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A GENERALIZATION OF THE NILPOTENT SPACE AND ITS APPLICATION

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ABSTRACT. For the generalized nilpotent spaces, e.g. the locally nilpotent space, the residually locally nilpotent space and the space satisfying the condition (T^*) or (T^{**}), we find the pullback property of them. Furthermore we investigate some fiber properties of the space satisfying the condition (T^*) or (T^{**}), especially locally nilpotent space.

1. Introduction

In this paper we find the pullback property of the locally nilpotent space and furthermore we find the fiber property of the space satisfying the condition (T^{**}). Especially we focus on studying the fiber property of the locally nilpotent space.

In this paper we work in the category of topological spaces having the homotopy type of connected CW-complexes with the base point and continuous maps preserving the base point. Now we denote the category by T.

2. Preliminaries

In this section we define the residually locally nilpotent space and investigate the finite product property of the residually locally nilpotent spaces. Furthermore we study the pullback and fiber property of the locally nilpotent space.

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The category of nilpotent spaces [1, 4] and continuous maps is denoted by T_N.

Now we recall the concept of a locally nilpotent space as an extension of a nilpotent space as follows [4]: a space $X(\in T)$ is said to be a locally nilpotent space if $\pi_1(X)$ is a locally nilpotent group and, in addition, there exists a nilpotent $\pi_1(X)$ - action on $\pi_n(X)$ for all $n \geq 2$ [1].

And we adopt the notation T_{LN} for the category of locally nilpotent spaces and continuous maps. Obviously, the category T_N is a full subcategory of T_{LN}. We know that the finite product space of locally nilpotent spaces is also locally nilpotent.

DEFINITION 2.1. We say that a space X satisfies the condition (T^*) if for all $g, t \in \pi_1(X)$ either $g[g, \pi_1(X)] = t[t, \pi_1(X)]$ or $g[g, \pi_1(X)] \bigcap t[t, \pi_1(X)] = \phi$. Furthermore we say that X satisfies the condition (T^{**}) if for all $g(\neq 1) \in \pi_1(X)$, $g \notin [g, \pi_1(X)]$ [4].

LEMMA 2.2 ([4]). *For $X \in T_{LN}$, X satisfies the condition (T^*).*

Now we need the residual property in order to make new category which we extend the category of nilpotent spaces.

We recall that a group G has the property χ residually if for every element $g(\neq 1) \in G$, there is a nontrivial normal subgroup N of G such that $g \notin N$ and G/N has the property χ [6].

DEFINITION 2.3. X is called a residually locally nilpotent space, if

(1) $\pi_1(X)$ is residually locally nilpotent, and
(2) there is a residually nilpotent $\pi_1(X)$-action on $\pi_n(X)$ for $n \geq 2$.

The above condition (2) means that for any element $g(\neq 1) \in \pi_1(X)$ and the selected nontrivial normal subgroup $(g \notin)N \subset \pi_1(X)$, $\pi_1(X)/N$ acts nilpotently on $\pi_n(X)$. Let T_{RLN} be the category of residually locally nilpotent spaces and continuous maps.

Furthermore the category T_{RLN} has a finite product property as the following.

THEOREM 2.4. *Let M be a finite set of indices and $\{X_\alpha | \alpha \in M$: finite$\}$. If $X_\alpha \in T_{RLN}$ for any α then $\prod_{\alpha \in M} X_\alpha \in T_{RLN}$.*

Proof. It is enough to show that for any $X_\alpha, X_\beta \in \{X_\alpha | \alpha \in M :$ finite$\}$ such that $X_\alpha, X_\beta \in T_{RLN}$, $X_\alpha \times X_\beta$ is a residually locally nilpotent space.

Since X_α and X_β are residually locally nilpotent spaces, for any $g_\alpha(\neq 1) \in \pi_1(X_\alpha)$ and any $g_\beta(\neq 1) \in \pi_1(X_\beta)$ there are nontrivial normal subgroups $(g_\alpha \notin)H_\alpha \lhd \pi_1(X_\alpha)$ and $(g_\beta \notin)H_\beta \lhd \pi_1(X_\beta)$, respectively such that $\pi_1(X_\alpha)/H_\alpha$ and $\pi_1(X_\beta)/H_\beta$ are locally nilpotent groups. Thus for any $(g_\alpha, g_\beta)(\neq 1) \in \pi_1(X_\alpha) \times \pi_1(X_\beta)$ we have a nontrivial normal subgroup $((g_\alpha, g_\beta) \notin)H_\alpha \times H_\beta$ in $\pi_1(X_\alpha) \times \pi_1(X_\beta)$ such that $\pi_1(X_\alpha)/H_\alpha \times \pi_1(X_\beta)/H_\beta$ is a locally nilpotent group.

