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FIXED POINT THEORY FOR MAPS HAVING CONVEXLY TOTALLY BOUNDED RANGES

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Abstract. Three new fixed point theorems are presented for the set valued maps of Idzik. Moreover a continuation theorem for such maps is also given.

1. Introduction

Schauder's conjecture states that every continuous function from a compact, convex subset of a Hausdorff topological vector space into itself would have a fixed point. In 1988 Idzik [3] gave a partial solution to this conjecture using the concept of convexly totally bounded sets (c.t.b.). This paper presents three generalizations of the result of Idzik. In addition we present a homotopy result for the maps of Idzik which automatically yield a generalization of the Leray–Schauder alternative of Idzik and Park [4].

For the remainder of this section we present some definitions and known results. Let E be a Hausdorff topological vector space.

Definition 1.1. A set $K \subseteq E$ is convexly totally bounded (c.t.b.) if for every neighborhood V of $0 \in E$ there exists a finite set $\{x_i : i \in I\} \subseteq K$ (I finite) and a finite family of convex sets $\{C_i : i \in I\}$ with $C_i \subseteq V$ for each $i \in I$ and $K \subseteq \cup_{i \in I}(x_i + C_i)$.

Note that $\{x_i : i \in I\}$ can be chosen in E (see [4]). We know every compact set in a locally convex space is c.t.b. From [4] we have the following two results:

Proposition 1.1. *If a compact set K is c.t.b., then the set $[0,1]K$ is compact and c.t.b.*

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PROPOSITION 1.2. *Every (compact) subset of a c.t.b. set is c.t.b.*

Now Idzik's [3] partial solution to Schauder's conjecture can be stated as follows.

THEOREM 1.3. *Let X be a nonempty, convex subset of a Hausdorff topological vector space E and $F : X \to CK(X)$ a upper semicontinuous map (here $CK(X)$ denotes the family of nonempty, compact, convex subsets of X). If $\overline{F(X)}$ is a compact, c.t.b. subset of X, then there exists $x_0 \in X$ with $x_0 \in F(x_0)$.*

COROLLARY 1.4. *Let X be a nonempty, convex subset of a Hausdorff topological vector space E and $F : X \to CK(X)$ a closed map. If $\overline{F(X)}$ is a compact, c.t.b. subset of X, then there exists $x_0 \in X$ with $x_0 \in F(x_0)$.*

Proof. From [2 p. 465] we know $F : X \to CK(X)$ is u.s.c, so the result will follow from Theorem 1.3. □

2. Fixed point theory

We begin this section by extending a result of Idzik [3].

THEOREM 2.1. *Let Ω be a closed, convex subset of a Hausdorff topological vector space E with $x_0 \in \Omega$. Suppose $F : \Omega \to CK(\Omega)$ is closed with the following property holding:*

$$A \subseteq \Omega,\ A = \overline{co}\,(\{x_0\} \cup F(A)) \text{ implies } A \text{ is compact.} \tag{2.1}$$

Also assume $\overline{F(\Omega)}$ is c.t.b. Then F has a fixed point in Ω.

Proof. Consider $\mathcal{F}$ the family of all closed, convex subsets C of Ω with $x_0 \in C$ and $F(x) \subseteq C$ for all $x \in C$. Note $\mathcal{F} \neq \emptyset$ since $\Omega \in \mathcal{F}$. Let

$$C_0 = \cap_{C \in \mathcal{F}} C.$$

Notice C_0 is nonempty, closed and convex and $F : C_0 \to 2^{C_0}$ since if $x \in C_0$ then $F(x) \subseteq C$ for all $C \in \mathcal{F}$. Let

$$C_1 = \overline{co}\,(\{x_0\} \cup F(C_0)). \tag{2.2}$$

Notice $F : C_0 \to 2^{C_0}$ together with C_0 closed and convex implies $C_1 \subseteq C_0$. Also $F(C_1) \subseteq F(C_0) \subseteq C_1$ from (2.2). Thus C_1 is closed and convex with $F(C_1) \subseteq C_1$. As a result $C_1 \in \mathcal{F}$, so $C_0 \subseteq C_1$. Consequently

$$C_0 = \overline{co}\,(\{x_0\} \cup F(C_0)). \tag{2.3}$$

Now (2.1) guarantees that C_0 is compact, and notice (2.3) implies $F(C_0) \subseteq C_0$. Thus $F : C_0 \to CK(C_0)$ is a closed map. In addition $\overline{F(C_0)}$ is compact (since $\overline{F(C_0)}$ is a closed subset of the compact space C_0) and $\overline{F(C_0)}$ is c.t.b (Proposition 1.2; note $C_0 \subseteq \Omega$ so $F(C_0) \subseteq F(\Omega)$). Apply Corollary 1.4 to deduce that there exists $x_0 \in C_0$ with $x_0 \in F(x_0)$. □

It is possible to relax assumption (2.1) as the following theorem shows.

