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DETERMINANTAL MAPS AND ANNIHILATOR OF M-COSEQUENCES

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Abstract. Let A be a commutative ring with nonzero identity and let M be an A-module. In this note we show that if $x = x_1, \cdots, x_n$ and $y = y_1, \cdots, y_n$ both M-cosequence such that $Hx^T = y^T$ for some $n \times n$ lower triangular matrix H over A, then the map $\beta_H : \mathrm{Ann}_M(y_1, \cdots, y_n) \longrightarrow \mathrm{Ann}_M(x_1, \cdots, x_n)$ induced by multiplication by $|H|$ is surjective.

1. Introduction

Throughout this paper A is a commutative ring (with identity) and M an A-module. In [**8**, 8], Matlis showed that, for a given M-sequence $x_1, \cdots, x_n$ the map $M/\sum_{i=1}^{n} x_i^t M \longrightarrow M/\sum_{i=1}^{n} x_i^{t+1} M$ induced by multiplication by $x_1 \cdots x_n$ is a monomorphism for all $t \in \mathbb{N}$. Also he showed that if $x_1, \cdots, x_n$ is an M-consequence, then the map $\mathrm{Ann}_M(x_1^{t+1}, \cdots, x_n^{t+1}) \longrightarrow \mathrm{Ann}_M(x_1^t, \cdots, x_n^t)$ induced by multiplication by $x_1 \cdots x_n$ is an epimorphism for all $t \in \mathbb{N}$. As a generalization of the first result of Matlis, in [**9**, 3.2], O'Carroll described that, when $x = x_1, \cdots, x_n$ and $y = y_1, \cdots, y_n$ are two sequences of elements of A such that $H[x_1 \cdots x_n]^T = [y_1 \cdots y_n]^T$ and y is a poor M-sequence then the determinantal map

$$\alpha_H : M/\sum_{i=1}^{n} x_i M \longrightarrow M/\sum_{i=1}^{n} y_i M$$

induced by multiplication by $|H|$ is a monomorphism and x is also a poor M-sequence. Afterward, Gibson has proved, in [**3**, 3.3], for arbitrary $n \times n$ matrix H with $Hx^T = y^T$, that α_H is injective if x and y are M-sequences. Finally, Chung, in [**2**, 2.2], consider the dual case of the

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O'Carroll result. That is, let y be a poor M-cosequence and x be a sequence of elements of A such that $Hx^T = y^T$ for some $n \times n$ lower triangular matrix H over A, then the map

$$\beta_H : \mathrm{Ann}_M(y_1, \cdots, y_n) \longrightarrow \mathrm{Ann}_M(x_1, \cdots, x_n)$$

induced by multiplication by $|H|$ is surjective and x is also poor M-cosequence. In this note we show that if x and y both M-cosequence then the map β_H is surjective.

2. Preliminaries

Throughout this paper, we use ^{T} to denote matrix transpose, for any positive integer n, $M_n(A)$ to denote the set of $n \times n$ matrices over A and $D_n(A)$ to denote the set of $n \times n$ lower triangular matrices over A. For $H \in M_n(A)$, $|H|$ denotes the determinant of H. Let $(a_1, \cdots, a_i)$ be the ideal generated by $\{a_1, \cdots, a_i\}$ and $(a_1, \cdots, a_i)M$ the submodule of M generated by $\{a_j m : j = 1, \cdots, i, \ \ m \in M\}$. Whenever we can do so without ambiguity, we denote $(x_1, \cdots, x_n) \in A^n$ by x and $[x_1 \cdots x_n]^T$ by x^T. We use $\mathbb{N}$ to the set of positive integers.

Let $x_1, \cdots, x_n$ be a sequence of elements of A. Then $x_1, \cdots, x_n$ is said to be an M-sequence if multiplicaiton by x_i on $M/(x_1, \cdots, x_{i-1})M$ is a monomorphism for all $i = 1, \cdots, n$ (where $x_0 = 0$) and $M/ (x_1, \cdots, x_n) M \neq 0$.

