VOLUME PRODUCT FOR PEDAL BODIES

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Abstract. Let K be a convex body of constant relative breadth and let K* be its polar dual with respect to the Euclidean unit circle. In this paper we obtain the lower bound for the volume of the pedal body PK* of K*. Using this, we also obtain the lower bound for the volume product V(PK*)V(PK) for planar bodies.

1. Introduction

Let K be a convex body of constant relative breadth and let K* be its polar dual with respect to the Euclidean unit circle. Let $V_ε(\cdot)$ denote the exterior Lebesgue measure with respect to the associated Euclidean space $E^n$. Following H. Busemann [2], we define the Minkowski volume $V(\cdot)$ by

$$V(K) = \sigma V_ε(K),$$

where \( \sigma = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1) V_ε(\mathbb{B})} \).

In this paper, we obtain the lower bound for the volume of the pedal body PK* of K*. Using this, we also obtain the lower bound for the volume product $V(PK*)V(PK)$ for planar bodies.

2. Basic concepts

Let $M^n$ be a Minkowski space with the metric $m$. Then there is a centrisymmetric compact convex body $C$ in the $n$-dimensional Euclidean space $E^n$, which gives a metric $m$ as follows. For $x, y \in E^n$ let $\bar{x}, \bar{y}$ be points on the intersection of $U = \partial C$, the boundary of $C$, with the line

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through the origin $o$ which is parallel to the line passing through $x$ and $y$. Then the Minkowski distance $m(x, y)$ is given by

$$m(x, y) = \frac{2e(x, y)}{e(x, y)}$$

using the associated Euclidean metric $e$. We call this centrosymmetric hypersurface $U$ the indicatrix of the Minkowski space $M^n$ and denote by $M^n(U)$ the Minkowski space with the indicatrix $U$. One can define the volume $V(\cdot)$ and the surface area $S(\cdot)$ on $M^n$ [6]. When $V$ and $S$ are given, we call the solution of the isoperimetric problem for $V$ and $S$ the isoperimetrix of $M^n$ and denote it by $I$ from now on. In the Minkowski space $M^n$ a line $L_1$ is said to be perpendicular to $L_2$ if $L_1$ intersects $L_2$ at $x$ and for any point $y \in L_1$

$$m(x, y) = \min\{m(y, z) | z \in L_2\}.$$  

If $L_1$ is perpendicular to $L_2$, then we say that $L_2$ is transversal to $L_1$. In general, the perpendicularity is not symmetric in Minkowski spaces $M^n$. For $n \geq 3$, the perpendicularity is symmetric if and only if $M^n$ is Euclidean [1].

One can define the relative support function $h_U(K, \cdot)$ by

$$h_U(K, u) = \frac{h(K, u)}{h(U, u)},$$

where $u$ is in $S^{n-1}$, the Euclidean unit sphere, and $h(\cdot, \cdot)$ is the Euclidean support function [5] defined by

$$h(K, u) = \sup\{x \cdot u | x \in K\}.$$  

**Definition 1.** Let $K$ be a convex body in $E^n$. We call the number

$$b_U(K, u) = h_U(K, u) + h_U(K, -u)$$

the relative breadth of $K$ in the direction $u$. If $b_U(K, u)$ is constant for any direction $u \in S^{n-1}$, then we say that $K$ is a body of constant relative breadth.

For details about constant breadth bodies, see [3].
3. Volume of $PK^*$

From now on we assume that $K$ has $o$ in its interior.

**Definition 2.** Let $K \subset E^n$ be a convex body. Then the pedal body $PK$ of $K$ is defined by

$$\rho(PK, u) = h(K, u),$$

where $\rho(\cdot, \cdot)$ is the radial function of $K$ defined by

$$\rho(K, u) = \sup \{ \alpha > 0 \mid \alpha u \in K \}.$$

From now we denote the polar dual of $K$ with respect to the Euclidean unit circle by

$$K^* = \{ y \in E^n \mid x \cdot y \leq 1 \text{ for all } x \in K \}.\]$$

We have the following theorem with the Minkowski volume $V(\cdot)$ defined by

$$V(\cdot) = \sigma V_e(\cdot)$$

as in (1).

**Theorem 1.** Let $K$ be a body of constant relative breadth $\beta$ in $M^n$. Then we have $V(PK^*) \geq \left( \frac{\beta}{2} \right)^n V(PI)$. The equality holds if and only if $K$ is a unit ball with the center at the origin.

**Proof.** Note that the pedal body of a convex body is a star body, a body whose radial function is continuous. Since $\rho(PK^*, u)$ is continuous on $S^{n-1}$ we have the following.

