A NOTE ON CLARKSON'S INEQUALITIES

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ABSTRACT. It is proved that if for each \( n \), \( 1 \leq p_n \leq 2 \) and the \((p_n, p'_n)\) Clarkson inequality holds in each Banach space \( X_n \) then the \((t, t')\) Clarkson inequality holds in \((\sum_{n=1}^{\infty} X_n)_r\), the \(\ell_r\)-sum of \( X_n \)'s, where \( 1 \leq r < \infty \), \( t = \min\{p, r, r'\} \) and \( p = \inf\{p_n\} \). The \((p, p')\) Clarkson inequality is preserved by quotient maps and a new proof of a Takahashi-Kato theorem stating that the \((p, p')\) Clarkson inequality holds in a Banach space \( X \) if and only if it holds in its dual space \( X^* \) is given.

1. Introduction

In 1936, while proving the uniform convexity of \( \ell_p \) and \( L_p \) \((1 < p < \infty)\) Clarkson [2] proved that if \( X \) is either \( \ell_p \) or \( L_p \) and \( x, y \in X \) then for \( p \geq 2 \) \((\frac{1}{p} + \frac{1}{p'} = 1)\)

\[
(1) \quad \|x + y\|_p^p + \|x - y\|_p^p \leq 2(\|x\|_p^p + \|y\|_p^p)^{p-1},
\]

\[
(2) \quad 2(\|x\|_p^p + \|y\|_p^p)^{p-1} \leq \|x + y\|_p^p + \|x - y\|_p^p,
\]

\[
(3) \quad 2(\|x\|_p^p + \|y\|_p^p) \leq \|x + y\|_p^p + \|x - y\|_p^p \leq 2^{p-1}(\|x\|_p^p + \|y\|_p^p).
\]

For \( 1 < p \leq 2 \) these inequalities hold in the reverse sense.

Setting \( x + y = \xi, x - y = \eta \), we can see that (1) is equivalent to (2), and the right side of (3) is equivalent to the left side of (3). Moreover, (3) follows from (1) and inequality \( 2(a^p + b^p)^{p-1} \leq 2p-1(a^p + b^p) \) for positive real numbers \( a \) and \( b \). Therefore, any Banach space \( X \) satisfying inequality (1) satisfies the rest inequalities and the uniform convexity of \( X \) follows. For \( 1 < p \leq 2 \), the inequality corresponding to (2) is

\[
(4) \quad \|x + y\|_p^{p'} + \|x - y\|_p^{p'} \leq 2(\|x\|_p^p + \|y\|_p^p)^{p'-1},
\]

which is equivalent to (1).

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Later, Clarkson’s inequalities have been studied in various Banach spaces and the inequalities themselves have been generalized in various ways (Ch XVIII in [10], [3], [4], [5], [6], [7], [8], [12], [13]). Extensive studies of Clarkson’s inequalities were done by Kato and Miyazaki ([4], [5], [6], [11]), and also by Kato and Takahashi ([6], [7], [12]). Following Takahashi and Kato ([12]), let us say that for \( 1 \leq p \leq 2 \) the \((p, p')\) Clarkson inequality holds in a Banach space \( X \) if

\[
\left( \|x + y\|^{p'} + \|x - y\|^{p'} \right)^{1/p'} \leq 2^{1/p'} \left( \|x\|^p + \|y\|^p \right)^{1/p}
\]

holds for all \( x \) and \( y \) in \( X \). For \( p = 1 \), inequality (5) should be understood to be

\[
\max\{\|x + y\|, \|x - y\|\} \leq \|x\| + \|y\|.
\]

Therefore, the \((1, \infty)\) Clarkson inequality holds in every Banach space. Setting \( x + y = \xi \) and \( x - y = \eta \), we can see that (5) is equivalent to:

\[
\left( \|\xi\|^{p'} + \|\eta\|^{p'} \right)^{1/p'} \leq 2^{-1/p'} \left( \|\xi\|^p + \|\eta\|^p \right)^{1/p}.
\]

In 1997, Takahashi and Kato ([12]) proved that if for \( 1 \leq p \leq 2 \), the \((p, p')\) Clarkson inequality holds in a Banach space \( X \), then the \((t, t')\) Clarkson inequality holds in Lebesgue-Bochner space \( L_r(X) \) \((1 \leq r < \infty)\), where \( t = \min\{p, r, r'\} \).

