LOCALIZATION OF THE COHOMOLOGY OF THE HOMOTOPY ORBIT SPACE OF $p$-COMPACT GROUPS

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ABSTRACT. For a $p$-compact group $G$, if $G$-space $X$ is of finite $S$-type then we show that the localization $S^{-1}H^{*}(X_{hG})$ is zero. By using this result, we prove the localization theorem for the pair $G$-space $(X,A)$.

1. Introduction

A loop space is a triple $G = (G, BG, e)$, where $G$ is a topological space, $BG$ is a connected pointed classifying space of $G$ and $e : G \to \Omega BG$ is a homotopy equivalence from $G$ to the space $\Omega BG$ of based loops in $BG$. Such a loop space is called $p$-compact group if $G$ is $\mathbb{F}_p$-finite and $BG$ is $\mathbb{F}_p$-complete. Here the second condition is equivalent to that $G$ is $\mathbb{F}_p$-complete and $\pi_0(G)$ is a finite $p$-group. The main example of $p$-compact group is the $p$-completion of compact Lie group $G$, $(\hat{G}, \hat{BG}_p, e)$, where $\pi_0(G)$ is a finite $p$-group and $e : \Omega \hat{BG}_p \simeq \hat{G}_p$. Dwyer and Wilkerson defined these $p$-compact groups and proved a lot of their properties in [5]. Their work shows that a $p$-compact group has much of the rich internal structure of a compact Lie group. In particular, they showed that every $p$-compact group has a maximal torus, normalizer of the maximal torus and Weyl groups. More homotopy theories of $p$-compact groups are developed in [6], [7], and [8].

Let $G$ be a $p$-compact group. A $G$-space $X$ is defined to be a fibration $p_X : X_{hG} \to BG$ with $X$ as the fibre. Here we say $X_{hG}$ to be homotopy orbit space of a $p$-compact group $G$. In this paper we give localization properties of $H^{*}(X_{hG})$ for $p$-compact group $G$. This is the generalization of the localization theorem for the equivariant cohomology of compact.

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Lie group. In Section 2 we give basic definitions and properties regarding 
$p$-compact groups as preliminaries. Section 3 gives new definitions with 
respect to the $p$-compact groups and the proof of our main results.

All unspecified cohomology $H^*(\_)$ groups are assumed with coefficients 
in $\mathbb{F}_p$.

2. Preliminaries

A graded vector space $H^*$ over a field $F$ is finite dimensional if each 
$H^i$ is finite dimensional over $F$ and $H^i = 0$ for all but finite number of 
i. A space $X$ is $\mathbb{F}_p$-finite if $H^*X$ is finite dimensional over a finite field 
$\mathbb{F}_p$. Let $\epsilon_X : X \to X_p^\sim$ be a natural map for any space $X$ where $(\_)_p^\sim$ 
is $\mathbb{F}_p$-completion functor constructed by Bousfield and Kan [2]. If $\epsilon_X$ is 
homotopy equivalent, we say $X$ is $\mathbb{F}_p$-complete.

Now we give the basic definitions regarding the $p$-compact groups [5]. 

A homomorphism $f : K \to G$ of $p$-compact groups is a pointed map 
$Bf : BK \to BG$. The homogeneous space $G/K$ is defined to be the 
homotopy fiber of $Bf$ over the basepoint of $BG$. The homomorphism 
f is said to be monomorphism or equivalently $K \to G$ is a subgroup of 
$G$ if the homotopy fibre $G/K$ of $Bf$ is $\mathbb{F}_p$-finite, and an epimorphism 
if $\Omega(G/K)$ is a $p$-compact group. Two homomorphism $f_1, f_2 : K \to G$ 
are conjugate if the associated maps $Bf_1, Bf_2 : BK \to BG$ are freely 
homotopic. A short exact sequence $K \xrightarrow{f} H \xrightarrow{g} G$ of $p$-compact groups 
is sequence such that $BK \xrightarrow{Bf} BH \xrightarrow{Bg} BG$ is a fibration sequence where 
f is a monomorphism and $g$ is an epimorphism.

Let $G$ be a $p$-compact group. We define $G$-space $X$ to be the fibration 
p_X : X_{hG} \to BG$ with $X$ as the fibre. The $G$-equivariant map $X \to Y$ 
is defined to be a map of spaces together with an extension to a map 
$X_{hG} \to Y_{hG}$ of spaces over $BG$. A $G$-subspace $A$ of $X$ is a subspace 
with the fibration $A_{hG} \to BG$ with a homotopy fibre $A$. If $A$ is a 
$G$-subspace of $X$, then $A_{hG}$ is a subspace of $X_{hG}$. For $i = 1, 2$, let 
f_i : H_i \to G$ be subgroups of $G$. Then $H_1$ is subconjugate to $H_2$ if 
there exists a homomorphism $h : H_1 \to H_2$ such that $f_2 \circ h$ and $f_1$ are 
conjugate. We say that the $p$-compact subgroups $H_1 \to G$ and $H_2 \to G$ 
are conjugate in $G$, denoted by $H_1 \sim H_2$, if $H_1$ and $H_2$ are subconjugate 
to each other.

