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GENERALIZATIONS OF THE STRENGTHENED HARDY INEQUALITY

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ABSTRACT. In this article, using the properties of power mean, some new generalizations of the strengthened Hardy's inequalities are given.

If $a_n \geq 0$, $\lambda_n > 0$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$ $(n \in N)$ and $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$, then

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \lambda_n a_n,$$

which is called Hardy's Inequality, and it is well known (cf. [3, Theorem 349]). Recently, a strengthened following inequality is proved in [6].

THEOREM A. *If* $0 < \lambda_{n+1} \leq \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$, $a_n \geq 0$ $(n \in N)$ *and* $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$, *then*

$$(*) \qquad \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \Big[1 - \frac{\lambda_n}{2(\Lambda_n + \lambda_n)}\Big] \lambda_n a_n.$$

For any positive values $a_1, a_2, \ldots, a_n$ and positive weights $\alpha_1, \alpha_2, \ldots, \alpha_n$, $\sum_{i=1}^{n} \alpha_i = 1$, and for any real $p \neq 0$, we defined the power mean or the mean of order p of the value a with weights α by

$$M_p(a;\alpha) = M_p(a_1, a_2, \ldots, a_n; \alpha_1, \alpha_2, \ldots, \alpha_n) = \Big(\sum_{i=1}^{n} \alpha_i a_i^p\Big)^{1/p}.$$

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An easy application of L'Hospital's rule shows that

$$\lim_{p\to 0} M_p(a;\alpha) = \prod_{i=1}^{n} a_i^{\alpha_i},$$

the geometric mean. Accordingly, we define $M_0(a;\alpha) = \prod_{i=1}^{n} a_i^{\alpha_i}$. It is well known that $M_p(a;\alpha)$ is a nondecreasing function of p for $-\infty \leq p \leq \infty$ and is strictly increasing unless all the a_i are equal (cf. [1]). This result includes the arithmetic-mean and geometric-mean inequality as a special case.

In this article, using the strict monotonicity on the power mean of distinct positive numbers, we shall prove some theorems. To prove theorems, we introduce the following lemmas.

LEMMA 1. *If $a_1, a_2, \dots, a_n > 0$ and $\alpha_1, \alpha_2 \dots, \alpha_n > 0$ with $\sum_{i=1}^{n} \alpha_i = 1$, then we have the following inequality:*

$$\left(\prod_{i=1}^{n} a_i^{\alpha_i}\right)^k \leq \left(\sum_{i=1}^{n} \alpha_i a_i\right)^k \tag{1}$$

for $0 < k$ with the equality holding if and only if all a_i are same.

Note that Lemma 1 is easily deduced form the fact that $M_p(a;\alpha)$ is a continuous strictly increasing function of p.

LEMMA 2. *For all $x \geq 1$, we have*

$$\left(1+\frac{1}{x}\right)^x \left(\frac{5x+6}{5x+1}\right)^{1/2} < \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e. \tag{2}$$

Proof. We make the following auxiliary function

$$f(x) = x\ln\left(1+\frac{1}{x}\right) + \frac{1}{2}\ln\left(\frac{5x+6}{5x+1}\right), \quad x \in [1,\infty).$$

It is obvious that

$$f'(x) = -\frac{1}{x+1} + \ln\left(1+\frac{1}{x}\right) - \frac{1}{2}\frac{1}{(x+6/5)(x+1/5)}$$

and, for all $x \geq 1$, it can be shown that

$$\begin{aligned} f''(x) &= \frac{1}{(1+x)^2} - \frac{1}{x(1+x)} + \frac{1}{2(x+1/5)^2} - \frac{1}{2(x+6/5)^2} \\ &= \frac{-5x(25x^2+10x-7)}{1250x(1+x)^2(x+1/5)^2(x+6/5)^2} \\ &< 0. \end{aligned}$$

Therefore $f'(x)$ is decreasing on $[1,\infty)$. Then for any $x \in [1,\infty)$, we have $f'(x) > \lim_{x\to\infty} f'(x) = 0$ and so $f'(x)$ is increasing on $[1,\infty)$ and $f(x) < \lim_{x\to\infty} f(x) = 1$ for all $x \in [1,\infty)$. By the definition of $f(x)$, it follows that

$$\left(1+\frac{1}{x}\right)^x \left(\frac{5x+6}{5x+1}\right)^{1/2} < e.$$

Hence (2) is true for all $x \in [1,\infty)$. This completes the proof. □

REMARK 1. By the direct calculation, we have following inequality

$$\left(1+\frac{1}{x+\frac{1}{5}}\right)^{-1/2} < \left(1-\frac{1}{2(x+1)}\right) \tag{3}$$

for all $x \geq 1$.