THEOREM 2.2. *Let Ω be a closed, convex subset of a Hausdorff topological vector space E and $x_0 \in \Omega$. Suppose $F : \Omega \to CK(\Omega)$ is closed and satisfies the following property:*

$$A \subseteq \Omega,\ A = co\,(\{x_0\} \cup F(A)) \ \text{ implies } \overline{A} \text{ is compact.} \tag{2.4}$$

Also assume $\overline{F(\Omega)}$ is c.t.b. Then F has a fixed point in Ω.

Proof. Let

$$D_0 = \{x_0\},$$
$$D_n = co\,(\{x_0\} \cup F(D_{n-1})) \quad \text{for} \quad n = 1, 2, \dots\dots$$

and

$$D = \cup_{n=0}^{\infty} D_n.$$

Now for $n = 0, 1, \dots$ notice D_n is convex. Also by induction we see that

$$D_0 \subseteq D_1 \subseteq \dots\dots \subseteq D_{n-1} \subseteq D_n \dots\dots \subseteq \Omega.$$

Consequently D is convex. It is also immediate since (D_n) is increasing that

$$D = \bigcup_{n=1}^{\infty} co\,(\{x_0\} \cup F(D_{n-1})) = co\,(\{x_0\} \cup F(D)). \tag{2.5}$$

Now (2.4) implies that $\overline{D}$ is compact, and (2.5) implies $F(D) \subseteq D$. Let

$$F^\star(x) = F(x) \cap \overline{D}.$$

We first show $F^\star : \overline{D} \to 2^{\overline{D}}$ i.e. we show $F^\star(x) \neq \emptyset$ for each $x \in \overline{D}$. To see this it is enough to show $\overline{D} \subseteq F^{-1}(\overline{D})$. Indeed if $x \in \overline{D}$ then $x_\alpha \to x$ for some net (x_α) in D. Take any $y_\alpha \in F(x_\alpha)$. Since $F(D) \subseteq D$ we have $y_\alpha \in D \subseteq \overline{D}$. The compactness of $\overline{D}$ guarantees that we may assume without loss of generality that $y_\alpha \to y$ for some $y \in \overline{D}$. Since

$(x_\alpha, y_\alpha) \in graph\, F$ and $graph\, F$ is closed, we have $(x,y) \in graph\, F$. Thus $y \in F(x) \cap \overline{D}$ i.e. $x \in F^{-1}(\overline{D})$. As a result $\overline{D} \subseteq F^{-1}(\overline{D})$ so $F^\star : \overline{D} \to 2^{\overline{D}}$. Also notice $graph\, F^\star$ is closed so $F^\star : \overline{D} \to CK(\overline{D})$ is a closed map with $\overline{F^\star(\overline{D})}$ compact (since $\overline{F^\star(\overline{D})}$ is a closed subset of the compact set $\overline{D}$) and $\overline{F^\star(\overline{D})}$ is c.t.b. (Proposition 1.2; note $\overline{D} \subseteq \Omega$ so $F^\star(\overline{D}) \subseteq F^\star(\Omega) \subseteq F(\Omega)$). Corollary 1.4 implies that there exists $x_0 \in \overline{D}$ with $x_0 \in F^\star(x_0)$. As a result $x_0 \in F(x_0)$. □

Next we obtain a Mönch (O'Regan, Precup [5]) theorem for the maps of Idzik.

THEOREM 2.3. *Let Ω be a closed, convex subset of a Hausdorff topological vector space E and $x_0 \in \Omega$. Suppose $F : \Omega \to CK(\Omega)$ is closed and satisfies the following properties:*

(2.6) *F maps compact sets into relatively compact sets,*

$$\left\{ \begin{array}{l} A \subseteq \Omega,\ A = co\,(\{x_0\} \cup F(A)) \text{ with } \overline{A} = \overline{C} \\ \text{and } C \subseteq A \text{ countable, implies } \overline{A} \text{ is compact,} \end{array} \right. \tag{2.7}$$

$$\left\{ \begin{array}{l} \text{for any relatively compact subset } A \text{ of } \Omega \text{ there} \\ \text{exists a countable set } B \subseteq A \text{ with } \overline{B} = \overline{A}, \end{array} \right. \tag{2.8}$$

and

(2.9) *if A is a compact subset of Ω then $\overline{co}\,(A)$ is compact.*

Also assume $\overline{F(\Omega)}$ is c.t.b. Then F has a fixed point in Ω.