In the next section we extend the localization theorem for equivariant 
cohomology of compact Lie group to a theorem for $p$-compact group.
3. Localization of $H^*(X_{hG})$ for $p$-compact group $G$

Let $G$ be a $p$-compact group. We give the localization property of the cohomology of the homotopy orbit space of a $p$-compact group $G$.

The following result is known by W. G. Dwyer and C. W. Wilkerson.

**Theorem 3.1 ([5]).** If $G$ is a $p$-compact group, then $H^*(BG, \mathbb{F}_p)$ is finitely generated as an algebra.

From classical algebra, if $X$ is connected then $H^*X$ is finitely generated as an algebra if and only if $H^*X$ is Noetherian as a graded ring if and only if every graded ideal in $H^*X$ has a finite number of homogeneous generators if and only if every graded submodule of a graded finitely generated $H^*X$-module is itself finitely generated. Also a $H^*X$-module satisfies the ascending chain condition on submodules if and only if every submodule of $H^*X$-module is finitely generated.

Now we give the following definitions.

**Definition.** An isotropy family for the $p$-compact group $G$ is a set $\mathcal{F}$ of $p$-compact subgroups $H \to G$ such that if $H \to G$ belongs to $\mathcal{F}$ and $H \sim K$, then $K \to G$ also belongs to $\mathcal{F}$. The isotropy family $\mathcal{F}$ is said to be open if $H \to G$ belongs to $\mathcal{F}$ and $K \to H$, a subgroup of $H$, implies that $K \to G$ belongs to $\mathcal{F}$. The isotropy family $\mathcal{F}$ is said to be closed if $K \to G$ belongs to $\mathcal{F}$ and $K \to H$, a subgroup of $H$, implies that $H \to G$ belongs to $\mathcal{F}$.

**Definition.** A $G$-space $X$ is $\mathcal{F}$-numerable if there exists a covering $\mathcal{U} = \{U_i \mid i \in I\}$ of $X$ by $G$-subspaces with the following properties.

(i) For each $i \in I$, there exists a $G$-equivariant map $f_i : U_i \to G/G_i$, where $G_i \to G$ belongs to $\mathcal{F}$.

(ii) There exists a locally finite partition of unity $(t_i \mid i \in I)$ subordinate to $\mathcal{U}$ by $G$-functions $t_i : X \to [0, 1]$.

If $f : X \to Y$ is $G$-equivariant and $Y$ is $\mathcal{F}$-numerable, then $X$ is also $\mathcal{F}$-numerable. For this, we take open $G$-covering $\mathcal{U} = \{U_i \mid U_i = f^{-1}(V_i), V_i \in \mathcal{V}\}$ where $\mathcal{V}$ is a $G$-covering of $Y$ satisfying the $\mathcal{F}$-numerable condition. Then the composition $U_i \to V_i \to G/G_i$ is also $G$-equivariant for $G_i \to G$ in $\mathcal{F}$. If we set $t_i = s_i \circ f$ where $\{s_i \mid s_i : Y \to [0, 1], i \in I\}$ is a locally finite partition of unity subordinate to $\mathcal{V}$, then it is easy to
see that \(\{t_i \mid i \in I\} \) is also a locally finite partition of unity subordinate to \(\mathcal{U}\).

Let \(X\) be a \(G\)-space with a fibration \(X \xrightarrow{i} X_{hG} \xrightarrow{p_X} BG\).

We assume \(H^*(\_\_\_)\) has its usual multiplicative structure. This means that we are given product pairings (cup product)

\[
H^m(X, A) \otimes H^n(X, B) \to H^{m+n}(X, A \cup B)
\]

with the usual properties. This product yields a product pairing

\[
H^m(X_{hG}, A_{hG}) \otimes H^n(X_{hG}, B_{hG}) \to H^{m+n}(X_{hG}, A_{hG} \cup B_{hG}).
\]

Now \(H^*(X_{hG})\) is a graded module over \(H^*(BG)\) in a canonical way. The module structure is defined as follows. For \(b \in H^*(BG)\) and \(x \in H^*(X_{hG})\), consider \(p_X^*(b) \in H^*(X_{hG})\) and form the product \(p_X^*(b) \cup x\).

In particular \(H^*(X_{hG})\) becomes a graded algebra with unit.