We can deduce the following improvement result of Theorem A:

THEOREM 1. *If* $0 < \lambda_{n+1} \leq \lambda_n$, $\Lambda_n = \sum_{m=1}^n \lambda_m$, $a_n \geq 0 (n \in N)$ *and* $0 < \sum_{n=1}^\infty \lambda_n a_n < \infty$, *then*

$$\sum_{n=1}^\infty \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^\infty \left(\frac{5\Lambda_n + 6\lambda_n}{5\Lambda_n + \lambda_n}\right)^{-1/2} \lambda_n a_n.$$

Proof. By the arithmetic-geometric mean inequality, we have

$$\alpha_1^{q_1} \alpha_2^{q_2} \cdots \alpha_n^{q_n} \leq \sum_{m=1}^n q_m \alpha_m,$$

where $\alpha_m \geq 0$ and $q_m > 0$ $(m = 1, 2, \ldots, n)$ with $\sum_{m=1}^n q_m = 1$. Setting $c_m > 0$, $\alpha_m = c_m a_m$ and $q_m = \lambda_m / \Lambda_n$, we obtain

$$(c_1 a_1)^{\lambda_1/\Lambda_n} (c_2 a_2)^{\lambda_2/\Lambda_n} \cdots (c_n a_n)^{\lambda_n/\Lambda_n} \leq \frac{1}{\Lambda_n} \sum_{m=1}^n \lambda_m c_m a_m.$$

Using the above inequality, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{1/\Lambda_n}\\&=\sum_{n=1}^{\infty}\lambda_{n+1}\frac{(c_1a_1)^{\lambda_1/\Lambda_n}(c_2a_2)^{\lambda_2/\Lambda_n}\cdots(c_na_n)^{\lambda_n/\Lambda_n}}{(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{1/\Lambda_n}}\\&\le\sum_{n=1}^{\infty}\left[\frac{\lambda_{n+1}}{(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{1/\Lambda_n}}\right]\frac{1}{\Lambda_n}\sum_{m=1}^{n}\lambda_mc_ma_m\\&=\sum_{m=1}^{\infty}\lambda_mc_ma_m\sum_{n=m}^{\infty}\frac{\lambda_{n+1}}{\Lambda_n(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{1/\Lambda_n}}.\end{aligned}$$

Choosing $c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n}=(\Lambda_{n+1})^{\Lambda_n}$ $(n\in N)$ and setting $\Lambda_0=0$, from $\lambda_{n+1}\le\lambda_n$, it follows that

$$c_n=\left[\frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}\right]^{1/\lambda_n}=\left(1+\frac{\lambda_{n+1}}{\Lambda_n}\right)^{\Lambda_n/\lambda_n}\cdot\Lambda_n\le\left(1+\frac{\lambda_n}{\Lambda_n}\right)^{\Lambda_n/\lambda_n}\cdot\Lambda_n.$$

This implies that

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{1/\Lambda_n}&\le\sum_{m=1}^{\infty}\lambda_mc_ma_m\sum_{n=m}^{\infty}\frac{\lambda_{n+1}}{\Lambda_n\Lambda_{n+1}}\\&=\sum_{m=1}^{\infty}\lambda_mc_ma_m\sum_{n=m}^{\infty}\left(\frac{1}{\Lambda_n}-\frac{1}{\Lambda_{n+1}}\right)\\&=\sum_{m=1}^{\infty}\lambda_m\frac{1}{\Lambda_m}c_ma_m\\&\le\sum_{m=1}^{\infty}\left(1+\frac{1}{\Lambda_m/\lambda_m}\right)^{\Lambda_m/\lambda_m}\lambda_ma_m.\end{aligned}$$

Hence, by the above inequality and Lemma 2, we have

$$\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{1/\Lambda_n}<e\sum_{m=1}^{\infty}\left(\frac{5\Lambda_m+6\lambda_m}{5\Lambda_m+\lambda_m}\right)^{-1/2}\lambda_ma_m.$$

Thus Theorem 1 is proved. □

The main purpose of this paper is to prove some new generalizations of the above mentioned inequality.