REMARK 2.1. If E is metrizable then (2.8) holds.

REMARK 2.2. If F is countably Φ–condensing [1] with (2.6) and (2.9) holding, then (2.7) is satisfied.

Proof. Let

$$D_0 = \{x_0\},$$
$$D_n = co\,(\{x_0\} \cup F(D_{n-1})) \text{ for } n = 1, 2, ...$$

and

$$D = \cup_{n=0}^{\infty} D_n.$$

We know from Theorem 2.2 that D is convex and

$$D = co\,(\{x_0\} \cup F(D)). \tag{2.10}$$

We now show D_n is relatively compact for $n = 0, 1, \ldots$. Suppose D_k is relatively compact for some $k \in \{1, 2, \ldots\}$. Then (2.6) guarantees that $F(\overline{D_k})$ is relatively compact and this together with (2.9) implies $\overline{co}\,(\{x_0\} \cup \overline{F(\overline{D_k})})$ is compact. Consequently D_{k+1} is relatively compact.

Now (2.8) implies that for each $n \in \{0, 1, ...\}$ there exists C_n with C_n countable, $C_n \subseteq D_n$, and $\overline{C_n} = \overline{D_n}$. Let $C = \cup_{n=0}^{\infty} C_n$. Now since

$$\bigcup_{n=0}^{\infty} D_n \subseteq \bigcup_{n=0}^{\infty} \overline{D_n} \subseteq \overline{\bigcup_{n=0}^{\infty} D_n}$$

we have

$$\overline{\bigcup_{n=0}^{\infty} \overline{D_n}} = \overline{\bigcup_{n=0}^{\infty} D_n} = \overline{D} \quad \text{and} \quad \overline{\bigcup_{n=0}^{\infty} \overline{D_n}} = \overline{\bigcup_{n=0}^{\infty} \overline{C_n}} = \overline{\bigcup_{n=0}^{\infty} C_n} = \overline{C}.$$

Thus $\overline{C} = \overline{D}$. This together with (2.7) implies that $\overline{D}$ is compact. From (2.10) we have $F(D) \subseteq D$. Let

$$F^{\star}(x) = F(x) \cap \overline{D}.$$

Essentially the same reasoning as in Theorem 2.2 guarantees that $F^{\star} : \overline{D} \to CK(\overline{D})$ is a closed map with $\overline{F^{\star}(\overline{D})}$ compact and $\overline{F^{\star}(\overline{D})}$ is c.t.b. Now apply Corollary 1.4. □

Next we present a "homotopy" type result for the maps of Idzik. The result was motivated by the papers [4, 6]. For our next three definitions assume E is a Hausdorff topological vector space, with U an open subset of E and $0 \in U$.

DEFINITION 2.1. $F \in IP(\overline{U}, E)$ if $F : \overline{U} \to CK(E)$ is a closed map with $\overline{F(\overline{U})}$ a compact, c.t.b. subset of E; here $\overline{U}$ denotes the closure of U in E.

DEFINITION 2.2. $F \in IP_{\partial U}(\overline{U}, E)$ if $F \in IP(\overline{U}, E)$ with $x \notin F(x)$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

DEFINITION 2.3. $F \in IP_{\partial U}(\overline{U}, E)$ is essential in $IP_{\partial U}(\overline{U}, E)$ if for every $G \in IP_{\partial U}(\overline{U}, E)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in G(x)$.

First we give an example of an essential map.

THEOREM 2.4. *Let E be a Hausdorff topological vector space, U an open subset of E and $0 \in U$. Then the zero map is essential in $IP_{\partial U}(\overline{U}, E)$.*

Proof. Let $\theta \in IP_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = \{0\}$. We must show that there exists $x \in U$ with $x \in \theta(x)$. Let $\Omega = \overline{co}\,(\{0\} \cup \theta(\overline{U}))$ and let F be given by

$$F(x) = \begin{cases} \theta(x), & x \in \overline{U} \\ \{0\}, & \text{otherwise.} \end{cases}$$

Note $0 \in \Omega$. Also notice $F : \Omega \to CK(\Omega)$ is a closed map with $\overline{F(\Omega)}$ compact since $\overline{\theta(\overline{U})}$ is compact. Also $\overline{F(\Omega)}$ is c.t.b. since $\overline{\theta(\overline{U})}$ is c.t.b. Thus $F \in IP(\Omega, \Omega)$. Now Corollary 1.4 guarantees that there exists $x \in \Omega$ with $x \in F(x)$. If $x \notin U$ we have $x \in F(x) = \{0\}$, which is a contradiction since $0 \in U$. Thus we have $x \in U$ so $x \in F(x) = \theta(x)$. □

Finally we present a generalization of the Leray–Schauder alternative in [4] (i.e. we not only conclude that the map has a fixed point but in addition that it is essential).

