously on $\partial B^m$. The action of $\Gamma$ divides $\partial B^m$ into two sets. The ordinary set $\Omega(\Gamma)$ is the largest open subset of $\partial B^m$ on which $\Gamma$ acts discontinuously. The complement of $\Omega(\Gamma)$ in $\partial B^m$ is the limit set, denoted by $\Lambda(\Gamma)$ or simply $\Lambda$. The limit set $\Lambda(\Gamma)$ is the set of accumulation points of the orbit $\Gamma(x)$ for one, hence for every, point $x \in B^m$. Equivalently, the limit set is the smallest nonempty closed set in $\partial B^m$ on which $\Gamma$ does not act discontinuously. If $\Lambda$ contains two or fewer points, $\Gamma$ is elementary, and contains a free abelian subgroup of finite index. Otherwise, $\Gamma$ is nonelementary. In this paper, we always assume that $\Gamma$ is nonelementary.

It is easy to see that $\Lambda(\Gamma) = \Lambda(\Gamma')$ for any nontrivial normal subgroup $\Gamma'$ of $\Gamma$. Also, if $x$ is any point of $\partial B^m$, then the accumulation points of any orbit of $x$ under $\Gamma$ lie in $\Lambda(\Gamma)$. For a nonelementary group $\Gamma$, define $CH(\Lambda)$ to be the smallest nonempty convex set in $B^m$ which is invariant under the action of $\Gamma$; this is the convex hull of $\Gamma$. The

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boundary at infinity of $CH(\Lambda)$ is precisely $\Lambda$, and so $CH(\Lambda)$ contains every geodesic line in $B^m$ both of whose endpoints at infinity are in $\Lambda$. By a neighborhood of $p$, we will always mean an open neighborhood of $p$ in $\partial B^m$.

**Definition.** One says that a neighborhood $U$ of $p$ can be concentrated at $p$ if for every neighborhood $V$ of $p$, there exists an element $\gamma \in \Gamma$ such that $p \in \gamma(U)$ and $\gamma(U) \subset V$.

The limit point $p$ is called a **concentration point** for $\Gamma$ if there exists a neighborhood of $W$ of $p$ such that every neighborhood $U$ of $p$ with $U \subset W$ can be concentrated at $p$. If $\Gamma$ is a Fuchsian group, a slight weaker concept than concentration point turns out to be important.

**Definition.** A limit point $p$ is called a **geodesic separation point** for the Fuchsian group $\Gamma$ if for every sufficiently small connected neighborhood $U$ of $p$, either $U$ or $S^1 - U$ can be concentrated at $p$.

A **weak concentration point** which is the weakest reasonable concept of concentration is characterized in [5]. Namely, the limit point $p$ is a weak concentration point for $\Gamma$ if there exists a connected open set $U$ that can be concentrated at $p$. The main property is that if the limit set approaches a point from two different tangential directions in $\partial B^m$ then it is a weak concentration point. This implies every limit point of a geometrically finite group must be a weak concentration point. A limit point $p$ is called a **controlled concentration point** if it has a neighborhood $U$ such that for every neighborhood $V$ of $p$, there exists an element $\gamma \in \Gamma$ so that $p \in \gamma(U)$ and $\gamma(U) \subset V$. In the following sections, we will give characterizations for controlled concentration points and concentration points in the case of two generator Schottky groups.

**Definition.** One says that a pair of open sets $(U_1, U_2)$ in $\partial B^m$ is an **admissible pair** if
(a) $\overline{U_2} \subset U_1$,
(b) $U_2 \cap \Lambda \neq \emptyset$ and
(c) $\Lambda \not\subset U_1$.

One says that an admissible pair $(U_1, U_2)$ can be concentrated at $p$ if for every neighborhood $V$ of $p$, there exists an element $\gamma \in \Gamma$ such that $p \in \gamma(U_2) \subset \gamma(U_1) \subset V$.

**Definition.** A geodesic $\lambda$ is called a geodesic for $\Gamma$ if both endpoints of $\lambda$ are limit points of $\Gamma$. The limit point $p$ is called a **Myrberg-Agard point**.
density point for $\Gamma$ if whenever $\mu$ is an oriented geodesic for $\Gamma$ and $\alpha$ is a geodesic ray ending at $p$ in $CH(\Lambda)$ (convex hull of $\Lambda$), there is a sequence of elements $\{\gamma_i\}$ such that $\{\gamma_i(\alpha)\}$ converge to $\mu$ in an oriented sense.

