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M-IDEALS AND PROPERTY SU

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Abstract. Suppose X and Y are Banach spaces for which $K(X,Y)$, the space of compact operators from X to Y, is an M-ideal in $L(X,Y)$, the space of bounded linear operators from X to Y. If Z is a closed subspace of Y such that $L(X,Z)$ has property SU in $L(X,Y)$ and $d(T,K(X,Z)) = d(T,K(X,Y))$ for all $T \in L(X,Z)$, then $K(X,Z)$ is an M-ideal in $L(X,Z)$ if and only if it has property SU in $L(X,Z)$.

1. Introduction

A closed subspace J of a Banach space X is called an M-ideal in X if the annihilator $J^\perp$ of J is an L-summand in X^*, namely there exists a closed subspace J_* of X^* such that X^* is an algebraic direct sum of $J^\perp$ and J_*, and for all $g \in J^\perp$ and $h \in J_*$ the norm condition

$$\|g+h\| = \|g\| + \|h\| \tag{1}$$

holds. In this case, we write $X^* = J^\perp \oplus_{\ell_1} J_*$.

A weaker notion is an HB-subspace. According to Hennefeld [8], a subspace J of a Banach space X is called an HB-subspace of X if there exists a closed subspace J_* of X^* such that X^* is an algebraic direct sum of $J^\perp$ and J_*, and for all $0 \neq g \in J^\perp$ and $h \in J_*$ the norm conditions

$$\|g+h\| \geq \|g\| \quad \text{and} \quad \|g+h\| > \|h\| \tag{2}$$

hold. It is easy to see that an HB-subspace satisfies property U in the sense of Phelps. According to Phelps [20], a subspace J of a Banach space X is said to satisfy property U if every bounded linear functional on J has a unique norm preserving extension to X. Lima [14] and others called property U Hahn-Banach smooth.

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In 1988, Oja [16] defined and investigate strong uniqueness property (briefly, property SU) which is an intermediate property between property U and HB-subspace. A subspace J of a normed space X is said to satisfy property SU in X if there exists a closed subspace J_* of X^* such that X^* is an algebraic direct sum of $J^\perp$ and J_*, and for all $0 \neq g \in J^\perp$ and $h \in J_*$ the norm condition

$$\|h+g\| > \|h\| \tag{3}$$

holds.

An M-ideal is an HB-subspace and an HB-subspace satisfies property SU. But there is an example of an HB-subspace which is not an M-ideal [8]. Oja [16] gave an example of a subspace which is not an HB-subspace, but satisfies property SU, and an example of a subspace satisfying property U, but not property SU. It is easy to see that property SU implies property U.

If X and Y are Banach spaces, $L(X,Y)$(respectively, $K(X,Y)$) will denote the space of all bounded linear operators(respectively, compact operators) from X to Y. We will write $L(X)$(respectively, $K(X)$) for $L(X,X)$(respectively, $K(X,X)$). There are many Banach spaces X and Y for which $K(X,Y)$ is an M-ideal in $L(X,Y)$ ([2], [3], [5], [6], [7], [10], [13], [21], [23]).

In Theorem 4, we will prove that if X and Y are Banach spaces for which $K(X,Y)$ is an M-ideal in $L(X,Y)$, then for a closed subspace Z of Y such that $L(X,Z)$ has property SU in $L(X,Y)$ and $d(T,K(X,Z)) = d(T,K(X,Y))$ for all $T \in L(X,Z)$, $K(X,Z)$ is an M-ideal in $L(X,Z)$ if and only if $K(X,Z)$ has property SU in $L(X,Z)$. Therefore, in this case M-ideals, HB-subspaces and subspaces satisfying property SU are the same.

2. M-ideals and property SU

If Y is a subspace of a Banach space X, i will always denote the identity map from Y into X. Then the adjoint $i^* : X^* \to Y^*$ carries each $x^* \in X^*$ to i^*x^*, the restriction of x^* to Y. We will use the notation

$$Y^{\#} = \{x^* \in X^* : \|i^*x^*\| = \|x^*\|\}.$$

We need the following result of Oja [16].

