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ON ACTION SPECTRUM BUNDLE

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ABSTRACT. In this paper when (M, ω) is a compact weakly exact symplectic manifold with nonempty boundary satisfying $c_1|_{\pi_2(M)} = 0$, we construct an *action spectrum bundle* over the group of Hamiltonian diffeomorphisms of the manifold M generated by the time-dependent Hamiltonian vector fields, whose fibre is nowhere dense and invariant under symplectic conjugation.

1. Introduction

Let (M, ω) be a $2n$-dimensional symplectic manifold. Then we can associate to a smooth Hamiltonian function $H : M \to \mathbb{R}$ the Hamiltonian vector field X_H on M, which is defined by $\omega(X_H, \cdot) = -dH(\cdot)$. The vector field X_H generates the Hamiltonian flow φ_H^t via $\frac{d}{dt}\varphi_H^t = X_H \circ \varphi_H^t$, $\varphi_H^0 = id$. A T-periodic solution $x(t)$ of the Hamiltonian equation $\dot{x} = X_H(x)$ on M is a solution defined by $x(t) = \varphi_H^t(x(0))$ satisfying the boundary condition $x(T) = x(0)$ for some $T > 0$.

Now we consider a smooth time-periodic Hamiltonian function $H : S^1 \times M \to \mathbb{R}$ and the time-dependent Hamiltonian differential equation

$$\dot{x}(t) = X_H(t, x(t)). \tag{1.1}$$

Throughout we identify $S^1 = \mathbb{R}/\mathbb{Z}$. We denote by $\mathcal{D}$ the group of Hamiltonian diffeomorphisms of M generated by the time-dependent Hamiltonian vector fields X_H. Its Lie algebra $\mathcal{A}$ is the space of Hamiltonian vector fields, which is identified with the space of all smooth Hamiltonian functions on $S^1 \times M$ satisfying the following normalization condition, where in the case of a compact manifold M the function is only

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unique up to an additive constant. A function H is said to satisfy a normalization condition if H is compactly supported when M is open, $\int_M H(t,x)\omega^n = 0$ for every $t \in S^1$ when M is closed. For $H \in \mathcal{A}$, its norm is defined by

$$\|H\| := \int_0^1 [\sup_x H(t,x) - \inf_x H(t,x)]dt.$$

Denote by $\mathcal{H}$ the space of smooth time-periodic Hamiltonian functions $H : S^1 \times M \to \mathbb{R}$ which satisfy the normalization condition. Given a map $\varphi \in \mathcal{D}$, its energy $E(\varphi)$ is defined by

$$E(\varphi) := \inf\{\|H\| \,|\varphi = \varphi_H,\ H \in \mathcal{H}\}. \tag{1.2}$$

This defines a distinguished bi-invariant metric d, which is called Hofer's metric, by $d(\varphi,\psi) := E(\varphi^{-1}\psi)$. Note that

$$d(\varphi,\psi) = \inf\{\|H - K\| \mid H \text{ generates } \psi \text{ and } K \text{ generates } \varphi\}. \tag{1.3}$$

Assume that (M,ω) is a compact weakly exact symplectic manifold with nonempty boundary which satisfies $c_1|_{\pi_2(M)} = 0$. The weakly exactness of M means $[\omega]|_{\pi_2(M)} = 0$, i.e., the integral of the symplectic structure vanishes over every sphere

$$\int_{S^2} u^*\omega \;=\; 0, \qquad u : S^2 \to M. \tag{1.4}$$

We choose an almost complex structure J on M which is compatible with ω in the sense that

$$g(\xi,\eta) \;=\; \omega(\xi, J(x)\eta), \qquad \xi,\ \eta\ \in T_xM, \tag{1.5}$$

defines a Riemannian metric on M. Then $\mathcal{H}$ is the space of all smooth Hamiltonian functions $H : S^1 \times M \to \mathbb{R}$, periodic in time, $H(t,x) = H(t+1,x)$ for $t \in \mathbb{R}$ and $x \in M$ having compact support in $S^1 \times (M \setminus \partial M)$, and $\mathcal{D} = \{\varphi_H \mid H \in \mathcal{H}\}$.