In this paper we will obtain a result analogous to that of Takahashi and Kato ([12]). In Theorem 3, we will prove that if for each \( n \), \( 1 \leq p_n \leq 2 \) and the \((p_n, p'_n)\) Clarkson inequality holds in each Banach space \( X_n \) then the \((t, t')\) Clarkson inequality holds in \( \sum_{n=1}^{\infty} X_n \), where \( t = \min\{p, r, r'\}, p = \inf\{p_n\} \) and \( 1 \leq r < \infty \). In Theorem 4, we will prove that the \((p, p')\) Clarkson inequality is preserved by quotient maps.

2. Clarkson’s inequality

We begin with reviewing a few definitions. If \( \{X_n\}_{n=1}^{\infty} \) is a sequence of Banach spaces \( X_n \)’s, for \( 1 \leq p < \infty \) the \( \ell_p \)-sum \( \left( \sum_{n=1}^{\infty} X_n \right)_p \) of \( X_n \)’s is the space of all sequences \( x = \{x_n\}_{n=1}^{\infty} \), \( x_n \in X_n \) with \( \sum_{n=1}^{\infty} \|x_n\|^p < \infty \). This space is a Banach space under the norm defined by \( \|x\| = \left( \sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p} \). The dual space of \( \left( \sum_{n=1}^{\infty} X_n \right)_p \) is \( \left( \sum_{n=1}^{\infty} X_n^* \right)_{p'} \), where \( X_n^* \) is the dual space of \( X_n \) and \( p' \) is the conjugate exponent of \( p \). The \( \ell_p \)-sum of two Banach spaces \( X \) and \( Y \) is defined in an obvious manner and will be denoted by \( X \oplus_p Y \). The dual space of \( X \oplus_p Y \) is \( X^* \oplus_{p'} Y^* \).
The Rademacher functions \( r_n \)'s are defined by \( r_n(t) = \text{sgn}(\sin(2^n \pi t)) \), \( 0 \leq t \leq 1 \). Observe that if \( X \) is a Banach space and \( x, y \in X \) then
\[
\int_0^1 \| x + r_1(t)y \| \, dt = \frac{1}{2} \{ \| x + y \| + \| x - y \| \}.
\]

We need two useful theorems due to Kato and Takahashi ([7], [12]).

**Theorem 1** [12]. Suppose \( 1 \leq r < p \leq 2 \). If the \((p, p')\) Clarkson inequality holds in a Banach space \( X \), then the \((r, r')\) Clarkson inequality holds in \( X \).

Kato and Takahashi [7] gave an elegant proof of the following theorem. However, we will give a new proof of the theorem for its own interest.

**Theorem 2** [7]. For \( 1 \leq p \leq 2 \), the \((p, p')\) Clarkson inequality holds in a Banach space \( X \) if and only if it holds in \( X^* \).

*New proof.* Suppose the \((p, p')\) Clarkson inequality holds in \( X \), and \( x^*, y^* \in X^* \). We will prove the inequality (6) for \( x^* \) and \( y^* \). Let \( \delta > 0 \). We choose \((x, y) \in X \oplus_p X\) such that \( \|x\|^p + \|y\|^p = 1 \) and \( (\|x^*\|^{p'} + \|y^*\|^{p'})^{1/p'} \leq (1 + \delta)(x^*(x) + y^*(y)). \) Then we have
\[
\frac{1}{1 + \delta} (\|x^*\|^{p'} + \|y^*\|^{p'})^{1/p'}
\leq x^*(x) + y^*(y)
= \int_0^1 (x^* + r_1(t)y^*)(x + r_1(t)y) \, dt
\leq \left( \int_0^1 \|x^* + r_1(t)y^*\|^p \, dt \right)^{1/p} \left( \int_0^1 \|x + r_1(t)y\|^{p'} \, dt \right)^{1/p'}
= \left\{ \frac{1}{2} (\|x^* + y^*\|^p + \|x^* - y^*\|^p) \right\}^{1/p} \left( \frac{1}{2} (\|x + y\|^{p'} + \|x - y\|^{p'}) \right)^{1/p'}
\leq 2^{-\frac{1}{p}} (\|x^* + y^*\|^p + \|x^* - y^*\|^p)^{1/p} (\|x\|^p + \|y\|^p)^{1/p}.
\]

Since \( \delta > 0 \) is arbitrary and \( \|x\|^p + \|y\|^p = 1 \), inequality (6) holds in \( X^* \), and hence the \((p, p')\) Clarkson inequality holds in \( X^* \).