Next, for any $g_\alpha(\neq 1) \in \pi_1(X_\alpha)$ and any $g_\beta(\neq 1) \in \pi_1(X_\beta)$ there are subgroups $(g_\alpha \notin)H_\alpha \lhd \pi_1(X_\alpha)$ and $(g_\beta \notin)H_\beta \lhd \pi_1(X_\beta)$ such that $\pi_1(X_\alpha)/H_\alpha$ acts nilpotently on $\pi_n(X_\alpha)$ and $\pi_1(X_\beta)/H_\beta$ acts nilpotently on $\pi_n(X_\beta)$ for $n \geq 2$.

Then we have the finite lower central series in $\pi_n(X_\alpha)$ and $\pi_n(X_\beta)$ under the $\pi_1(X_\alpha)/H_\alpha$ and $\pi_1(X_\beta)/H_\beta$-action with the nilpotent class n, m respectively as followings;

$$\pi_n(X_\alpha) \supset G_2 \supset G_3 \supset \cdots \supset G_j \supset \cdots \supset G_n = \{e\},$$
$$\pi_n(X_\beta) \supset H_2 \supset H_3 \supset \cdots \supset H_i \supset \cdots \supset H_m = \{e\}.$$

Now we make the following new sequence from above;

$$\begin{aligned}\pi_n(X_\alpha) \times \pi_n(X_\beta) &\supset \pi_n(X_\alpha) \times H_2 \supset G_2 \times H_2 \\ &\supset \cdots \supset G_{j-1} \times H_i \supset G_j \times H_i \supset G_j \times H_{i+1} \\ &\supset \cdots \supset G_n \times H_m = \{e\} \times \{e\} \cdots (\sharp_1).\end{aligned}$$

Then the above sequence $(\sharp_1)$ is a finite lower central series of $\pi_n(X_\alpha) \times \pi_n(X_\beta)$ under the componentwise $\pi_1(X_\alpha)/H_\alpha \times \pi_1(X_\beta)/H_\beta$-action. Thus the finite product space $\prod_{\alpha \in M} X_\alpha \in T_{RLN}$. □

REMARK 1. The converse of the Theorem 2.4 does not hold in general. The category T_{LN} is a full subcategory of T_{RLN}. In general $X(\in T_{RLN})$ does not satisfy the condition (T^*).

The residually locally nilpotent space is very important in studying the fiber property in relation to the condition (T^{**}) [see $(\sharp_9)$ from Theorem 4.5].

3. Pullback property of the locally nilpotent space

In this section we make results on a fibration of the space satisfying the condition (T^{**}) and apply them to make the pullback diagram in the category of locally nilpotent spaces.

Let us recall the following:

LEMMA 3.1 ([3]). *Let X be finite. If $\pi_1(X)$ contains a torsion-free normal abelian subgroup $A \neq 1$ which acts nilpotently on $H_n(\tilde{X})$ where $n \geq 0$, then $\chi(X) = 0$.*

LEMMA 3.2 ([4]). *Let X be a space satisfying the condition (T^{**}). If*

(1) *the action $\pi_1(X)$ on $H_n(\tilde{X})$ is nilpotent for $n \geq 0$, and*
(2) *$\pi_1(X)(\neq 1)$ is finite,*

then $X \in T_N$, where $\tilde{X}$ is a universal covering space of X.

We know that the condition (T^*) is stronger than the condition (T^{**}) (see the proof of Theorem 4.4). Thus if the space X satisfies the condition (T^*) with (1) and (2) from Lemma 3.2 then $X \in T_N$.

We recall the Engel group G [6], i.e. the group which has a relation of the form $[\cdots[[x,y],y],\cdots,y] := [x,y,y,\cdots,y] := [x,_n y] = 1$, where $[x,y] := x^{-1}y^{-1}xy$, the commutator of any two elements x, $y(\in G)$. The number of entries of y's in the formula above depends on both x and y. We do not need to bound it uniformly.