For $H \in \mathcal{H}$, we denote the set of fixed points of the associated Hamiltonian diffeomorphism φ_H by

$$\mathrm{Fix}(\varphi_H) = \{x \in M \mid \varphi_H(x) = x\}.$$

On $\mathbb{R}^{2n}$ there exists the solution $x(t) = \varphi_H^t(x_0)$ for $0 \le t \le 1$, $x_0 \in \mathrm{Fix}(\varphi_H)$, of the Hamiltonian equation. It is contractible 1-periodic in $\mathbb{R}^{2n}$ and its action is defined by

$$A(x_0,H) = \int_0^1 \frac{1}{2}\langle -J\dot{x}(t), x(t)\rangle dt - \int_0^1 H(t,x(t))dt. \tag{1.6}$$

But one cannot say that there exist the contractible 1-periodic solutions of the Hamiltonian equation on any compact symplectic manifold. In

Section 2 by a relation between Floer homology and Maslov index we conclude the existence of contractible 1-periodic solutions of the Hamiltonian equation under some assumptions. In Section 3 we define the action of the contractible 1-periodic solution of the Hamiltonian equation (1.1) and construct the action spectrum bundle whose fibre is a nowhere dense subset of $\mathbb{R}$, and which is invariant under symplectic conjugation.

2. Floer homology and Maslov index

Let (M, ω) be a compact symplectic manifold with nonempty boundary which satisfies $[\omega]|_{\pi_2(M)} = 0$ and $c_1|_{\pi_2(M)} = 0$. Let $\mathcal{B}_1$ denote the Hilbert manifold of contractible $W^{1,2}$ loops $x : S^1 \to M$. Now to a periodic function $H \in \mathcal{H}$, we associate the action functional $\Phi_H : \mathcal{B}_1 \to \mathbb{R}$ defined by

$$\Phi_H(x) = -\int_D \bar{x}^*\omega + \int_0^1 H(t, x(t))\, dt. \tag{2.1}$$

Here $D \subset \mathbb{C}$ denotes the closed unit disc and $\overline{x} : D \to M$ is a smooth function extending x, $\overline{x}|_{\partial D} = x$. Such an extension exists since x is assumed to be contractible, and it follows from (1.4) that the first integral on the right hand side of (2.1) does not depend on the choice of the extension and hence depends only on the loop x. Computing the derivative of Φ_H at $x \in \mathcal{B}_1$ in the direction of $\xi \in T_x\mathcal{B}_1$, we find that

$$\begin{aligned} d\Phi_H(x)\xi &= \int_0^1 \{\omega(\dot{x}, \xi) + dH(t, x)\xi\} dt \\ &= \int_0^1 g(J(x)\dot{x} + \nabla H(t, x), \xi) dt. \end{aligned}$$

Consequently, $d\Phi_H(x)\xi = 0$ for all $\xi \in T_x\mathcal{B}_1$ if and only if the loop x satisfies that

$$\nabla\Phi_H(x) = J(x)\dot{x} + \nabla H(t, x) = 0, \tag{2.2}$$

and hence $\Phi_H(x)$ is a critical value of Φ_H if and only if x is a solution of the equation (1.1). The critical points of Φ_H are the zeros of the gradient function, $x \mapsto \nabla\Phi_H(x)$. Now using a dynamical approach, we interpret the critical points of Φ_H as the equilibrium points of the gradient equation given by

$$\dot{x} = -\nabla\, \Phi_H(x). \tag{2.3}$$

In order to proceed to familiar Cauchy-Riemann operator, by using (2.2) we view a solution x of (2.3) as a $W^{1,p}$ map $u : \mathbb{R} \times S^1 \to M$,

$u(s,t) = u(s,t+1)$, $p > 2$, which solves the following perturbation of a Cauchy-Riemann equation

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(t,u) = 0. \tag{2.4}$$

By (2.3) we get

$$\begin{aligned} \tfrac{d}{ds}\Phi_H(u(s)) &= d\Phi_H(u(s))(\tfrac{d}{ds}u(s)) \\ &= g(\nabla\Phi_H(u(s)), \tfrac{d}{ds}u(s)) \\ &= g(\nabla\Phi_H(u(s)), -\nabla\Phi_H(u(s))) \\ &= -\|\nabla\Phi_H(u(s))\|^2_{J,L^2} \end{aligned} \tag{2.5}$$

which implies that the map $s \mapsto \Phi_H(u(s))$ is decreasing. For simplicity, we use $\|\cdot\| = \|\cdot\|_{J,L^2}$ in the following.