Using the characterization of Theorem 4.1 in [3], one can easily see that every Myrberg-Agard density point is a controlled concentration point. The next theorem will be useful to get a characterization of Myrberg-Agard density points for a two generator Schottky group.

**Theorem 1.1.** A limit point $p$ is a Myrberg-Agard density point for $\Gamma$ if and only if every admissible pair $(U_1, U_2)$ can be concentrated at $p$.

**Proof.** See Theorem 3.1 in [2]. \hfill \Box

2. Schottky groups and limit points

We will work with a 2-generator $m$-dimensional Schottky group $\Gamma$, although it will be apparent that the same phenomena occur for other examples (in particular, with more generators). The limit set of $\Gamma$ is a Cantor set which can be understood quite explicitly using the sequence of crossings of a geodesic ray (ending at the limit point) with the translates of two fixed sides of a fundamental domain.

To define $\Gamma$, we work in the Poincaré unit disc $B^m$. Let $a$ and $a'$ be the geodesic hyperplanes in $B^m$ which lie in the spheres in $\mathbb{R}^m$ with centers at the points $(1.1,0,...,0)$ and $(-1.1,0,...,0)$, say. Similarly, let $b$ and $b'$ lie in the spheres with centers at the points $(0,...,0,1.1)$ and $(0,...,0,-1.1)$. Choose $a, a', b, b'$ so that they are mutually disjoint. As the generators of $\Gamma$, select two orientation-preserving hyperbolic isometries: one carrying $a$ to $a'$ and one carrying $b$ to $b'$. Fix one of the direction normal to $a$ as the positive direction. It determines a positive normal direction for each translate of $a$. Similarly, we label $b$ and its translates. A crossing of an oriented geodesic of geodesic ray in $B^m$ with a translate of $a$ or $b$ will be called a positive crossing when it agrees with the selected direction; otherwise it will be called a negative crossing.

Suppose $\alpha$ is a geodesic ray in $B^m$, which does not lie in a translate of $a$ and $b$. Then $\alpha$ crosses a sequence (finite or infinite, possibly of length 0) of translates of $a$ and $b$. (When a geodesic ray starts in a translate, we count that intersection as a crossing.) To $\alpha$, we associate a sequence $S(\alpha) = x_1 x_2 x_3...$ of elements in the set $\{a, \bar{a}, b, \bar{b}\}$ in the following way.
If the $n$th crossing of $\alpha$ with the union of the translates of $a$ and $b$ is a positive crossing with a translate of $a$, then $x_n = a$. If the $n$th crossing is a negative crossing with a translate of $a$, then $x_n = \bar{a}$. For crossings with translates of $b$, the elements $b$ and $\bar{b}$ are assigned similarly. Note that $S(\alpha)$ is an infinite sequence if and only if $\alpha$ ends at a limit point of $\Gamma$, and note that, for each sequence $S = x_1x_2x_3\ldots$ of elements of the set $\{a, \bar{a}, b, \bar{b}\}$ (with the property that for no $n$ is $x_n, x_{n+1}$ in the set $\{a\bar{a}, \bar{a}a, b\bar{b}, \bar{b}b\}$), there exists a geodesic ray $\alpha$ with $S(\alpha) = S$.

Using these sequences, the controlled concentration points of $\Gamma$ can be characterized. The following characterization appears in [1], but we reproduce its proof here for the convenience of the reader.

**PROPOSITION 2.1.** Let $p$ be a limit point of $\Gamma$ which is the endpoint of a geodesic ray $\alpha$ with $S(\alpha) = x_1x_2x_3\ldots$. Then $p$ is a controlled concentration point for $\Gamma$ if and only if $S(\alpha)$ has the following property. There exists $N$ such that for all $n \geq N$, for all positive $k$, and for all $M$, there exists $m \geq M$ such that $x_{n+i} = x_{m+i}$ for all $i$ with $0 \leq i \leq k$.