When the case $\pi_1(X)$ is finite Engel group, $\pi_1(X)$ is trivially nilpotent. If $\pi_1(X)$ is infinite Engel group with the maximal condition, then $\pi_1(X)$ is also nilpotent.

REMARK 2. Since the Engel group is a kind of generalization of a locally nilpotent group, we can represent our following assertion regardless of the (finite or infinite) cardinality of $\pi_1(X)$ for the space X satisfying the following two conditions with $\pi_1(X)$ non-trivial:

(1) $\pi_1(X)$ is an Engel group with the maximal condition
(2) $\pi_1(X)$ acts nilpotently on $H_n(\tilde{X})$ for $n \geq 0$

then we get $\chi(X) = 0$.

In an exact sequence $A' \to A \to A''$ of Q-modules with Q-actions w', w, w'' for a group Q respectively, w is nilpotent if w', w'' are nilpotent [5, 4.3].$\cdots$ ($\sharp_2$)

When the sequence above is short exact, w is nilpotent if and only if w', w'' are nilpotent [5, 4.3].

We recall that the extension of nilpotent groups does not need to be nilpotent [7]. In the following short exact sequence

$$1 \to N \to H \to G \to 1 \cdots (\sharp_3)$$

with N, G nilpotent, H is not necessary nilpotent.

Furthermore even though groups N and G are locally nilpotent H does not need to be locally nilpotent.

However, a locally nilpotent group has a hereditary property, i.e. if the group G is locally nilpotent, so is the subgroup H. Furthermore a factor group of G is also locally nilpotent but the converse need not be true [7].

In the following pullback diagram ♣:

$$\begin{array}{ccc} W & \xrightarrow[u_0]{} & X \\ {\scriptstyle v_0}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & B \end{array}$$

suppose that $X, Y \in T_N$ in the diagram ♣ above, then $W \in T_N$ if and only if $\pi_1(W)$ operates nilpotently on $\pi_n(B)$ for $n \geq 2$, via $f \circ u_0$ [5, 7.4].$\cdots$ ($\sharp_4$)

Theorem 3.3. *In the above pullback diagram ♣, suppose that X, $Y \in T_{LN}$ and $\pi_2(B) = 0$. Then the pullback $W \in T_{LN}$ if and only if $\pi_1(W)$ operates nilpotently on $\pi_n(B)$ for $n \geq 2$ via $f \circ u_o$.*

Proof. Let us take the Mayer-Vietoris sequence from the pullback diagram ♣ above:

$$\begin{aligned} \pi_{n+1}(B) &\to \pi_n(W) \to \pi_n(X) \oplus \pi_n(Y) \to \cdots \\ &\to \pi_2(B) \to \pi_1(W) \xrightarrow{p_*} K \to 0 \cdots (\sharp_5) \end{aligned}$$

with $\pi_1(W)$-action where K is the pullback of the following diagram [5, 7.4]:

$$\begin{array}{ccc} & & \pi_1(X) \\ & & \big\downarrow{\scriptstyle \pi_1(f)} \\ \pi_1(Y) & \xrightarrow{\pi_1(g)} & \pi_1(B) \end{array}$$

While, we get $\pi_1(W)/\mathrm{Ker}p_* \cong K$ from ($\sharp_5$). Since $\pi_1(X) \oplus \pi_1(Y)$ is locally nilpotent, the subgroup K of $\pi_1(X) \oplus \pi_1(Y)$ is also locally nilpotent. Since $\pi_1(W)/\mathrm{Ker}p_*$ is locally nilpotent and $\mathrm{Ker}p_*$ is trivial, $\pi_1(W)$ is locally nilpotent. Next, since the action of $\pi_1(W)$ on $\pi_n(B)$ is nilpotent via $f \circ u_0$ for $n \geq 2$, $\pi_1(W)$ acts nilpotently on both $\pi_n(X)$ and $\pi_n(Y)$ from ($\sharp_2$). Thus $\pi_1(W)$ acts nilpotently on $\pi_n(W)$ for $n \geq 2$ from ($\sharp_2$) and ($\sharp_4$). Therefore $W \in T_{LN}$.

Conversely, suppose $W \in T_{LN}$. Then from an application of ($\sharp_2$) to ($\sharp_5$) we get our assertion easily. □

4. Fiber property of the space satisfying the conditions (T^*) or (T^{**}) and locally nilpotent space

From the above pullback diagram ♣ with $X, Y (\in T_N)$, recall that if X or Y is 1-connected then $W \in T_N$ [5, 7.7].