Suppose that $(\Phi_H(u(s)))_{s\in\mathbb{R}}$ is bounded in $\mathbb{R}$, where $u \in W^{1,p}(\mathbb{R} \times S^1, M)$ satisfies (2.4). Then, using (2.5) we obtain

$$\int_{-\infty}^{\infty} \|\nabla\Phi_H(u(s))\|^2\, ds = -\int_{-\infty}^{\infty} \frac{d}{ds}\Phi_H(u(s)) < \infty,$$

and hence there exists a sequence $\{s_n\}$ in $\mathbb{R}$ with $s_n \to \infty$ such that $\|\nabla\Phi_H(u(s_n))\| \to 0$ and $\|\Phi_H(u(s_n))\| \to \alpha$ for some $\alpha \in \mathbb{R}$. Set $x_n = u(s_n)$. Then its limit, say x_∞, is a critical point of Φ_H and $\alpha = \Phi_H(x_\infty)$ is a critical value of Φ_H. Thus in order to guarantee the existence of critical points of Φ_H, from now we assume that $(\Phi_H(u(s)))_{s\in\mathbb{R}}$ is bounded.

DEFINITION 2.1. A periodic solution $x(t)$ of the Hamiltonian equation is said to be nondegenerate if for $\varphi = \varphi_H \in \mathcal{D}$

$$\det(1 - d\varphi(x(0))) \neq 0. \tag{2.6}$$

We shall assume that all the contractible 1-periodic solutions $x(t)$ of the equation (1.1) are nondegenerate, and denote by $\mathcal{P}_H$ the set of these periodic solutions of the Hamiltonian equation

$$\mathcal{P}_H = \{x \in \mathcal{B}_1 | x \simeq 0, x(t) = x(t+1), x \text{ satisfies (1.1) and (2.6)}\}.$$

This is a finite set since M is compact and the nondegenerate 1-periodic solutions are isolated in M. We denote by $\mathcal{M} = \mathcal{M}(H,J)$ the set of solutions of (2.4) which satisfies the boundedness of $(\Phi_H(u(s)))_{s\in\mathbb{R}}$. In [1, 3] it was shown that the set $\mathcal{M}$ is compact in the topology of uniform convergence with all derivatives on compact sets, provided that M is compact and $[\omega]|_{\pi_2(M)} = 0$, and that for every bounded orbit u, there exists a pair x, $y \in \mathcal{P}_H$ of periodic solutions such that u is a connecting orbit from x to y, i.e.,

$$\lim_{s \longrightarrow -\infty} u(s,t) = x(t), \quad \lim_{s \longrightarrow +\infty} u(s,t) = y(t). \tag{2.7}$$

Given two 1-periodic solutions x, $y \in \mathcal{P}_H$ we denote by $\mathcal{M}(x,y) = \mathcal{M}(x,y;\, H,J)$ the set of bounded orbits which are the solutions of (2.4) satisfying the asymptotic boundary condition (2.7). Then

$$\mathcal{M} \;=\; \mathcal{M}(H,J) \;=\; \cup_{x,y \in \mathcal{P}_H} \mathcal{M}(x,y).$$

Now we denote by $\mathcal{B}_2$ the Banach manifold of $W^{1,p}$ maps $u : \mathbb{R} \times S^1 \longrightarrow M$ which satisfy the condition (2.7) in the $W^{1,p}$ sense with $p > 2$ (see [2]). Consider the bundle $\mathcal{T} \longrightarrow \mathcal{B}_2$ whose fibre at $u \in \mathcal{B}_2$ is the Banach space of L^p-vector fields along u. Let the section $s : \mathcal{B}_2 \longrightarrow \mathcal{T}$ be defined by $s(u) = \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(t,u)$. Then the elements of $\mathcal{M}(x,y)$ are to be found as the zeros of the section s of the Banach space bundle $\mathcal{T} \longrightarrow \mathcal{B}_2$ which satisfy the condition (2.7) in the $W^{1,p}$ sense. The linearization of $s(u)$ in the direction $\xi \in W^{1,p}(u^*TM)$ along $u \in \mathcal{M}(x,y)$ defines a linear first-order differential operator

$$D_u : W^{1,p}(u^*TM) \longrightarrow L^p(u^*TM)$$

which is given by

$$D_u\xi := \nabla_s \xi + J(u) \nabla_t \xi + \nabla_\xi J(u) \frac{\partial u}{\partial t} + \nabla_\xi \nabla H(t,u), \tag{2.8}$$

where ∇_s, ∇_t and ∇_ξ are covariant derivatives with respect to the metric (1.5).