If $\mathfrak{a}$ is an ideal of A, we set $\mathrm{Ann}_M \mathfrak{a} = \{m \in M : \mathfrak{a}m = 0\}$. We have a dual definition; $x_1, \cdots, x_n$ is said to be an M-consequence if the multiplication by x_i on $\mathrm{Ann}_M(x_1, \cdots, x_{i-1})$ is an epimorphism for all $i = 1, \cdots, n$ (where $x_0 = 0$) and $\mathrm{Ann}_M(x_1, \cdots, x_n) \neq 0$ (See [**5**]).

In order to prove the main result of this paper we need a lemma. The following remark is dual to [**6**, pp. 100–101].

REMARK 2.1. Let N and M be A-modules. Suppose that there exists an $x \in A$ such that x is M-consequence and $xN = 0$. Then it is routine to check that $M \otimes_A N = 0$.

The following lemma, which is proved in [**10**, 3.6(2)] under the assumption that M is Artinian, is dual to [**6**, p. 101].

LEMMA 2.2. *Let M and N be A-modules. Suppose that the elements $x_1, \cdots, x_n$ constitute an M-consequence and $(x_1, \cdots, x_n)N = 0$. Then*

$$Tor_n^A(M, N) \cong \mathrm{Ann}_M(x_1, \cdots, x_n) \otimes_A N.$$

Proof. We prove this by induction on n. For the case in which $n = 1$, consider the exact sequence $0 \longrightarrow \mathrm{Ann}_M(x_1) \longrightarrow M \xrightarrow{x_1} M \longrightarrow 0$ to deduce the exact sequence

$$0 \to Tor_1^A(M,N) \longrightarrow \mathrm{Ann}_M(x_1)\otimes_A N \longrightarrow M\otimes_A N \xrightarrow{x_1\otimes \mathrm{id}_N} M\otimes_A N \to 0.$$

Hence, by 2.1, $Tor_1^A(M,N) \cong \mathrm{Ann}_M(x_1) \otimes_A N$. Now, suppose inductively that $n > 1$ and the result has been proved for smaller values of n. It follows from inductive hypothesis and 2.1 that $Tor_{n-1}^A(M,N) \cong \mathrm{Ann}_M(x_1, \cdots, x_{n-1}) \otimes_A N = 0$. Again, consider the exact sequence $0 \longrightarrow \mathrm{Ann}_M(x_1) \longrightarrow M \xrightarrow{x_1} M \longrightarrow 0$. We get from the induced long exact sequence of Tor

$$Tor_n^A(M,N) \cong Tor_{n-1}^A(\mathrm{Ann}_M(x_1), N).$$

Since $x_2, \cdots, x_n$ is $\mathrm{Ann}_M(x_1)$-consequence, by inductive hypothesis we obtain that $Tor_{n-1}^A(\mathrm{Ann}_M(x_1), N) \cong \mathrm{Ann}_M(x_1, \cdots, x_n) \otimes_A N$ and the result follows. □

3. The results

Let us, firstly, review the construction of Koszul complex and establish our notations. Let $x = x_1, \cdots, x_n$ be a sequence of elements of A. For $j \in \mathbb{N}$ with $1 \leq j \leq n$, we write

$$I(j,n) = \{(\alpha_1, \cdots, \alpha_j) \in \mathbb{N}^j : 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_j \leq n\}.$$

We denote by e_α the exterior product $e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_j}$ for all $\alpha = (\alpha_1, \cdots, \alpha_j) \in I(j,n)$. The (ascending) Koszul complex of A with respect x has the form

$$K^\bullet(x,A) : 0 \longrightarrow K^0(x) \xrightarrow{d^0(x)} K^1(x) \xrightarrow{d^1(x)} \cdots \xrightarrow{d^{n-1}(x)} K^n(x) \longrightarrow 0,$$

where, for $0 \leq j \leq n$, $K^j(x) = \wedge^j A^n$, with basis $e_\alpha, \alpha \in I(j,n)$, and the homomorphisms $d^j(x)$ (for $0 \leq j \leq n-1$) given by $d^j(x)(e_\alpha) = (\sum_{k=1}^{n} x_k e_k) \wedge e_\alpha$ for all $\alpha \in I(j,n)$. Also, set $K_\bullet(x,A) = \mathrm{Hom}_A(K^\bullet(x,A), A)$ Let $x = x_1, \cdots, x_n$ and $y = y_1, \cdots, y_n$ be sequences of elements of A such that $y^T = Hx^T$ for some $H = [H_{ij}] \in M_n(A)$. One can define, for any j with $0 \leq j \leq n$, a map $\wedge^j H : \wedge^j A^n \longrightarrow \wedge^j A^n$ given by $e_\alpha \mapsto He_{\alpha_1} \wedge \cdots \wedge He_{\alpha_j}$, or, equivalently by $e_\alpha \mapsto \sum_{\beta \in I(j,n)} \Delta_{\beta,\alpha} e_\beta$ for