We recall the fiber property of the nilpotent space in a fibration [6], furthermore we get the following:

THEOREM 4.1. *In a fibration* $F \xrightarrow{i} E \xrightarrow{p} B$ *with* $\pi_2(B) = 0$, *if* $E \in T_{LN}$ *then* $F \in T_{LN}$.

Proof. In the pullback diagram from Theorem 3.3, put Y as one point. From the homotopical exact sequence ($\sharp_5$) above and by the similar method of the proof from Theorem 3.3, we get $\pi_1(F)$ locally nilpotent.

Next, since the action $\pi_1(F)$ on $\pi_n(B)$ is nilpotent via $p \circ i$ for $n \geq 2$, $\pi_1(F)$ acts nilpotently on $\pi_n(F)$ for $n \geq 2$ by ($\sharp_2$) above. Therefore $F \in T_{LN}$. □

THEOREM 4.2. *In a fibration* $F \xrightarrow{i} E \xrightarrow{p} B$, *if* $E \in T_{LN}$ *then* B *also satisfies the condition* (T^*).

Proof. Since E satisfies the condition (T^*) by Lemma 2.2, then the space B such that $\pi_1(B) \cong \pi_1(E)/\mathrm{Ker}p_*$ also satisfies the condition (T^*). The reason why is that the followings are equivalent [4]:

(1) E satisfies the condition (T^*).

(2) For each $a \in \pi_1(E)$,

$$h \in [a, \pi_1(E)] \Rightarrow [ah, \pi_1(E)] = [a, \pi_1(E)]. \quad \cdots (\natural_6)$$

Thus for any $\bar{a} \in \pi_1(E)/L$, with $\bar{h} \in [\bar{a}, \pi_1(E)/L]$ where $L = \mathrm{Ker}p_*$ and $\bar{a} = aL, \bar{h} = hL$. Since E satisfies the condition (T^*) we get $[ah, \pi_1(E)] = [a, \pi_1(E)]$ by $(\natural_6)$. And $[ah, \pi_1(E)]L = [a, \pi_1(E)]L \Rightarrow [\bar{a}\bar{h}, \pi_1(E)/L] = [\bar{a}, \pi_1(E)/L]$. Thus B satisfies the condition (T^*). □

We remind that a map $f : X \to X$ is called a fixed point free deformation if f has no fixed point and is homotopic to 1_X [2].

THEOREM 4.3. *For finite $X (\in T_{LN})$ such that $\pi_1(X) \neq 0$ and X satisfies one of the followings:*

(1) $\pi_1(X)$ is finite

(2) $\pi_1(X)$ is infinite with the maximal condition on normal subgroups of $\pi_1(X)$

(3) $\pi_1(X)$ is torsion-free with all proper subgroups of $\pi_1(X)$ nilpotent,

then $X \in T_N$ and X admits a fixed point free deformation. Furthermore in the case (1) above, the universal covering space of X with $\pi_1(X)$ nontrivial also admits a fixed point free deformation.

Proof. For the cases of (1) and (2): we already proved that X is a nilpotent space [4].

For case (3): If $X (\in T_{LN})$ and $\pi_1(X)$ is torsion-free with all proper subgroups of $\pi_1(X)$ nilpotent [8, 2.1] then $X \in T_N$. Hence $\chi(X) = 0$. Thus X has a fixed point free deformation.

Next, from the following property: $\chi(\tilde{X}) = |\pi_1(X)|\chi(X)$ for $\pi_1(X)$ non-trivial, obviously we get $\chi(\tilde{X}) = 0$ for the above case (1), thus our proof is completed. □

THEOREM 4.4. *For the fibration* $F \xrightarrow{i} E \xrightarrow{p} B$ *if* $E \in T_{LN}$ *then* B *satisfies the condition* (T^{**}). *Furthermore, if* $\pi_1(B)$ *acts nilpotently on* $H_n(\tilde{B})$ *where* $n \geq 0$, *and satisfies one of the following cases:*

(1) $\pi_1(B)$ *is finite,*
(2) $\pi_1(B)$ *is torsion-free with all proper subgroups of* $\pi_1(B)$ *nilpotent,*
(3) $\pi_1(B)$ *is infinite with the maximal condition on normal subgroups of* $\pi_1(B)$

then $B \in T_N$.