DEFINITION 2.2. A pair (H,J) with an almost complex structure J satisfying (1.5) is said to be regular, if every contractible 1-periodic solution of the equation (1.1) is nondegenerate and D_u is onto for every $u \in \mathcal{M}$.

We introduce two lemmas needed to define the Maslov index on $\mathcal{P}_H$.

LEMMA 2.3. ([4]) *For any smooth map $\phi : D \longrightarrow M$ there exists a trivialization*

$$D \times \mathbb{R}^{2n} \xrightarrow{\ \Phi\ } \phi^*TM \;:\; (z,\zeta) \mapsto \Phi(z)\zeta$$

such that

$$J\Phi = \Phi J_0, \;\; \Phi^*\omega = \omega_0, \;\; g(\Phi\zeta, \Phi\zeta') = \zeta^T \zeta'.$$

Any two such trivializations are homotopic.

This lemma shows that if for $x \in \mathcal{P}_H$ we choose a smooth function $\bar{x} : D \longrightarrow M$ such that $\bar{x}(e^{2\pi it}) = x(t)$, then there exists a unitary trivialization of $\bar{x}^*TM$ and this gives rise to a unitary trivialization

$$\Phi_x(t) = \Phi(e^{2\pi it}) \; : \; \mathbb{R}^{2n} \longrightarrow T_{x(t)}M$$

of x^*TM such that

$$\Phi_x(t+1) \;=\; \Phi_x(t).$$

LEMMA 2.4. ([4]) *If the first Chern class c_1 vanishes over $\pi_2(M)$ then the homotopy class of Φ_x is independent of the choice of the extension $\bar{x} : D \to M$.*

Given a trivialization Φ_x of x^*TM as above we consider the linearized flow along $x(t)$ and define the loop

$$\Psi_x(t) = \Phi_x(t)^{-1} d\varphi^t(x(0))\Phi_x(0), \;\; 0 \le t \le 1$$

which is contained in $\mathrm{Sp}(2n;\mathbb{R})$ for every t. We set $\Psi = \Psi_x(t)$ for notational conveniency and define the map $\rho : \mathrm{Sp}(2n;\mathbb{R}) \to S^1$ by $\rho(\Psi) = \det(X(t) + iY(t))$, where

$$Q = \begin{pmatrix} X(t) & -Y(t) \\ Y(t) & X(t) \end{pmatrix} = (\Psi\Psi^T)^{-1/2}\Psi \in \mathrm{Sp}(2n;\mathbb{R}) \cap O(2n) = U(n)$$

is the orthogonal part of Ψ in the polar decomposition $\Psi = PQ$. Let $\tau : \mathbb{R}/\mathbb{Z} \longrightarrow \mathrm{Sp}(2n;\mathbb{R})$ be a map which represents a loop of symplectic matrices $\Psi_x(t)$. Then the Maslov index $\mu(H,x)$ of the loop $\Psi_x(t)$ is defined by the degree of the composition $\rho \circ \tau \; : \; \mathbb{R}/\mathbb{Z} \longrightarrow S^1$:

$$\mu(H,x) \;=\; \deg(\rho \circ \tau).$$

In other words,

$$\mu(H,x) \;=\; \alpha(1) - \alpha(0)$$

where $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ is a lift of $\rho \circ \tau$:

$$\det(X(t) + iY(t)) \;=\; e^{2\pi i\alpha(t)}.$$

Define a map $\mu_H : \mathcal{P}_H \to \mathbb{Z}$ by $\mu_H(x) = \mu(H,x)$. In [4], it was shown that the operator D_u which was given by (2.8) is a Fredholm operator and

$$\mathrm{Index} D_u = \mu_H(x) - \mu_H(y),$$

and hence for every regular pair (H,J) and every pair of contractible 1-periodic solutions $x,\ y \in \mathcal{P}_H$ the space $\mathcal{M}(x,y) = \mathcal{M}(x,y;H,J)$ is a manifold whose local dimension near $u \in \mathcal{M}(x,y)$ is $\dim \mathcal{M}(x,y) = \mu_H(x) - \mu_H(y)$.