Conversely, if the \((p, p')\) Clarkson inequality holds in \( X^* \), then it holds in \( X^{**} \) and hence in \( X (\subseteq X^{**}) \). \( \Box \)

Combining the Clarkson's original proof of a Clarkson's inequality for \( \ell_p \) [2] and Takahashi-Kato proof of the \((t, t')\) Clarkson inequality for \( L_r(X) \) [12], we have the following:
THEOREM 3. Suppose $1 \leq r < \infty$, $1 \leq p_n \leq 2$ for each $n$ and $p = \inf \{p_n\}$. If for each $n$, the $(p_n, p'_n)$ Clarkson inequality holds in $X_n$, then the $(t, t')$ Clarkson inequality holds in $(\sum_{n=1}^{\infty} X_n)_r$, where

$$t = \begin{cases} 
  r & \text{if } 1 \leq r \leq p \\
  p & \text{if } p \leq r \leq p' \\
  r' & \text{if } p' \leq r < \infty
\end{cases}$$

Proof. Since the $(1, \infty)$ Clarkson inequality for $(\sum_{n=1}^{\infty} X_n)_r$ is trivial, we assume that $r > 1$ and $p > 1$.

Let $p \leq r \leq p'$, and let $x = (x_n), y = (y_n) \in (\sum_{n=1}^{\infty} X_n)_r$ with $x_n, y_n \in X_n$. Then

$$\|x + y\|^{p'} + \|x - y\|^{p'}
= (\sum_{n=1}^{\infty} \|x_n + y_n\|^{r'/r'})^{p'/r'} + (\sum_{n=1}^{\infty} \|x_n - y_n\|^{r'/r'})^{p'/r'}
= (\sum_{n=1}^{\infty} (\|x_n + y_n\|^r)^{r'/r})^{p'/r'} + (\sum_{n=1}^{\infty} (\|x_n - y_n\|^r)^{r'/r})^{p'/r'}
\leq (\sum_{n=1}^{\infty} (\|x_n + y_n\|^r)^{r'/r})^{p'/r'} + (\sum_{n=1}^{\infty} (\|x_n - y_n\|^r)^{r'/r})^{p'/r'}
\quad \text{(by Minkowski's inequality for } r/p' \leq 1)
\leq 2(\sum_{n=1}^{\infty} (\|x_n\|^p + \|y_n\|^p)^{r'/r})^{p'/r'}
\quad \text{(by the } (p_n, p'_n) \text{ Clarkson inequalities in } X_n's)
\leq 2((\sum_{n=1}^{\infty} \|x_n\|^r)^{p'/r} + (\sum_{n=1}^{\infty} \|y_n\|^r)^{p'/r})^{p'/p'}
\quad \text{(by Minkowski's inequality for } r/p \geq 1)
= 2(\|x\|^p + \|y\|^p)^{p'/p'}.

Therefore, the $(p, p')$ Clarkson inequality holds in $(\sum_{n=1}^{\infty} X_n)_r$.

If $1 < r \leq p$, then by Theorem 1 the $(r, r')$ Clarkson inequality holds in every $X_n$ and hence in $(\sum_{n=1}^{\infty} X_n)_r$ by the preceding part. If $p' < r < \infty$, then $1 < r' < p$ and the $(r', r)$ Clarkson inequality holds in every $X_n^*$ and hence in $(\sum_{n=1}^{\infty} X_n^*)_r = ((\sum_{n=1}^{\infty} X_n)_r)^*$. Therefore, the $(r', r)$ Clarkson inequality holds in $(\sum_{n=1}^{\infty} X_n)_r$. \qed
Recall that a linear map $T$ from a Banach space $X$ to a Banach space $Y$ is called a quotient map if $T$ carries the open unit ball of $X$ onto the open unit ball of $Y$. The $(p,p')$ Clarkson inequality is preserved by quotient maps. More specifically we have:

**Theorem 4.** Suppose $X$ and $Y$ are Banach spaces, and the $(p,p')$ Clarkson inequality holds in $X$. If there exists a quotient map $T : X \to Y$, then the $(p,p')$ Clarkson inequality holds in $Y$. In particular, if $Z$ is a closed subspace of $X$, then the $(p,p')$ Clarkson inequality holds in $X/Z$.

**Proof.** Suppose the $(p,p')$ Clarkson inequality holds in $X$ and $T : X \to Y$ is a quotient map. Then $T^* : Y^* \to X^*$ is an isometry into $X^*$. Since the $(p,p')$ Clarkson inequality holds in $X^*$. It holds in $T^*(Y^*)$ and hence in $Y^*$. By Theorem 2, the $(p,p')$ Clarkson inequality holds in $Y$. The $(p,p')$ Clarkson inequality for $X/Z$ is obvious. $\square$

**References**

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