Proof. In the classical homotopical exact sequence of the above fibration, we put $\pi_1(B) \cong \pi_1(E)/\mathrm{Ker}p_*$. Since E satisfies the condition (T^*) from Lemma 2.2, B also satisfies the condition (T^*) from Theorem 4.2.

Suppose B does not satisfy the condition (T^{**}). Then there is $g(\neq 1) \in \pi_1(B)$ such that $g \in [g, \pi_1(B)]$. Thus $g^{-1} \in [g, \pi_1(B)]$ and $1 \in g[g, \pi_1(B)]$. This implies $g[g, \pi_1(B)] \bigcap 1[1, \pi_1(B)] \neq \phi$. Since B satisfies the condition (T^*), $g[g, \pi_1(B)] = 1$. But $g(\neq 1) \in g[g, \pi_1(B)]$, we have a contradiction. Thus B must satisfy the condition (T^{**}).

Now we check the space B according to the three cases above.

For case (1): if $\pi_1(B)$ is finite, since B satisfies the condition (T^{**}) we get $B \in T_N$ by Lemma 3.2.

For case (2): since $B(\in T_{LN})$ has the property that $\pi_1(B)$ is torsion-free with all proper subgroups of $\pi_1(B)$ nilpotent, $B \in T_N$.

For case (3): we get $\pi_1(B)$ finitely generated nilpotent, and $\pi_1(B)$ contains a torsion-free normal abelian subgroup $A \neq 1$ which acts nilpotently on $H_*(\tilde{B})$. Therefore we get $B \in T_N$ from the implication of Lemma 3.1. At any cases above, B is nilpotent. □

The condition (T^{**}) is very convenient tool in checking the nilpotent structure for the given locally nilpotent space. Thus let's check the following:

THEOREM 4.5. *In a fibration* $F \xrightarrow{i} E \xrightarrow{p} B$, *if* E *satisfies the condition* (T^{**}) *then* B *also satisfies the condition* (T^{**}). $\cdots (\sharp_7)$

But the converse of the statement $(\sharp_7)$ *does not need to be true.* $\cdots (\sharp_8)$

In a while, the converse statement of $(\sharp_7)$ *is valid if* $E \in T_{RLN}$. $\cdots (\sharp_9)$

Proof. Since E satisfies the condition (T^{**}), for any $g(\neq 1) \in \pi_1(E)$ we get $g \notin [g, \pi_1(E)]$. Let us remind that $\pi_1(B) \cong \pi_1(E)/\mathrm{Ker}p_*$. Moreover for any $\bar{g}(\neq 1) \in \pi_1(E)/\mathrm{Ker}p_*, \bar{g} \notin [\bar{g}, \pi_1(E)/\mathrm{Ker}p_*]$. Thus B satisfies the condition (T^{**}).

Next, we get a following example of the statement $(\sharp_8)$. Let us use the further property that a group extension $N \to G \to Q$ induces a fibration $BN \to BG \to BQ$ where B means the classifying functor. In this case for a given group extension $\mathbb{Z}_2 \to S_3 \xrightarrow{p} \mathbb{Z}_3$ we get the fibration $B\mathbb{Z}_2 \to BS_3 \xrightarrow{p} B\mathbb{Z}_3$. From the fibration we take an exact sequence: $\to \pi_1(B\mathbb{Z}_2) \to \pi_1(BS_3) \xrightarrow{p_*} \pi_1(B\mathbb{Z}_3) \to 0$.

From the exact sequence we get $\pi_1(B\mathbb{Z}_3) \cong \mathbb{Z}_3 \cong \pi_1(BS_3)/\mathrm{Ker}p_*$. As we know that $B\mathbb{Z}_3$ satisfies the condition (T^{**}) but BS_3 does not satisfies the condition (T^{**}). Thus we get the above statement $(\sharp_8)$.

Finally if $E \in T_{RLN}$, for $g(\neq 1) \in \pi_1(E)$, there exists a normal subgroup $(g \in)H$ of $\pi_1(E)$ such that $\pi_1(E)/H$ is locally nilpotent. If $\bar{g}(\neq 1) \in \pi_1(E)/H$ then $\bar{g} \notin [\bar{g}, \pi_1(E)/H]$. Hence $g \notin g[g, \pi_1(E)]$. Thus we get the statement $(\sharp_9)$ for $E \in T_{RLN}$. □

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