Now we set

$$\begin{aligned} C &= C(M,H) = \oplus_{k\in\mathbb{Z}} C_k \\ C_k &= C_k(M,H) = \oplus_{x\in\mathcal{P}_H,\mu_H(x)=k}\mathbb{Z}_2\langle x\rangle. \end{aligned}$$

If $x, y \in \mathcal{P}_H$ satisfy $\mu_H(x) - \mu_H(y) = 1$, then $\mathcal{M}(x,y)$ is a compact one dimensional manifold. Hence if we denote by $\hat{\mathcal{M}}(x,y)$ the quotient of $\mathcal{M}(x,y)$ by the time s shift, then $\hat{\mathcal{M}}(x,y)$ has only finitely many orbits u, which implies that $\mathcal{M}(x,y)$ has only finitely many components. The boundary operator $\partial_k : C_k \to C_{k-1}$ is defined by the formula

$$\partial_k x = \sum_{\{y\in\mathcal{P}_H|\mu_H(y)=k-1\}} \langle\partial x, y\rangle y,$$

for $x \in \mathcal{P}_H$ satisfying $\mu_H(x) = k$, where $\langle\partial x, y\rangle$ is the number of components of $\mathcal{M}(x,y)$ counted modulo 2. In [1] Floer proved that $\partial \circ \partial = 0$, so that (C,∂) defines a chain complex. Its homology

$$HF_*(M,H,J) := \frac{\ker \partial}{\mathrm{im}\partial}$$

is called the Floer homology for the regular pair (H,J). In [4] it was shown that it is independent of the regular pair (H,J), and that there exists a natural isomorphism between the Floer homology of the pair (H,J) and the singular homology of M with $\mathbb{Z}_2$ coefficients

$$HF_k(M,H,J) \cong H_{n+k}(M,\mathbb{Z}_2), \quad -n \le k \le n.$$

Until now we have found that the index formula for the Fredholm operator plays an important role in Floer homology and the formula involves the Maslov index of nondegenerate contractible 1-periodic solutions of the Hamiltonian equation (1.1). Moreover, Floer's connecting orbits of those solutions can serve as a model for the homology of the underlying manifold, and hence $HF_*(M,H,J) \neq 0$, $\mathcal{M} \neq \emptyset$, and $\mathcal{P}_H \neq \emptyset$, which imply the existence of contractible 1-periodic solutions of the Hamiltonian equation.

3. The action spectrum of a Hamiltonian diffeomorphism

Let (M,ω) be a compact weakly exact symplectic manifold with nonempty boundary which satisfies $c_1|_{\pi_2(M)} = 0$. Recall from Section 2 that there exist contractible 1-periodic solutions of Hamiltonian equation (1.1) provided that every 1-periodic solution is nondegenerate. Define the action $A(x_0, H) \in \mathbb{R}$ of the contractible 1-periodic solution

$x(t) = \varphi_H^t(x_0)$ for $0 \leq t \leq 1$, $x_0 \in \mathrm{Fix}(\varphi_H)$, $x \in \mathcal{B}_1$, of the Hamiltonian equation (1.1) by

$$A(x_0, H) = -\int_D \overline{x}^*\omega + \int_0^1 H(t, x(t))dt, \tag{3.1}$$

where $D \subset \mathbb{C}$ denotes the closed unit disc and $\overline{x} : D \to M$ is a smooth function such that $\overline{x}|_{\partial D} = x$.

LEMMA 3.1. *If $H, K \in \mathcal{H}$ generate the same map, i.e., $\varphi_H = \varphi_K$ then*

$$A(x_0, H) = A(x_0, K)$$

for every $x_0 \in Fix(\varphi_H) = Fix(\varphi_K)$.

Proof. We construct, for every $x \in M$ and contractible Hamiltonian flows φ_H^t and φ_K^t generated by H and K, respectively, a contractible loop $x(t) = \psi^t(x)$, for $t \in [0,2]$, as follows:

$$x(t) = \psi^t(x) = \begin{cases} \varphi_H^t(x) & \text{for} \quad t \in [0,1] \\ \varphi_K^{2-t}(x) & \text{for} \quad t \in [1,2]. \end{cases}$$

Then $x(t)$ can be extended to a smooth function $\overline{x} : D \to M$ satisfying $\overline{x}|_{\partial D} = x$. We set, for $x(t) = \psi^t(x)$,

$$\Delta(x) = -\int_D \overline{x}^*\omega \ + \ \int_0^1 H(t, x(t))dt \ + \ \int_1^2 K(2-t, x(t))dt.$$