Let \(S \subset H^*(BG)\) be a multiplicatively closed subset of homogeneous elements. We assume \(S\) is contained in the center of \(H^*(BG)\). Let \(M\) be a graded \(H^*(BG)\)-module and consider the localization of \(M\) with respect to \(S\), denoted by \(S^{-1}M\). Regard \(S^{-1}M\) as \(S^{-1}H^*(BG)\)-module and

\[
S^{-1}M \cong S^{-1}H^*(BG) \otimes_{H^*(BG)} M.
\]

The following result is from the commutative algebra.

**Proposition 3.2 ([1]).**

(i) If \(M \to N \to P\) is an exact sequence of \(H^*(BG)\)-modules, then \(S^{-1}M \to S^{-1}N \to S^{-1}P\) is an exact sequence of \(S^{-1}H^*(BG)\)-modules.

(ii) The kernel of the canonical map \(\phi: M \to S^{-1}M\) consists of those \(m \in M\) which are annihilated by some element of \(S\).

(iii) Localization commutes with colimits.

We consider the following set of \(p\)-compact subgroups of \(G\).

\[
\mathcal{F}(S) = \{H \to G \mid S \cap \ker[H^*(BG) \xrightarrow{p_H} H^*((G/H)_{hG})] \neq \emptyset\}
\]

where \(p_H^*\) is the induced homomorphism on the cohomology from the fibration \((G/H)_{hG} \xrightarrow{p_H} BG\).

**Proposition 3.3.** The set of \(p\)-compact subgroups of \(G\), \(\mathcal{F}(S)\) is an isotropy family.

**Proof.** We need to show that if \(H \to G\) belongs to \(\mathcal{F}\) and \(H \sim K\), then \(K \to G\) belongs to \(\mathcal{F}\). We consider the fibrations \(p_H : (G/H)_{hG} \to BG\) and \(p_K : (G/K)_{hG} \to BG\). Since \(H \sim K\), \((G/H)_{hG} \simeq BH\) and \((G/K)_{hG} \simeq BK\), there exist \(\tilde{f} : (G/H)_{hG} \to (G/K)_{hG}\) and \(\tilde{g} : H \to K\).
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\((G/K)_{hG} \to (G/H)_{hG}\) such that \(p_K \circ \tilde{f} \simeq p_H\) and \(p_H \circ \tilde{g} \simeq p_K\). Then \(\tilde{f}^* \circ p_K^* = p_H^*\) and \(\tilde{g}^* \circ p_H^* = p_K^*\). Let \(x\) belong to \(S \cap \ker p_H^*\). For such \(x \in S, p_H^* (x) = 0\) and \(p_K^* (x) = \tilde{g}^* \circ p_H^* (x) = 0\). Thus \(x \in \ker p_K^*\), and hence \(x \in S \cap \ker p_K^*\). This implies \(S \cap \ker p_K^* \neq \emptyset\). Therefore \(K \to G\) belongs to \(\mathcal{F}\).

**Definition.** A \(G\)-space \(X\) is of finite \(S\)-type if there is a numerable, finite dimensional \(G\)-covering \(\{U_\alpha \mid \alpha \in A\}\) of \(X\), a finite number of \(p\)-compact subgroups \(H_1 \to G, \ldots, H_r \to G\) in \(\mathcal{F}(S)\) and \(G\)-equivariant maps \(f_\alpha : U_\alpha \to G/H_{n(\alpha)}\), \(n(\alpha) \in \{1, 2, \ldots, r\}\).

It is easily checked that if \(X\) is of finite \(S\)-type and \(Y \subset X\), then \(Y\) is of finite \(S\)-type.

**Lemma 3.4.** Suppose \(U_1, U_2, \ldots, U_r\) is an open \(G\)-covering of \(X\). If we are given elements \(x_1 \in H^*(X_{hG})\) whose restriction to \(U_i\) is zero, then the product \(x_1 \cdots x_r\) is zero.

**Proof.** Since \(U_i\) is a \(G\)-subspace of \(X\), \((U_i)_{hG}\) is a subspace of \(X_{hG}\) for \(i = 1, 2, \ldots, r\). For each pair space \((X_{hG}, (U_i)_{hG})\), there exists a long exact sequence

\[
\cdots \to H^n(X_{hG}, (U_i)_{hG}) \overset{j^*}{\to} H^n(X_{hG}) \overset{i^*}{\to} H^n((U_i)_{hG}) \overset{\delta^*_i}{\to} H^{n+1}(X_{hG}, (U_i)_{hG}) \to \cdots.
\]

Given element \(x_i \in H^*(X_{hG})\), there exists \(y_i \in H^n(X_{hG}, (U_i)_{hG})\) such that \(i^* \circ j^*(y_i) = i^*(x_i) = 0\) by exactness. Then the product \(y_1 \cdots y_r\) is defined and contained in \(H^*(X_{hG}, (U_1)_{hG} \cup \cdots \cup (U_r)_{hG})\). However

\[H^*(X_{hG}, (U_1)_{hG} \cup \cdots \cup (U_r)_{hG}) = 0.\]