THEOREM 2.5. *Let E be a Hausdorff topological vector space, U an open subset of E with $0 \in U$. Suppose $F \in IP(\overline{U}, E)$ satisfies*

$$x \notin \lambda F(x) \quad \text{for } x \in \partial U \text{ and } \lambda \in (0,1]. \tag{2.11}$$

Then F is essential in $IP_{\partial U}(\overline{U}, E)$.

Proof. Let $H \in IP_{\partial U}(\overline{U}, E)$ with $H|_{\partial U} = F|_{\partial U}$. We must show H has a fixed point in U. Consider

$$B = \left\{x \in \overline{U} :\ x \in t\,H(x) \ \text{ for some } \ t \in [0,1]\right\}.$$

Now $B \neq \emptyset$ since $0 \in U$. Also B is closed since H is a closed map. In fact B is compact since $\overline{H(\overline{U})}$ is compact. Also $B \cap \partial U = \emptyset$ since (2.11) holds and $H|_{\partial U} = F|_{\partial U}$. Now since Hausdorff topological spaces are completely regular, there exists a continuous $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(B) = 1$. Define a map R_μ by $R_\mu(x) = \mu(x)\,H(x)$. Clearly $R_\mu : \overline{U} \to CK(E)$ is a closed map with $\overline{R_\mu(\overline{U})}$ compact since $\overline{H(\overline{U})}$ is compact. In addition from Proposition 1.1 we know the compact set $\overline{[0,1]\,\overline{H(\overline{U})}}$ is c.t.b., and as a result from Proposition 1.2 we have that $\overline{R_\mu(\overline{U})}$ is c.t.b. Consequently $R_\mu \in IP(\overline{U}, E)$. Note $R_\mu|_{\partial U} = \{0\}$ so $R_\mu \in IP_{\partial U}(\overline{U}, E)$ with $R_\mu|_{\partial U} = \{0\}$. Now Theorem 2.4 guarantees that the zero map is essential in $IP_{\partial U}(\overline{U}, E)$, so as a result there exists $x \in U$ with $x \in R_\mu(x)$. Consequently $x \in B$ and so $\mu(x) = 1$. This implies $x \in H(x)$. □

In fact it is also possible to improve Theorem 2.5 as follows.

DEFINITION 2.4. $F \in IPC(\overline{U}, E)$ if $F : \overline{U} \to CK(E)$ is a closed map, takes compact sets into relatively compact sets, with $\overline{F(\overline{U})}$ a c.t.b. subset of E, and F satisfies condition (C) (i.e. if $A \subseteq \overline{U}$ and $A \subseteq \overline{co}\,(\{0\} \cup F(A))$ then $\overline{A}$ is compact).

DEFINITION 2.5. $F \in IPC_{\partial U}(\overline{U}, E)$ if $F \in IPC(\overline{U}, E)$ with $x \notin F(x)$ for $x \in \partial U$.

DEFINITION 2.6. $F \in IPC_{\partial U}(\overline{U}, E)$ is essential in $IPC_{\partial U}(\overline{U}, E)$ if for every $G \in IPC_{\partial U}(\overline{U}, E)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in G(x)$.

THEOREM 2.6. *Let E be a Hausdorff topological vector space, U an open subset of E with $0 \in U$. Suppose $F \in IPC(\overline{U}, E)$ and assume the following conditions are satisfied:*

$$\text{the zero map is essential in } IPC_{\partial U}(\overline{U}, E), \tag{2.12}$$

$$x \notin \lambda F(x) \quad \text{for } x \in \partial U \text{ and } \lambda \in (0, 1], \tag{2.13}$$

and

$$\begin{cases} \text{for any continuous map } \mu : \overline{U} \to [0,1] \text{ with } \mu(\partial U) = 0 \\ \text{and for any map } H \in IPC(\overline{U}, E) \text{ with } H|_{\partial U} = F|_{\partial U}, \\ \text{then } \overline{R_\mu(\overline{U})} \text{ is c.t.b.; here } R_\mu(x) = \mu(x)\, H(x). \end{cases} \tag{2.14}$$

Then F is essential in $IPC_{\partial U}(\overline{U}, E)$.