$\alpha \in I(j, n)$. Here

$$\Delta_{\beta,\alpha} = \begin{vmatrix} h_{\beta_1\alpha_1} & \cdots & h_{\beta_1\alpha_j} \\ \vdots & & \vdots \\ h_{\beta_j\alpha_1} & \cdots & h_{\beta_j\alpha_j} \end{vmatrix}$$

is the $j \times j$-minor of H determined by α and β (see for example [**4**]). Now by using the functor $\mathrm{Hom}_A(-, A)$ and by [**1**, 1.6.8] the map $\wedge^\bullet(H, A)$ induces a map of Koszul complexes $K_\bullet(y, A)$ to $K_\bullet(x, A)$.

We now come to the main theorem in this paper.

THEOREM 3.1. *Let A be a ring and let M be an A-module. Suppose that there exist M-consequences $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ (denoted by x and y respectively), and $H = [h_{ij}] \in M_n(A)$, such that $y^T = Hx^T$. Then the A-homomorphism*

$$\beta : \mathrm{Ann}_M(y_1, \cdots, y_n) \longrightarrow \mathrm{Ann}_M(x_1, \cdots, x_n)$$

such that $\beta(m) = |H|m$ is surjective.

Proof. Let K be the adjoint matrix of H. Then by using the fact that $Ky^T = KHx^T = |H|x^T$, it is easy to see that the map β is well defined. Let $R = \mathbb{Z}[X_1, \cdots, X_n, H_{11}, H_{12}, \cdots, H_{ij}, \cdots, H_{nn}]$, where the elements X_i and H_{ij} are all indeterminates over $\mathbb{Z}$. Hence $X_1, \cdots, X_n$ is an R-sequence. Let $Y_1, \cdots, Y_n$ be a sequence of elements of R such that $[Y_1 \cdots Y_n]^T = [H_{ij}][X_1 \cdots X_n]^T$. Then, by [**5**, p. 1038, Proposition 21], $Y_1, \cdots, Y_n$ is an R-sequence. Now, let $\varphi : R \longrightarrow A$ be the ring homomorphism such that $\varphi(X_i) = x_i$ and $\varphi(H_{ij}) = h_{ij}$ for all i and j with $1 \leq i, j \leq n$. Then M can be given an R-module structure by defining $r.m$ to be $\varphi(r)m$ where $r \in R, m \in M$. Also, for $m \in M$,

$$Y_i m = \varphi(\sum_{j=1}^{n} H_{ij}X_j).m = (\sum_{j=1}^{n} h_{ij}x_j)m = y_i m$$

for all i with $1 \leq i \leq n$. Similarly $|[H_{ij}]|.m = |H|m$. Therefore $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ are M-consequences. For simplicity of notation we identify elements of R with their images in A, and relabel X_i as x_i, Y_i as y_i and H_{ij} as h_{ij} for all i and j with $1 \leq i, j \leq n$. Furthermore, we denote $(x_1, \cdots, x_n)M$ and $(y_1, \cdots, y_n)M$ by xM and yM respectively, and use obvious extensions of this notation. For this point forward, $\beta : \mathrm{Ann}_M(y_1, \cdots, y_n) \longrightarrow \mathrm{Ann}_M(x_1, \cdots, x_n)$ will denote the R-module homomorphism induced by multiplication by $|H|$, which is surjective if and only if the A-homomorphism in the statement of the theorem is surjective. Now, consider the exact sequence of R-modules

$$0 \longrightarrow \frac{xR}{yR} \xrightarrow{f} \frac{R}{yR} \xrightarrow{g} \frac{R}{xR} \longrightarrow 0,$$

where f is the inclusion map and g the canonical projection map. This yields the exact sequence

$$Tor_n^R(\frac{R}{yR}, M) \xrightarrow{\theta} Tor_n^R(\frac{R}{xR}, M) \longrightarrow Tor_{n-1}^R(\frac{xR}{yR}, M).$$

