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VOLUME PRODUCT FOR PEDAL BODIES

Y. D. Chai†, Yong-Il Kim‡, and Doohann Lee

Abstract. Let K be a convex body of constant relative breadth and let K^* be its polar dual with respect to the Euclidean unit circle. In this paper we obtain the lower bound for the volume of the pedal body PK^* of K^*. Using this, we also obtain the lower bound for the volume product $V(PK^*)V(PK)$ for planar bodies.

1. Introduction

Let K be a convex body of constant relative breadth and let K^* be its polar dual with respect to the Euclidean unit circle. Let $V_e(\cdot)$ denote the exterior Lebesgue measure with respect to the associated Euclidean space E^n. Following H. Busemann [2], we define the Minkowski volume $V(\cdot)$ by

$$V(K) = \sigma V_e(K), \tag{1}$$

where $\sigma = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)V_e(U)}$.

In this paper, we obtain the lower bound for the volume of the pedal body PK^* of K^*. Using this, we also obtain the lower bound for the volume product $V(PK^*)V(PK)$ for planar bodies.

2. Basic concepts

Let M^n be a Minkowski space with the metric m. Then there is a centrisymmetric compact convex body C in the n-dimensional Euclidean space E^n, which gives a metric m as follows. For $x, y \in E^n$ let $\overline{x}, \overline{y}$ be points on the intersection of $U = \partial C$, the boundary of C, with the line

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through the origin o which is parallel to the line passing through x and y. Then the Minkowski distance $m(x,y)$ is given by

$$m(x,y) = \frac{2e(x,y)}{e(\overline{x},\overline{y})}$$

using the associated Euclidean metric e. We call this centrisymmetric hypersurface U the indicatrix of the Minkowski space M^n and denote by $M^n(U)$ the Minkowski space with the indicatrix U. One can define the volume $V(\cdot)$ and the surface area $S(\cdot)$ on M^n [6]. When V and S are given, we call the solution of the isoperimetric problem for V and S the isoperimetrix of M^n and denote it by I from now on. In the Minkowski space M^n a line L_1 is said to be perpendicular to L_2 if L_1 intersects L_2 at x and for any point $y \in L_1$

$$m(x,y) = \min\{m(y,z)|z \in L_2\}.$$

If L_1 is perpendicular to L_2, then we say that L_2 is transversal to L_1. In general, the perpendicularity is not symmetric in Minkowski spaces M^n. For $n \geq 3$, the perpendicularity is symmetric if and only if M^n is Euclidean [1].

One can define the relative support function $h_U(K,\cdot)$ by

$$h_U(K,u) = \frac{h(K,u)}{h(U,u)},$$

where u is in S^{n-1}, the Euclidean unit sphere, and $h(\cdot,\cdot)$ is the Euclidean support function [5] defined by

$$h(K,u) = \sup\{x \cdot u|x \in K\}.$$

DEFINITION 1. Let K be a convex body in E^n. We call the number

$$b_U(K,u) = h_U(K,u) + h_U(K,-u)$$

the relative breadth of K in the direction u. If $b_U(K,u)$ is constant for any direction $u \in S^{n-1}$, then we say that K is a body of constant relative breadth.

For details about constant breadth bodies, see [3].

3. Volume of PK^*

From now on we assume that K has o in its interior.

DEFINITION 2. Let $K \subset E^n$ be a convex body. Then the pedal body PK of K is defined by

$$\rho(PK, u) = h(K, u),$$

where $\rho(\cdot, \cdot)$ is the radial function of K defined by

$$\rho(K, u) = \sup\{\alpha > 0 \mid \alpha u \in K\}.$$

From now we denote the polar dual of K with respect to the Euclidean unit circle by

$$K^* = \{y \in E^n | x \cdot y \leq 1 \text{ for all } x \in K\}.$$

We have the following theorem with the Minkowski volume $V(\cdot)$ defined by

$$V(\cdot) = \sigma V_e(\cdot)$$

as in (1).