$$\frac{n}{\sigma} V(PK^*) = \int_{S^{n-1}} \rho^n(PK^*, u) du$$

$$= \int_{S^{n-1}} h^n(K^*, u) du$$

$$= \int_{S^{n-1}} \frac{h^n(K^*, u) + h^n(K^*, -u)}{2} du$$

$$= \int_{S^{n-1}} \frac{h^n(K^*, u) + h^n(-K^*, u)}{2} du$$

$$= \int_{S^{n-1}} \frac{h^n(K^*, u) + h^n((-K)^*, u)}{2} du.$$
where \( du \) is the surface area element on \( S^{n-1} \). Since the real valued function \( f(t) = t^n \) is convex and increasing on the interval \([0, \infty)\),
\[
\int_{S^{n-1}} \frac{h^n(K^*, u) + h^n((-K)^*, u)}{2} du \\
\geq \int_{S^{n-1}} \left( \frac{h(K^*, u) + h((-K)^*, u)}{2} \right)^n du \\
\geq \int_{S^{n-1}} \left( \frac{\frac{1}{h(K^*, u)} + \frac{1}{h((-K)^*, u)}}{2} \right)^n du \\
= \int_{S^{n-1}} \left( \frac{2}{\rho(K, u) + \rho(-K, u)} \right)^n du \\
= \int_{S^{n-1}} \left( \frac{\rho(K, u) + \rho(-K, u)}{2} \right)^{-n} du.
\]
Because \( \rho(\cdot, \cdot) \) satisfies that \( \rho(K_1 + K_2, u) \geq \rho(K_1, u) + \rho(K_2, u) \), we get
\[
\int_{S^{n-1}} \left( \frac{\rho(K, u) + \rho(-K, u)}{2} \right)^{-n} du \geq \int_{S^{n-1}} \rho^{-n} \left( \frac{K + (-K)}{2}, u \right) du.
\]
Since \( K + (-K) = \beta U \),
\[
\int_{S^{n-1}} \rho^{-n} \left( \frac{K + (-K)}{2}, u \right) du = \int_{S^{n-1}} h^n \left( \left( \frac{K + (-K)}{2} \right)^*, u \right) du \\
= \left( \frac{\beta}{2} \right)^n \int_{S^{n-1}} h^n(U^*, u) du \\
= \left( \frac{\beta}{2} \right)^n \int_{S^{n-1}} \rho^n(\rho(U^*, u)) du.
\]
We know that \( I \) is the polar dual of \( U \). So
\[
\int_{S^{n-1}} \rho^n(\rho(U^*, u)) du = \int_{S^{n-1}} \rho^n(\rho(U^*, u)) du \\
= nV_e(PI).
\]
This proves the result. \( \square \)

**Corollary 1.** If \( U \) is the Euclidean unit circle with the center \( o \), then
\[
\Gamma \left( \frac{n}{2} + 1 \right) V(OK) \geq \left( \frac{\beta \sqrt{2^n}}{2} \right)^n.
\]

**Proof.** Trivial from Theorem 1. \( \square \)
4. Planar bodies

Let $K$ be a plane convex body in $M^2(U)$. Let $u = u(\xi) = (\cos \xi, \sin \xi)$ be in $S^1$, the unit circle with center o. Let $T(u,q,p)$ be a supporting tangent line at $q = q(\xi)$ on $\partial K$, the boundary of $K$, which is perpendicular to the ray $R(u)$ emanating from $o$ and passing through $u$. And $T(u,q,p)$ meets $R(u)$ at the point $p = p(\xi) \in M^2$. Then the pedal body $PK$ is a trace of points $p$. We know that the circle containing the points $o,p,q$ has common tangent $T(PK,p)$ at $p$ with $PK$ [4]. From this we can compute the angle $\angle(T(PK,p),e_1)$ between $T(PK,p)$ and the $x_1$-axis, where $e_1 = (1,0)$. We get

$$\angle(T(PK,p),e_1) = 2\xi - \alpha - \frac{\pi}{2},$$

where $\alpha = \angle(q,o,e_1)$. Thus the Minkowski length $L(PK)$ of $PK$ is

$$L(PK) = \int_{-\pi}^{\pi} \sqrt{\rho^2(PK,\xi) + \rho^2(PK,\xi)\rho^{-1} \left(U, 2\xi - \alpha - \frac{\pi}{2}\right)} \, d\xi,$$

where $'$ denotes $\frac{d}{d\xi}$. We call the Minkowski space with the indicatrix $I$, the dual space of $M^n(U)$ and denote it by $M^n$. Now we have the following.

**Theorem 2.** Let $K$ be a body of constant relative breadth $\beta$ in $M^2$. Then

$$V(PK)V(PK^*) \geq \frac{\beta^2V(PI)\left[r_i^2L^2(PK) + 4\pi V(K)\right]}{32\pi},$$

where $r_i$ is the inradius of $K$, the maximal radius of circles contained in $K$.

**Proof.** We know that

$$L(PK) = \int_{-\pi}^{\pi} \sqrt{h^2(K,\xi) + h'^2(K,\xi)\rho^{-1} \left(U, 2\xi - \alpha - \frac{\pi}{2}\right)} \, d\xi$$

$$\leq \sqrt{\int_{-\pi}^{\pi} [h^2(K,\xi) + h'^2(K,\xi)]d\xi \int_{-\pi}^{\pi} \rho^{-2} \left(U, 2\xi - \alpha - \frac{\pi}{2}\right) \, d\xi},$$

(2)

Here (2) is less than $\frac{2\pi}{r_i}$ times the square root of

$$\beta^2 \int_{0}^{\pi} \left[\rho^{-2}(U,\xi) + \rho'^{-2}(U,\xi)\right]d\xi + 2V(K,-K) - 4\int_{0}^{\pi} h(K,\xi)h(K,\xi + \pi)d\xi.$$
Here $V(\cdot, \cdot)$ denotes the mixed volume. Since
\[
\int_0^\pi h(K, \xi)h(K, \xi + \pi) d\xi = \frac{\beta^2}{2} \int_0^\pi \rho^2(U, \xi) d\xi - \frac{1}{2} \int_{-\pi}^\pi h^2(K, \xi) d\xi
\]
we have
\[
L(PK) \leq \frac{1}{r_i} \sqrt{4\pi(2V(PK) - V(K))}.
\]
This completes the proof. \(\square\)

References