THEOREM 1 [16]. *For a closed subspace $Y \neq \{0\}$ of X, the following statements are equivalent.*

(i) *Y satisfies property SU in X.*

(ii) $Y^{\#}$ *is the only complement of the annihilator* $Y^{\perp}$ *in* X^* *such that for all* $f \in X^*$, $f = g + h$ *with* $g \in Y^{\#}$, $h \in Y^{\perp}$ *and* $\|g + h\| > \|g\|$ *if* $h \neq 0$.
(iii) $Y^{\#}$ *is the algebraic complement of* $Y^{\perp}$ *in* X^*.
(iv) Y *satisfies property U in* X *and* $Y^{\#}$ *is a subspace.*
(v) $Y^{\#}$ *is a subspace.*

Notice that if J is an M-ideal in a Banach space X then by Theorem 1, $X^* = J^{\perp} \oplus_{\ell_1} J^{\#}$. If Y is a subspace of X and $x \in X$, $d(x, Y)$ will denote the distance from x to Y.

The following lemma plays a key role in the proof of Theorem 4.

LEMMA 2. *Suppose* X *and* Y *are Banach spaces for which* $K(X,Y)$ *is an M-ideal in* $L(X,Y)$, *and suppose* Z *is a closed subspace of* Y. *If* $i : L(X,Z) \to L(X,Y)$ *denotes the identity map, then for each* $f \in L(X,Z)^*$, *there exist* $g_1 \in K(X,Z)^{\perp}$ *and* $g_2 \in i^*(K(X,Y)^{\#})$ *such that* $f = g_1 + g_2$ *and*

$$\|g_1 + g_2\| = \|g_1\| + \|g_2\|.$$

Proof. For each $f \in L(X,Z)^*$, we choose a norm preserving extension $\tilde{f} \in L(X,Y)^* = K(X,Y)^{\perp} \oplus_{\ell_1} K(X,Y)^{\#}$. We write $\tilde{f} = f_1 + f_2$ for $f_1 \in K(X,Y)^{\perp}$ and $f_2 \in K(X,Y)^{\#}$. Then $\|\tilde{f}\| = \|f_1 + f_2\| = \|f_1\| + \|f_2\|$ and $f = i^* f_1 + i^* f_2$. Since i^* is norm decreasing, we have

$$\|i^* f_1 + i^* f_2\| = \|f\| = \|\tilde{f}\| = \|f_1\| + \|f_2\| \geq \|i^* f_1\| + \|i^* f_2\|.$$

Therefore, we have $\|i^* f_1 + i^* f_2\| = \|i^* f_1\| + \|i^* f_2\|$, $\|f_1\| = \|i^* f_1\|$, $\|f_2\| = \|i^* f_2\|$, $i^* f_1 \in K(X,Z)^{\perp}$ and $i^* f_2 \in i^*(K(X,Y)^{\#})$. Put $g_1 = i^* f_1$ and $g_2 = i^* f_2$. □

LEMMA 3. *If* X *is a Banach space and* Y, Z_1 *and* Z_2 *are closed subspaces of* X *such that* $Z_1 \subseteq Y$, $Z_1 \subseteq Z_2$ *and* $d(y, Z_1) = d(y, Z_2)$ *for all* $y \in Y$, *then every* $f \in Z_1^{\perp}$ *has a norm preserving extension which belongs to* $Z_2^{\perp}$.

Proof. Let $\pi : Y \to Y/Z_1$ be the canonical projection. Then $\pi^* : (Y/Z_1)^* \to Z_1^{\perp}(\subseteq Y^*)$ is an isometric isomorphism by which $f \in (Y/Z_1)^*$ can be identified with $\pi^*(f) = f \circ \pi \in Z_1^{\perp}(\subseteq Y^*)$. Therefore, we have $(Y/Z_1)^* = Z_1^{\perp}$. Similarly, $(X/Z_2)^* = Z_2^{\perp}$.

On the other hand, the map $\phi : Y/Z_1 \to X/Z_2$ defined by $\phi(y + Z_1) = y + Z_2$ for $y \in Y$ is an isometric isomorphism into X/Z_2 and $\phi^* : (X/Z_2)^* \to (Y/Z_1)^*$ is a quotient map. Observe that if $f \in Z_2^{\perp}$, then $\phi^*(f) = f|_Y$, the restriction of f to Y. Since every functional in $(\phi(Y/Z_1))^*$ has a norm preserving extension in $(X/Z_2)^* = Z_2^\perp$, for every $g \in Z_1^\perp (\subseteq Y^*)$ there exists $\tilde{g} \in Z_2^\perp (\subseteq X^*)$ such that $\tilde{g}|_Y = g$ and $\|\tilde{g}\| = \|g\|$. □