THEOREM 2. *If* $0 < \lambda_{n+1} \le \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$ $(\Lambda_n \ge 1)$, $a_n \ge 0$ $(n \in N)$ *and* $0 < \sum_{n=1}^{\infty} \lambda_n (a_n)^t < \infty$ *for* $1 \le t < \infty$, *then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{t/\Lambda_n} < te^{1/t} \sum_{n=1}^{\infty} \Big(\frac{5\Lambda_n + 6\lambda_n}{5\Lambda_n + \lambda_n}\Big)^{-1/2t} \lambda_n a_n \Lambda_n^{(1-t)/t} \Big(\sum_{k=1}^{n} \lambda_k c_k a_k\Big)^{t-1}. \tag{4}$$

Proof. By Lemma 1, we have

$$(\alpha_1^{q_1} \alpha_2^{q_2} \cdots \alpha_n^{q_n})^t \le \Big(\sum_{m=1}^{n} q_m \alpha_m\Big)^t, \quad t \ge 1,$$

where $\alpha_m \ge 0$ and $q_m > 0$ $(m = 1, 2, \ldots, n)$ with $\sum_{m=1}^{n} q_m = 1$. Setting $c_m > 0$, $\alpha_m = c_m a_m$ and $q_m = \lambda_m/\Lambda_n$, we obtain

$$\big((c_1 a_1)^{\lambda_1/\Lambda_n} (c_2 a_2)^{\lambda_2/\Lambda_n} \cdots (c_n a_n)^{\lambda_n/\Lambda_n}\big)^t \le \Big(\frac{1}{\Lambda_n} \sum_{m=1}^{n} \lambda_m c_m a_m\Big)^t.$$

Using the above inequality, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{t/\Lambda_n} \\
&= \sum_{n=1}^{\infty} \lambda_{n+1} \Big(\frac{(c_1 a_1)^{\lambda_1/\Lambda_n} (c_2 a_2)^{\lambda_2/\Lambda_n} \cdots (c_n a_n)^{\lambda_n/\Lambda_n}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}}\Big)^t \\
&\le \sum_{n=1}^{\infty} \Big[\frac{\lambda_{n+1}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{t/\Lambda_n}}\Big] \Big(\frac{1}{\Lambda_n} \sum_{m=1}^{n} \lambda_m c_m a_m\Big)^t \\
&\le \sum_{n=1}^{\infty} \Big[\frac{\lambda_{n+1}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{t/\Lambda_n}}\Big] \frac{1}{\Lambda_n} \Big(\sum_{m=1}^{n} \lambda_m c_m a_m\Big)^t
\end{aligned} \tag{5}$$

for $\Lambda_n \ge 1$ and $t \ge 1$. Choosing $c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n} = (\Lambda_{n+1})^{\Lambda_n/t}$ $(n \in N)$ and setting $\Lambda_0 = 0$, from $\lambda_{n+1} \le \lambda_n$, we have

$$\begin{aligned}
c_n &= \Big[\frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}\Big]^{1/t\lambda_n} \\
&= \Big(1 + \frac{\lambda_{n+1}}{\Lambda_n}\Big)^{\Lambda_n/t\lambda_n} \cdot \Lambda_n^{1/t} \le \Big(1 + \frac{\lambda_n}{\Lambda_n}\Big)^{\Lambda_n/t\lambda_n} \cdot \Lambda_n^{1/t}.
\end{aligned} \tag{6}$$

By using the following inequality (see [2], [5]),

$$\left(\sum_{m=1}^{n} z_m\right)^t \le t\sum_{m=1}^{n} z_m\left(\sum_{k=1}^{m} z_m\right)^{t-1},$$

where $t \ge 1$ is constant and $z_m \ge 0$ $(m = 1, 2, \cdots)$, it is easy to observe that

$$\left(\sum_{m=1}^{n} \lambda_m c_m a_m\right)^t \le t\sum_{m=1}^{n} \lambda_m c_m a_m\left(\sum_{k=1}^{m} \lambda_m c_m a_m\right)^{t-1}. \tag{7}$$