THEOREM 1. *Let K be a body of constant relative breadth β in M^n. Then we have $V(PK^*) \geq \left(\frac{\beta}{2}\right)^n V(PI)$. The equality holds if and only if K is a unit ball with the center at the origin.*

Proof. Note that the pedal body of a convex body is a star body, a body whose radial function is continuous. Since $\rho(PK^*, u)$ is continuous on S^{n-1} we have the following.

$$\begin{aligned}
\frac{n}{\sigma} V(PK^*) &= \int_{S^{n-1}} \rho^n(PK^*, u) du \\
&= \int_{S^{n-1}} h^n(K^*, u) du \\
&= \int_{S^{n-1}} \frac{h^n(K^*, u) + h^n(K^*, -u)}{2} du \\
&= \int_{S^{n-1}} \frac{h^n(K^*, u) + h^n(-K^*, u)}{2} du \\
&= \int_{S^{n-1}} \frac{h^n(K^*, u) + h^n((-K)^*, u)}{2} du,
\end{aligned}$$

where du is the surface area element on S^{n-1}. Since the real valued function $f(t) = t^n$ is convex and increasing on the interval $[0, \infty)$,

$$\begin{aligned}
&\int_{S^{n-1}} \frac{h^n(K^*, u) + h^n((-K)^*, u)}{2} du \\
\geq &\int_{S^{n-1}} \left(\frac{h(K^*, u) + h((-K)^*, u)}{2}\right)^n du \\
\geq &\int_{S^{n-1}} \left(\frac{2}{\frac{1}{h(K^*,u)} + \frac{1}{h((-K)^*,u)}}\right)^n du \\
= &\int_{S^{n-1}} \left(\frac{2}{\rho(K, u) + \rho(-K, u)}\right)^n du \\
= &\int_{S^{n-1}} \left(\frac{\rho(K, u) + \rho(-K, u)}{2}\right)^{-n} du.
\end{aligned}$$

Because $\rho(\cdot, \cdot)$ satisfies that $\rho(K_1 + K_2, u) \geq \rho(K_1, u) + \rho(K_2, u)$, we get

$$\int_{S^{n-1}} \left(\frac{\rho(K, u) + \rho(-K, u)}{2}\right)^{-n} du \geq \int_{S^{n-1}} \rho^{-n} \left(\frac{K + (-K)}{2}, u\right) du.$$

Since $K + (-K) = \beta U$,

$$\begin{aligned}
\int_{S^{n-1}} \rho^{-n} \left(\frac{K + (-K)}{2}, u\right) du &= \int_{S^{n-1}} h^n \left(\left(\frac{K + (-K)}{2}\right)^*, u\right) du \\
&= \left(\frac{\beta}{2}\right)^n \int_{S^{n-1}} h^n(U^*, u) du \\
&= \left(\frac{\beta}{2}\right)^n \int_{S^{n-1}} \rho^n(PU^*, u) du.
\end{aligned}$$

We know that I is the polar dual of U. So

$$\begin{aligned}
\int_{S^{n-1}} \rho^n(PU^*, u) du &= \int_{S^{n-1}} \rho^n(PI, u) du \\
&= nV_e(PI).
\end{aligned}$$

This proves the result. □

COROLLARY 1. *If U is the Euclidean unit circle with the center o, then*

$$\Gamma\left(\frac{n}{2} + 1\right) V(PK^*) \geq \left(\frac{\beta\sqrt{\pi}}{2}\right)^n.$$