THEOREM 4. *Suppose X and Y are Banach spaces for which $K(X,Y)$ is an M-ideal in $L(X,Y)$. For a closed subspace Z of Y such that $L(X,Z)$ has property SU in $L(X,Y)$ and $d(T,K(X,Z)) = d(T,K(X,Y))$ for every $T \in L(X,Z)$, the following statements are equivalent.*

(i) *$K(X,Z)$ is an M-ideal in $L(X,Z)$.*
(ii) *$K(X,Z)$ is an HB-subspace of $L(X,Z)$.*
(iii) *$K(X,Z)$ satisfies property SU in $L(X,Z)$.*

Proof. The implications (i) ⇒ (ii) ⇒ (iii) are obvious.
(iii) ⇒ (i) : Assume statement (iii) holds. By Theorem 1, we have

$$L(X,Z)^* = K(X,Z)^\perp \oplus K(X,Z)^\#.$$

Therefore, every $f \in L(X,Z)^*$ has a unique expression $f = g + h$ with $g \in K(X,Z)^\perp$ and $h \in K(X,Z)^\#$. Now it suffices to show that $\|g+h\| = \|g\| + \|h\|$.

Fix $f \in L(X,Z)$ and the above expression $f = g + h$. As in Lemma 2, let $\tilde{f}$ be a norm preserving extension of f to $L(X,Y)$, and we write $\tilde{f} = f_1 + f_2$ with $f_1 \in K(X,Y)^\perp$ and $f_2 \in K(X,Y)^\#$. Then by Lemma 2, $f = i^*f_1 + i^*f_2$ and $\|f\| = \|i^*f_1\| + \|i^*f_2\|$. Since $i^*f_1 \in K(X,Z)^\perp$, if $i^*f_2 \in K(X,Z)^\#$ then $i^*f_1 = g$ and $i^*f_2 = h$, and we are done. Therefore, it remains to show that $i^*f_2 \in K(X,Z)^\#$.

Let $\tilde{h} \in L(X,Y)$ be the unique norm preserving extension of h. Since f_2 is the unique norm preserving extension of i^*f_2 and $L(X,Z)^\#$ is a subspace, $f_2 - \tilde{h}$ is the unique norm preserving extension of $i^*f_2 - h = g - i^*f_1 \in K(X,Z)^\perp (\subseteq L(X,Z))$. By Lemma 3 applied to the space $L(X,Y)$, and its subspaces $L(X,Z)$, $K(X,Z)$ and $K(X,Y)$, we have $f_2 - \tilde{h} \in K(X,Y)^\perp (\subseteq L(X,Y)^*)$. Since $\|f_2\| = \|f_2|_{K(X,Y)}\|$, $\|\tilde{h}\| = \|\tilde{h}|_{K(X,Y)}\|$ and $f_2 - \tilde{h} \in K(X,Y)^\perp$, $\|f_2\| = \|\tilde{h}\|$ and so $\|i^*f_2\| = \|h\|$. If $i^*f_2 - h = g - i^*f_1 \in K(X,Z)^\perp$ were a nonzero functional, then $\|i^*f_2\| = \|(g - i^*f_1) + h\| > \|h\|$, which contradicts to the fact that $\|i^*f_2\| = \|h\|$. Therefore, $i^*f_2 - h = 0$ and $i^*f_2 = h \in K(X,Z)^\#$. □

As mentioned earlier, there are many Banach spaces X and Y for which $K(X,Y)$ is an M-ideal in $L(X,Y)$. In particular, for every closed subspace X of ℓ_p $(1 < p < \infty)$, $K(X, \ell_p)$ is an M-ideal in $L(X, \ell_p)$ [3]. Thus we have the following corollary.

COROLLARY 5. *For a closed subspace X of ℓ_p $(1 < p < \infty)$ such that $L(X)$ has property SU in $L(X, \ell_p)$ and $d(T, K(X)) = d(T, K(X, \ell_p))$ for all $T \in L(X)$, the following statements are equivalent.*

(i) *$K(X)$ is an M-ideal in $L(X)$.*
(ii) *$K(X)$ is an HB-subspace in $L(X)$.*
(iii) *$K(X)$ satisfies property SU in $L(X)$.*

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