**Proof.** Denote by $\lambda_n$ the translate of $a$ or $b$ whose crossing with $\alpha$ determines $x_n$, and by $U_n$ the neighborhood of $p$ bounded by the endpoints of $\lambda_n$. Suppose the condition in the Proposition holds. By truncating $\alpha$, we may assume that every subsequence reappears infinitely often. Let $m_n$ be an integer so that $x_{m_n+i} = x_i$ for $1 \leq i \leq n$. Let $\gamma_{m_n}$ be the element of $\Gamma$ that translates $\lambda_1$ to $\lambda_{m_n+1}$. Note that this element translates $\lambda_{k+1}$ onto $\lambda_{m_n+k+1}$ for all $1 \leq k < n$. Given a neighborhood $V$ of $p$, choose $n$ so large that $\lambda_n$ has endpoints in $V$. Then $\gamma_{m_n}(U_1) \subseteq V$ and $p \in \gamma_{m_n}(V)$, showing that $U_1$ can be concentrated with control. Conversely, suppose $p$ is a controlled concentration point and choose $N$ large enough so that $U_N$, and hence every neighborhood of $p$ inside $U_N$, can be concentrated with control. For any $n, k > N$ and any $M$, there exists $\gamma$ so that $\gamma(U_n) \subseteq U_M$ and $\gamma^{-1}(p) \subseteq U_{n+k}$. This $\gamma$ must move $\lambda_n, \lambda_{n+1}, \ldots, \lambda_{n+k}$ onto a sequence of translates of $a$ and $b$ crossed by $\alpha$, with endpoints in $U_M$. Thus the condition of Proposition 2.1 holds.

The next theorem which is originally stated in [2] (Theorem 3.3) is revised. The author would like to thank Darryl McCullough for improvement of the original statement.

We say a sequence $y_1y_2\cdots y_n$ is **admissible** if no pair $y_i, y_{i+1}$ is in $\{a\bar{a}, \bar{a}a, b\bar{b}, \bar{b}b\}$.
Theorem 2.2. The limit point \( p \) is a Myrberg-Agdard density point if and only if for every \( \alpha \) ending at \( p \), every admissible sequence appears as a subsequence of \( S(\alpha) \).

Proof. Denote by \( W_n \) the neighborhood of \( p \) determined by the half space bounded by translates of \( a \) or \( b \) whose crossing with \( \alpha \) determines \( x_n \) of the sequence \( S(\alpha) = x_1 x_2 x_3 \cdots \).

Let \( p \) be a Myrberg-Agdard density point and let \( \alpha \) be a geodesic ray ending at \( p \). For each admissible sequence \( y_1 y_2 \cdots y_n \), we have an admissible pair \((U_1, U_2)\) determined by the half spaces bounded by translates of \( a \) or \( b \) such that \( U_1 = W_{y_1} \) and \( U_2 = W_{y_n} \). For a neighborhood \( V \) of \( p \), we choose an \( x_j \) with \( W_{x_j} \subset V \). Since the admissible pair \((U_1, U_2)\) can be concentrated at \( p \) from Theorem 1.1, there exists an element \( \gamma \in \Gamma \) such that \( p \in \gamma(U_2) \subset \gamma(U_1) \subset W_{x_j} \). Therefore every admissible sequence \( y_1 y_2 \cdots y_n \) appears as a subsequence of \( S(\alpha) \).

Conversely, let \((U_1, U_2)\) be an admissible pair at \( p \) and let \( V \) be a neighborhood of \( p \). Choose a pair \((W_1, W_2)\) which are half spaces bounded by translates of \( a \) or \( b \) such that \( W_2 \subset U_2 \) and \( W_1 \supset U_1 \). If there is an element \( \gamma \in \Gamma \) that concentrates \((W_1, W_2)\) will concentrate \((U_1, U_2)\).

For a geodesic ray \( \alpha \) ending at \( p \) and meeting \( W_1 \), form a sequence \( y_1 y_2 \cdots y_m \) of \( S(\alpha) \) by crossing of \( \alpha \) with translates of \( a \) or \( b \) so that

\[
W_1 = W_{y_1} \supset \cdots \supset W_{y_m} = W_2.
\]

Then the sequence \( y_1 y_2 \cdots y_m \) of \( S(\alpha) \) is admissible.

Now choose an \( x_j \) with \( W_{x_j} \subset V \), and past \( x_j \) there must have an appearance \( x_k x_{k+1} \cdots x_{k+m-1} = y_1 y_2 \cdots y_m \), so there exists an element \( \gamma \in \Gamma \) that moves \( W_1 \supset W_2 \) onto \( W_{x_k} \supset W_{x_{k+m-1}} \). Therefore \((U_1, U_2)\) can be concentrated at \( p \). By using Theorem 1.1, \( p \) is a Myrberg-Agdard density point.