Then, by (5) $\sim$ (7), we obtain

(8)

$$\begin{aligned}
&\sum_{n=1}^{\infty} \lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{t/\Lambda_n}\\
&\le \sum_{n=1}^{\infty}\left[\frac{\lambda_{n+1}}{(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{t/\Lambda_n}}\right]\frac{1}{\Lambda_n}\left(\sum_{m=1}^{n} \lambda_m c_m a_m\right)^t\\
&\le \sum_{n=1}^{\infty}\left[\frac{\lambda_{n+1}}{(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{t/\Lambda_n}}\right]\frac{1}{\Lambda_n}t\sum_{m=1}^{n} \lambda_m c_m a_m\left(\sum_{k=1}^{m} \lambda_m c_m a_m\right)^{t-1}\\
&= t\sum_{m=1}^{\infty} \lambda_m c_m a_m \sum_{n=m}^{\infty}\left(\frac{\lambda_{n+1}}{\Lambda_n\Lambda_{n+1}}\right)\left(\sum_{k=1}^{m} \lambda_m c_m a_m\right)^{t-1}\\
&= t\sum_{m=1}^{\infty} \lambda_m c_m a_m \sum_{n=m}^{\infty}\left(\frac{1}{\Lambda_n}-\frac{1}{\Lambda_{n+1}}\right)\left(\sum_{k=1}^{m} \lambda_m c_m a_m\right)^{t-1}\\
&= t\sum_{m=1}^{\infty} \lambda_m c_m a_m \frac{1}{\Lambda_m}\left(\sum_{k=1}^{m} \lambda_m c_m a_m\right)^{t-1}\\
&\le t\sum_{m=1}^{\infty}\left[\left(1+\frac{1}{\Lambda_m/\lambda_m}\right)^{\Lambda_m/\lambda_m}\right]^{1/t}\lambda_m a_m \Lambda^{(1-t)/t}\left(\sum_{k=1}^{m} \lambda_m c_m a_m\right)^{t-1}.
\end{aligned}$$

Hence, by the above inequality (8) and Lemma 2, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{t/\Lambda_n}\\
&< te^{1/t}\sum_{m=1}^{\infty}\left(\frac{5\Lambda_m+6\lambda_m}{5\Lambda_m+\lambda_m}\right)^{-1/2t}\lambda_m a_m \Lambda^{(1-t)/t}\left(\sum_{k=1}^{m} \lambda_m c_m a_m\right)^{t-1}.
\end{aligned}$$

Thus the inequality (4) is proved. □

REMARK 2. Setting $t \equiv 1$ in Theorem 2, then from (4), we obtain the inequality in Theorem 1 and, letting $\lambda_n \equiv 1$ in Theorem 2, then (4) becomes

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}\Big(1+\frac{1}{n+\frac{1}{5}}\Big)^{-1/2}a_n. \tag{9}$$

The inequality (9) is a better improvement of the following Carleman's inequality (cf. [3, Chap. 9.12]):

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}a_n.$$

Moreover, we can consider a generalization version of Theorem A.

THEOREM 3. *If* $0 < \lambda_{n+1} \le \lambda_n$, $\Lambda_n = \sum_{m=1}^{n}\lambda_m$ $(\Lambda_n \ge 1)$, $a_n \ge 0$ $(n \in N)$ *and* $0 < \sum_{n=1}^{\infty}\lambda_n(a_n)^t < \infty$ *for* $1 \le t < \infty$, *then*

$$\begin{aligned}&\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{t/\Lambda_n}\\ &< te^{1/t}\sum_{n=1}^{\infty}\Big(1-\frac{\lambda_n}{2(\lambda_n+\Lambda_n)}\Big)^{1/t}\lambda_n a_n\Lambda_n^{(1-t)/t}\Big(\sum_{k=1}^{n}\lambda_k c_k a_k\Big)^{t-1}.\end{aligned}\tag{10}$$

Proof. The proof is immediate. In fact, by the inequalities (3), (8) and Lemma 2, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{t/\Lambda_n}\\ &< te^{1/t}\sum_{m=1}^{\infty}\Big(1-\frac{\lambda_m}{2(\lambda_m+\Lambda_m)}\Big)^{1/t}\lambda_m a_m\Lambda^{(1-t)/t}\Big(\sum_{k=1}^{m}\lambda_m c_m a_m\Big)^{t-1}.\end{aligned}$$

Thus the inequality (10) is proved. □

REMARK 3. Setting $t \equiv 1$ in Theorem 3, then, form (10) we have the inequality (∗). Of course, we know that the inequality (4) is a better improvement of the inequality (10).

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