Therefore \(y_1 \cdots y_r\) is zero. This implies \(x_1 \cdots x_r\) is zero. \(\square\)

**Theorem 3.5.** Let \(X\) be of finite \(S\)-type. Then \(S^{-1}H^*(X_{hG})\) is zero.

**Proof.** Since \(X\) is of finite \(S\)-type, there exist finite number of \(p\)-compact subgroups \(H_1 \to G, H_2 \to G, \ldots, H_r \to G\) in \(\mathcal{F}(S)\) and \(G\)-equivariant maps \(f_\alpha : U_\alpha \to G/H_{n(\alpha)}, n(\alpha) \in \{1, 2, \ldots, r\}\). Let \(A_i = \cup\{U_\alpha \mid n(\alpha) = i\}\). Then \(A_1, A_2, \ldots, A_r\) is open \(G\)-covering of \(X\). If we show that for each \(i\) there exists \(s_i \in S\) with image in \(H^*((A_i)_{hG})\) being zero, then the product \(s = s_1 s_2 \cdots s_r\) is zero in \(H^*(X_{hG})\) by Lemma 3.4. Hence each element in \(H^*(X_{hG})\) is annihilated by \(s\). This implies \(\ker \phi\) is equal to \(H^*(X_{hG})\) for the canonical map \(\phi : H^*(X_{hG}) \to S^{-1}H^*(X_{hG})\), and hence \(S^{-1}H^*(X_{hG})\) is zero. Since the covering \(\{U_\alpha \mid n(\alpha) = i\}\) of
$A_i$ is finite dimensional and numerable, it is sufficient to consider the case $r = 1$. Let $H = H_1$, $A = \cup \{U_\alpha \mid n(\alpha) = 1\}$ and there exists $H \to G \in \mathcal{F}(S)$ with $G$-equivariant maps $f_\alpha : U_\alpha \to G/H$. Then there exists a covering $V_0, \ldots, V_n$ of $X$ such that each $V_i$ is a disjoint union of open $G$-sets which are contained in at least one of $U_\alpha$. In particular, each $V_i$ has a $G$-equivariant map $h_i : V_i \to G/H$. Since $H \to G$ belongs to $\mathcal{F}(S)$, there exists $s \in S$ in the kernel of $p_H^*: H^*(BG) \to H^*((G/H)_{hG})$. Then $s$ belongs to the kernel of the composition

$$H^*(BG) \to H^*((G/H)_{hG}) \to H^*((V_i)_{hG})$$

where the second map is $(h_i)^*_{hG}$. Thus $H^*((V_i)_{hG})$ is annihilated by $s$. Therefore $H^*(X_{hG})$ is annihilated by $s^{n+1}$. This completes the proof. □

**Definition.** The $G$-subspace $A$ of $X$ is **taut** in $X$ with respect to $H^*(\_)$ if the canonical map

$$\text{colim}_V H^*(X_{hG}, V_{hG}) \to H^*(X_{hG}, A_{hG})$$

is an isomorphism where the colimit is taken over the open $G$-neighborhoods $V$ of $A$ in $X$.

**Theorem 3.6.** Let $A$ be taut in $X$ and closed. Let $X \setminus A$ be of finite $S$-type. Then the inclusion map $A \hookrightarrow X$ induces an isomorphism

$$S^{-1}H^*(X_{hG}) \approx S^{-1}H^*(A_{hG}).$$

**Proof.** Since localization preserves exactness, it is sufficient to show that $S^{-1}H^*(X_{hG}, A_{hG})$ is zero. Since localization commutes with colimits, it suffices to show that $S^{-1}H^*(X_{hG}, V_{hG})$ is zero for open $G$-neighborhoods $V$ of $A$. Since $A$ is closed in $X$, we have the following excision isomorphism

$$S^{-1}H^*(X_{hG}, V_{hG}) \approx S^{-1}H^*((X \setminus A)_{hG}, (V \setminus A)_{hG}).$$

However $X \setminus A$ is of finite $S$-type, and hence $V \setminus A$ is of finite $S$-type. Thus $S^{-1}H^*((X \setminus A)_{hG})$ and $S^{-1}H^*((V \setminus A)_{hG})$ are zeros by Theorem 3.5. By exact cohomology sequence for the pair space $((X \setminus A)_{hG}, (V \setminus A)_{hG})$, $S^{-1}H^*((X \setminus A)_{hG}, (V \setminus A)_{hG})$ is zero. Therefore $S^{-1}H^*(X_{hG}, V_{hG})$ is zero. This completes the proof. □
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References


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