By 2.1 and 2.2,

$$Tor_{n-1}^R(\frac{xR}{yR}, M) \cong \frac{xR}{yR} \otimes_R \mathrm{Ann}_M(y_1, \cdots, y_n) = 0.$$

Thus the map θ is surjective. By making use of the Koszul complex we now calculate this induced map. The reader is referred to [**1**] for basic properties of the Koszul complex. Since $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ are R-sequences, the Koszul complexes $K^\bullet(x, R)$ and $K^\bullet(y, R)$ provide projective resolution of R/xR and R/yR respectively. Also the map $\wedge^\bullet(H, R)$ is a map of $K_\bullet(y, R)$ to $K^\bullet(x, R)$. (Note that the existence of $\wedge^\bullet(H, R)$ is explained in the beginning of this section.) Applying the functor $- \otimes M$ to the corresponding double complex gives in particular the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & R \otimes_R M & \xrightarrow{d^0(y)^\star} & (\bigoplus_{i=1}^{n} R) \otimes_R M & \longrightarrow \\ & & \Big\downarrow \Lambda^0(H, M) & & \Big\downarrow \Lambda^1(H, M) & \\ 0 & \longrightarrow & R \otimes_R M & \xrightarrow{d^0(x)^\star} & (\bigoplus_{i=1}^{n} R) \otimes_R M & \longrightarrow \end{array}$$

where $\wedge^\bullet(H, M) := \wedge^\bullet(H, R) \otimes \mathrm{id}_M$. It is easy to see that, in the top row, $\ker d^0(y)^\star = \mathrm{Ann}_M(y_1, \cdots, y_n)$ and, in the bottom row, $\ker d^0(x)^\star = \mathrm{Ann}_M(x_1, \cdots, x_n)$. We also note that there exists a natural isomorphism $\gamma : R \otimes_R M \longrightarrow M$. Thus we have the commutative diagram

$$\begin{array}{ccc} R\otimes_R M & \xrightarrow{\ \gamma\ } & M \\ \Big\downarrow \Lambda^0(H,M) & & \Big\downarrow \psi \\ R\otimes_R M & \xrightarrow{\ \gamma\ } & M \end{array}$$

where ψ is the homorphism induced by $\wedge^{\bullet}(H,M)$. Now it is easy to see that ψ must likewise be multiplication by $|H|$. Since $\ker d^0(x)^\star = \operatorname{Ann}_M(x_1,\cdots,x_n)$ and $\ker d^0(y)^\star = \operatorname{Ann}_M(y_1,\cdots,y_n)$, we have the following commutative diagram, where β is the map induced by multiplication by $|H|$.

$$\begin{array}{ccc} \operatorname{Tor}_n^R(\frac{R}{yR},M) & \cong & \operatorname{Ann}_M(y_1,\cdots,y_n) \\ \Big\downarrow \theta & & \Big\downarrow \alpha \\ \operatorname{Tor}_n^R(\frac{R}{xR},M) & \cong & \operatorname{Ann}_M(x_1,\cdots,x_n) \end{array}$$

It has already shown that θ is surjective, and so the map β must be surjective. □

Theorem 3.1 has some consequences which we record here.

CONSEQUENCES 3.2. *Let $x = x_1,\cdots,x_n$ and $y = y_1,\cdots,y_n$ be two sequences of elements of A such that $y^T = Hx^T$ for some $H \in D_n(A)$.*

(1) [**8**, 8]. *If x is M-cosequence then the map*

$$\operatorname{Ann}_M(x_1^{t+1},\cdots,x_n^{t+1}) \longrightarrow \operatorname{Ann}_M(x_1^t,\cdots,x_n^t)$$

induced by multiplication by $x_1\cdots x_n$ is an epimorphism for all $t \in \mathbb{N}$.

(2) (Compare [**2**, 2.2]). *If x and y are M-cosequences then the map*

$$\operatorname{Ann}_M(y_1,\cdots,y_n) \longrightarrow \operatorname{Ann}_M(x_1,\cdots,x_n)$$

induced by multiplication by $|H|$ is surjective.

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