Proof. Let $H \in IPC_{\partial U}(\overline{U}, E)$ with $H|_{\partial U} = F|_{\partial U}$. Let B, μ and R_μ be as in Theorem 2.5. Clearly $R_\mu : \overline{U} \to CK(E)$ is a closed map, takes compact sets into relatively compact sets, and $\overline{R_\mu(\overline{U})}$ is c.t.b. by (2.14). In addition R_μ satisfies condition (C). To see this let $A \subseteq \overline{U}$ and $A \subseteq \overline{co}\,(\{0\} \cup R_\mu(A))$. Then since $R_\mu(A) \subseteq co\,(\{0\} \cup H(A))$ we have

$$A \subseteq \overline{co}\,(co\,(\{0\} \cup H(A))) = \overline{co}\,(\{0\} \cup H(A)).$$

Now since H satisfies condition (C), we have that $\overline{A}$ is compact. Thus $R_\mu \in IPC(\overline{U}, E)$. Moreover since $R_\mu|_{\partial U} = \{0\}$ we have $R_\mu \in IPC_{\partial U}(\overline{U}, E)$. Now (2.12) implies that there exists $x \in U$ with $x \in R_\mu(x)$. Consequently $x \in B$ and so $\mu(x) = 1$. This implies $x \in H(x)$. □

REMARK 2.3. If condition (C) in Definition 2.4 is replaced by, if $A \subseteq \overline{U}$ and $A \subseteq co\,(\{0\} \cup F(A))$ then $\overline{A}$ is compact, then the result in Theorem 2.6 is again true.

REMARK 2.4. If condition (C) in Definition 2.4 is replaced by, if $A \subseteq \overline{U}$, $A \subseteq co(\{0\} \cup F(A))$ with $\overline{A} = \overline{C}$ and $C \subseteq A$ countable, then $\overline{A}$ is compact, then the result in Theorem 2.6 is again true if one of the following conditions hold:

$$E \text{ is a normal space} \tag{2.15a}$$

or

$$\begin{aligned} &E \text{ is such that any closed subset is compact} \\ &\text{if and only if it is sequentially compact.} \end{aligned} \tag{2.15b}$$

One can easily put conditions to guarantee (2.12).

THEOREM 2.7. *Let E be a Hausdorff topological vector space, U an open subset of E with $0 \in U$. Assume the following condition is satisfied:*

(2.16)

$$\begin{cases} \text{for any map } \theta \in IPC_{\partial U}(\overline{U}, E) \text{ with } \theta|_{\partial U} = \{0\}, \text{ and for any set} \\ A \subseteq \Omega = \overline{co}\,(\theta(\overline{U}) \cup \{0\}) \text{ with } A \subseteq \overline{co}\,(\{0\} \cup \theta(A \cap U)) \text{ and} \\ \theta(\overline{A \cap U}) \text{ relatively compact, we have that } \overline{A} \text{ is compact.} \end{cases}$$

Then the zero map is essential in $IPC_{\partial U}(\overline{U}, E)$.

Proof. Let θ, Ω and F be as in Theorem 2.4. Notice $F : \Omega \to CK(\Omega)$ is a closed map and $\overline{F(\Omega)}$ is c.t.b. since $\overline{\theta(\overline{U})}$ is c.t.b. Also note F takes compact sets into relatively compact sets. Next we show that F satisfies condition (C). To see this notice if $A \subseteq \Omega$ with $A \subseteq \overline{co}\,(\{0\} \cup F(A))$, then

$$A \subseteq \overline{co}\,(\{0\} \cup \theta(A \cap U)) \text{ so } A \cap U \subseteq \overline{co}\,(\{0\} \cup \theta(A \cap U)). \tag{2.17}$$

Now since θ satisfies condition (C) we have that $\overline{A \cap U}$ is compact, and so since $\theta \in IPC(\overline{U}, E)$ we have that $\theta(\overline{A \cap U})$ is relatively compact. This together with (2.16) and (2.17) yields that $\overline{A}$ is compact, so F satisfies condition (C). Thus $F \in IPC(\Omega, \Omega)$. Now Theorem 2.1 guarantees that there exists $x \in \Omega$ with $x \in F(x)$. As before $x \in U$ so $x \in \theta(x)$. □

REMARK 2.5. If condition (C) is as in Remark 2.3 then Theorem 2.7 again holds (in the proof we use Theorem 2.2). If condition (C) is as in Remark 2.4 then Theorem 2.7 again holds (in the proof we use Theorem 2.3).

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