This map $\Delta : M \to \mathbb{R}$ is smooth and by differentiation in x we get

$$\begin{aligned} \Delta'(x)\xi &= \textstyle\int_0^2 \omega(\dot{x}, \xi)dt + \int_0^1 dH(t,x)\xi dt + \int_1^2 dK(2-t, x)\xi dt \\ &= \textstyle\int_0^2 g(J(x)\dot{x}, \xi)dt + \int_0^1 g(\nabla H(t,x), \xi)dt + \int_1^2 g(\nabla K(2-t,x), \xi)dt \\ &= \textstyle\int_0^1 g(J(x)\dot{x} + \nabla H(t,x), \xi)dt + \int_1^2 g(J(x)\dot{x} + \nabla K(2-t,x), \xi)dt \\ &= 0, \end{aligned}$$

since $x(t) = \psi^t(x)$ is a solution of the Hamiltonian equation associated to $H(t, x(t))$ for $t \in [0,1]$ and to $K(2-t, x(t))$ for $t \in [1,2]$. This shows that $\Delta : M \to \mathbb{R}$ is a constant. Now let $U_1, U_2 \subset S^1 \times (M \setminus \partial M)$ be open sets satisfying $\mathrm{supp}(H) \subset U_1$ and $\mathrm{supp}(K) \subset U_2$, respectively, and let $U \subset S^1 \times (M \setminus \partial M)$ be a large open set containing U_1 and U_2 such that $x(t) \equiv x$ for $t \in [0,2]$ and $x \in M \setminus U$. Then for $x \in M \setminus U$, $\Delta(x) = 0$, and hence $\Delta \equiv 0$. Therefore, if $x_0 \in \mathrm{Fix}(\varphi_H) = \mathrm{Fix}(\varphi_K)$, then $A(x_0, H) - A(x_0, K) = \Delta(x_0) = 0$. □

This lemma shows that $A(x_0, H)$ depends only on the fixed point x_0 and the map φ_H, and is independent of the choice of the function H

generating the map, and hence we can associate with a fixed point x_0 of the map $\varphi = \varphi_H \in \mathcal{D}$ the action $A(x_0, \varphi)$ via

$$A(x_0, \varphi) = A(x_0, H) \quad \text{if} \quad \varphi = \varphi_H. \tag{3.2}$$

DEFINITION 3.2. The action spectrum of $\varphi \in \mathcal{D}$ is the set $\sigma(\varphi) \subset \mathbb{R}$ defined by

$$\sigma(\varphi) = \{A(x, \varphi) \mid x \in \mathrm{Fix}(\varphi)\}.$$

PROPOSITION 3.3. *The action spectrum $\sigma(\varphi)$ of $\varphi \in \mathcal{D}$ is nowhere dense.*

Proof. Assume $\varphi = \varphi_H$. We prove the nowhere density of $\sigma(\varphi)$ by constructing a smooth function on M whose critical values contain $\sigma(\varphi)$ and then by using Sard's theorem. As mentioned before, a solution $x(t) = \varphi^t(x_0) \in \mathcal{B}_1$ of the Hamiltonian equation is a critical point of the action functional Φ_H. Hence $A(x_0, \varphi) = \Phi_H(x)$ with $x(t) = \varphi^t(x_0)$ and $x_0 \in \mathrm{Fix}(\varphi)$, is a critical value of Φ_H and $\sigma(\varphi)$ is a set of critical values of Φ_H. Define a smooth function $\psi : S^1 \times M \to M$ by

$$\psi(t, x) = \begin{cases} \varphi^t(x) & \text{if } \ x \in \mathrm{Fix}(\varphi) \\ x & \text{if } \ x \in M \setminus \mathrm{Fix}(\varphi). \end{cases}$$

Then for every $x \in M$, the map $t \mapsto \psi(t, x)$ represents a contractible loop contained in $\mathcal{B}_1$, and we can define the smooth function $\Psi : M \to \mathcal{B}_1$ by $\Psi(x)(t) = \psi(t, x)$. Consider the composition $\Phi_H \circ \Psi : M \to \mathbb{R}$. Since $A(x_0, \varphi) = \Phi_H(x)$ with $x(t) = \varphi^t(x_0)$ and $x_0 \in \mathrm{Fix}(\varphi)$, is a critical value of Φ_H, $\Phi_H \circ \Psi(x_0) = \Phi_H(\varphi^t(x_0)) = A(x_0, \varphi)$ shows that $\Psi(x_0)$ and $x_0 \in \mathrm{Fix}(\varphi)$ are critical points of Φ_H and $\Phi_H \circ \Psi$, respectively, and hence $A(x_0, \varphi) = \Phi_H \circ \Psi(x_0)$ is a critical value of $\Phi_H \circ \Psi$. This map $\Phi_H \circ \Psi$ is the desired smooth map on M whose critical values contain $\sigma(\varphi)$. By Sard's theorem, the set of critical values of $\Phi_H \circ \Psi$ is nowhere dense, and hence $\sigma(\varphi)$ is nowhere dense. □