Proof. Trivial from Theorem 1. □

4. Planar bodies

Let K be a plane convex body in $M^2(U)$. Let $u = u(\xi) = (\cos\xi, \sin\xi)$ be in S^1, the unit circle with center o. Let $T(u,q,p)$ be a supporting tangent line at $q = q(\xi)$ on ∂K, the boundary of K, which is perpendicular to the ray $R(u)$ emanating from o and passing through u. And $T(u,q,p)$ meets $R(u)$ at the point $p = p(\xi) \in M^2$. Then the pedal body PK is a trace of points p. We know that the circle containing the points o, p, q has common tangent $T(PK,p)$ at p with PK [4]. From this we can compute the angle $\angle(T(PK,p), e_1)$ between $T(PK,p)$ and the x_1-axis, where $e_1 = (1,0)$. We get

$$\angle(T(PK,p), e_1) = 2\xi - \alpha - \frac{\pi}{2},$$

where $\alpha = \angle(q, o, e_1)$. Thus the Minkowski length $L(PK)$ of PK is

$$L(PK) = \int_{-\pi}^{\pi} \sqrt{\rho^2(PK,\xi) + \rho'^2(PK,\xi)}\rho^{-1}\left(U, 2\xi - \alpha - \frac{\pi}{2}\right) d\xi,$$

where $'$ denotes $\frac{d}{d\xi}$. We call the Minkowski space with the indicatrix I, the dual space of $M^n(U)$ and denote it by M^{n*}. Now we have the following.

THEOREM 2. *Let K be a body of constant relative breadth β in M^{2*}. Then*

$$V(PK)V(PK^*) \geq \frac{\beta^2 V(PI)[r_i^2 L^2(PK) + 4\pi V(K)]}{32\pi},$$

where r_i is the inradius of K, the maximal radius of circles contained in K.

Proof. We know that

$$L(PK) = \int_{-\pi}^{\pi} \sqrt{h^2(K,\xi) + h'^2(K,\xi)}\rho^{-1}\left(U, 2\xi - \alpha - \frac{\pi}{2}\right) d\xi$$

$$(2) \qquad \leq \sqrt{\int_{-\pi}^{\pi} [h^2(K,\xi) + h'^2(K,\xi)] d\xi \int_{-\pi}^{\pi} \rho^{-2}\left(U, 2\xi - \alpha - \frac{\pi}{2}\right) d\xi}.$$

Here (2) is less than $\frac{2\pi}{r_i}$ times the square root of

$$\beta^2 \int_0^{\pi} [\rho^{-2}(U,\xi) + \rho'^{-2}(U,\xi)] d\xi + 2V(K,-K) - 4\int_0^{\pi} h(K,\xi)h(K,\xi+\pi) d\xi.$$

Here $V(\cdot,\cdot)$ denotes the mixed volume. Since

$$\begin{aligned} &\int_0^{\pi} h(K,\xi)h(K,\xi+\pi)d\xi \\ = &\ \frac{\beta^2}{2}\int_0^{\pi}\rho^2(U,\xi)d\xi - \frac{1}{2}\int_{-\pi}^{\pi} h^2(K,\xi)d\xi \\ = &\ \frac{\beta^2}{2}V(PI) - V(PK), \end{aligned}$$

we have

$$L(PK) \le \frac{1}{r_i}\sqrt{4\pi(2V(PK) - V(K))}.$$

This completes the proof. □

References

[1] W. Blaschke, *Räumliche Variationsprobleme mit symmetrischer Transversalitätsbedingung*, Ber. Verh. Sä chs. Akad. Wiss. Leipzig Math.-Naturw. Kl. **68** (1916), 50–55.

[2] H. Busemann, *The isoperimetric problem for Minkowski area*, Amer. J. Math. **71** (1949), 743–762.

[3] G. D. Chakerian and H. Groemer, *Convex bodies of constant width*, In Convexity and Its Applications, ed. by P. M. Gruber and J. M. Wills. Birkhäuser, Basel, 1983.

[4] P. J. Kelly, *Curves with a kind of constant width*, Amer. Math. Monthly **64** (1957), 333–336.

[5] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, Cambridge, 1993.

[6] A. C. Thomson, *Minkowski geometry*, Cambridge University Press, 1996.

Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea

E-mail: ydchai@yurim.skku.ac.kr
yikim@math.skku.ac.kr
dhlee@math.skku.ac.kr