If \((U_1, U_2)\) is not an admissible pair at \( p \), then because the orbit of any limit point is dense in the limit set, there is \( \tau \in \Gamma \) so that \( \tau^{-1}(p) \) is \( U_2 \). Therefore \( p \in \tau(U_2) \subset \tau(U_1) \) hence \((\tau(U_1), \tau(U_2))\) is an admissible pair at \( p \). Now we apply the same argument as in the above to show that \((\tau(U_1), \tau(U_2))\) can be concentrated at \( p \). This also implies the pair \((U_1, U_2)\) can be concentrated at \( p \). Again by using Theorem 1.1, \( p \) is a Myrberg-Agdard density point. This completes the proof of the theorem 2.2.

\( \square \)

The next theorems work only for Schottky groups acting on \( B^2 \). Let \( \Gamma \) be a Schottky group with 2 generators as is described in the beginning
of section 2 but we need to choose the geodesics in $B^2$ which lie in the
spheres in $\mathbb{R}^2$ with centers at the points (1,1,0), (-1,1,0), (0,1,1)
and (0,-1,1).

Denote by $a_n$ a sequence of n a’s, and by $\overline{a_n}$ a sequence of n $\overline{a}$’s. The
following theorem which is a slight modification of Theorem 4.2 in [4]
gives examples of concentration points but not Myberg-Agard density
points for a two generator Schottky group. Other interesting phenomena
of concentration points and related properties can be found in [4].

**Theorem 2.3.** For each increasing sequence of positive integers $1 \leq
i_1 < j_1 < i_2 < j_2 < i_3 \cdots$, if p is a limit point which is the endpoint of
a geodesic ray whose crossing is

$$ba_i, b\overline{a}_i, ba_i, b\overline{a}_j, ba_i, b\overline{a}_j, \cdots$$

then p is a concentration point but not a Myberg-Agard density point.

The next theorem characterizes limit points for finitely generated
Fuchsian groups.

**Theorem 2.4.** (Theorem 3.2 in [4]) Let $\Gamma$ be a Fuchsian group. If
$\Gamma$ is finitely generated, then every limit point of $\gamma$ is either a parabolic
fixed point or a geodesic separation point.

Now we will give an example which shows that in Theorem 2.4 the
hypothesis that $\Gamma$ is finitely generated is necessary.

**Example 2.5.** A conical limit point for an infinitely generated 2-
dimensional Schottky group which is not a geodesic separation point.

For each positive integer $k$, let $z_k = \exp \left( \frac{\pi}{2} (1 - \frac{1}{k}) \right) \in S^1 \subset \overline{B^2}$.
Denote by $\ell_n$ the geodesic in $B^2$ through the origin with one end limiting
to $z_n$. Choose geodesics $\lambda_n$ perpendicular to $\ell_n$, near $z_n$, with diameters
small and limiting to 0 sufficiently fast so that each $\lambda_n$ separates $z_n$ from
all other $z_i$, and the $\lambda_n$ are pairwise disjoint. Let $\lambda'_n$ be the image of $\lambda_{n+1}$
under the reflection across the line through the origin perpendicular to
$\ell_{n+1}$. For $n \geq 1$ let $\gamma_n$ be the hyperbolic isometry which moves $\lambda_n$
on to $\lambda'_n$, carries the complementary region of $\lambda_n$ containing $z_n$ to the
complementary region of $\lambda'_n$ containing $z_n$, and carries $\ell_n$ to $\ell_{n+1}$. The
group generated by the $\gamma_n$ is a Schottky group of infinite rank. For
\( n \geq 1 \) let \( \tau_n = \gamma_1^{-1}\gamma_2^{-1}\cdots\gamma_n^{-1} \) and \( \mu_1 = \lambda_1 \) and \( \mu_n = \tau_{n-1}(\lambda_n) \) for \( n \geq 2 \). Then the \( \mu_n \) form a nested sequence limiting toward \( z_1 \), and each \( \mu_n \) is the only translate of \( \lambda_n \) that crosses \( \ell_1 \). Therefore \( z_1 \) is not a geodesic separation point. But \( \tau_n^{-1}(\ell_1) = \ell_{n+1} \), and the images of the origin under \( \tau_n \) form a sequence of points on \( \ell_1 \) limiting to \( z_1 \). Therefore \( z_1 \) is a conical limit point for \( \Gamma \).

References


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