Denote by $\mathcal{G}$ the group of conformally symplectic diffeomorphisms, i.e., $g^*\omega = \alpha\omega$ for all $g \in \mathcal{G}$ and for some constant $\alpha = \alpha(g) \in (0, \infty)$. Note that if $\varphi = \varphi_H \in \mathcal{D}$, then

$$g \circ \varphi_H^t \circ g^{-1} = \varphi_{\alpha H_g}^t, \tag{3.3}$$

where $H_g(t, x) = H(t, g^{-1}(x))$.

The following Proposition 3.4 shows that the action spectrum of $\varphi \in \mathcal{D}$ is invariant under symplectic conjugation.

PROPOSITION 3.4. *If $\varphi \in \mathcal{D}$ and $g \in \mathcal{G}$, then*

$$A(g(x), g\varphi g^{-1}) = \alpha A(x, \varphi),$$

and hence $\sigma(g\varphi g^{-1}) = \alpha\sigma(\varphi)$, where α is the conformally symplectic constant of g. In particular, if g is a symplectic diffeomorphism, i.e., $g^*\omega = \omega$, then $\sigma(g\varphi g^{-1}) = \sigma(\varphi)$.

Proof. Assume that $\varphi = \varphi_H$ and $x \in \mathrm{Fix}(\varphi)$. Then $g(x) \in \mathrm{Fix}(g\varphi g^{-1})$ for $g \in \mathcal{G}$ since $g\varphi g^{-1}(g(x)) = g\varphi(x) = g(x)$. From (3.2) and (3.3) we have to show that $A(g(x), \alpha H_g) = \alpha A(x, H)$. Set

$$\begin{array}{rcl} x(t) & = & \varphi_H^t(x) \\ y(t) & = & g(x(t)) = g \circ \varphi_H^t \circ g^{-1}(g(x)) = \varphi_{\alpha H_g}^t(g(x)). \end{array}$$

Then, by using the weakly exactness of M (1.4), we find that

$$\begin{array}{rcl} A(g(x), \alpha H_g) & = & -\int_D \bar{y}^*\omega + \int_0^1 \alpha H_g(t, y(t))dt \\ & = & -\int_D \bar{x}^* g^*\omega + \alpha \int_0^1 H(t, g^{-1}y(t))dt \\ & = & -\alpha \int_D \bar{x}^*\omega + \alpha \int_0^1 H(t, x(t))dt \\ & = & \alpha[-\int_D \bar{x}^*\omega + \int_0^1 H(t, x(t))dt] \\ & = & \alpha A(x, H). \end{array}$$

□

The constant solution $x(t) = \varphi^t(x_0) \equiv x_0$, for $x_0 \in \mathrm{Fix}(\varphi)$ with $x_0 \in M \backslash \mathrm{supp(H)}$, $\varphi_H = \varphi$, of the Hamiltonian equation satisfies $A(x_0, \varphi) = 0$, and hence $0 \in \sigma(\varphi)$ for every $\varphi \in \mathcal{D}$. From this fact and Propositions 3.3 and 3.4 we get the following:

THEOREM 3.5. *Let (M, ω) be a compact weakly exact symplectic manifold with nonempty boundary which satisfy $c_1|_{\pi_2(M)} = 0$. Assume that every contractible 1-periodic solution of the Hamiltonian equation is nondegenerate. Then there exists an action spectrum bundle $\mathcal{B} \to \mathcal{D}$ defined by*

$$\mathcal{B} = \cup_{\varphi \in \mathcal{D}} \{\varphi\} \times \sigma(\varphi),$$

equipped with the metric induced from $(\mathcal{D}, d) \times \mathbb{R}$, where d denotes the Hofer's metric. Every fibre is a nowhere dense subset of $\mathbb{R}$ which is invariant under symplectic conjugation and the bundle $\mathcal{B}$ contains a trivial continuous section.

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