NOTES ON THE MCSHANE-STIELTJES INTEGRABILITY

BYONG IN SEUNG

ABSTRACT. In this paper, we define the McShane-Stieltjes integral for Banach-valued functions, and will investigate some of its properties and comparison with the Pettis integral.

1. INTRODUCTION


Fremlin and Mendoza [5] improved some results of Gordon as follows: A function $\phi : [0, 1] \to X$ is McShane integrable whenever a tagged partition $P$ is sub $\delta$ on $[0, 1]$ if and only if it is Pettis integrable.

We are concerned with the McShane-Stieltjes integral for Banach-valued functions which is a generalization of the McShane-Stieltjes integral for real-valued functions.

In this paper, for Banach-valued functions we shall introduce the McShane-Stieltjes integral that is encouraged naturally by the idea of Riemann-Stieltjes or Lebesgue-Stieltjes integral. Also, we will investigate some properties of McShane-Stieltjes integrability and comparison with the Pettis integral.

2. PRELIMINARIES

Throughout this paper, $(\Omega, \Sigma, \mu)$ is a finite measure space and $X, Y$ will denote Banach spaces with dual $X^*, Y^*$ and unless otherwise stated, $\alpha$ is an increasing function on $[0, 1]$ into $\mathbb{R}$.

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Definition 2.1 (Diestel and Uhl [1]). A function \( f : \Omega \to X \) is called \textit{simple} if there exist \( x_1, x_2, \ldots, x_n \in X \) and \( E_1, E_2, \ldots, E_n \in \Sigma \) such that \( f = \sum_{i=1}^{n} x_i \chi_{E_i} \), where \( \chi_{E_i}(\omega) = 1 \) if \( \omega \in E_i \) and \( \chi_{E_i}(\omega) = 0 \) if \( \omega \notin E_i \).

A function \( f : \Omega \to X \) is called \textit{\( \mu \)-measurable} if there exists a sequence of simple functions \( \{f_n\} \) with \( \lim_n \|f_n - f\| = 0 \) \( \mu \)-almost everywhere.

A function \( f : \Omega \to X \) is called \textit{weakly \( \mu \)-measurable} if for each \( x^* \in X^* \) the numerical function \( x^* f \) is \( \mu \)-measurable. More generally, if \( \Gamma \subseteq X^* \) and \( x^* f \) is measurable for each \( x^* \in \Gamma \), then \( f \) is called \textit{\( \Gamma \)-measurable}. If \( f : \Omega \to X^* \) is \( X \)-measurable (when \( X \) is identified with its image under the natural imbedding of \( X \) into \( X^{**} \)), then \( f \) is called \textit{weak\(^*\)-measurable}.

Theorem 2.2 (Pettis’s Measurability Theorem). A function \( f : \Omega \to X \) is \( \mu \)-measurable if and only if

(i) \( f \) is \( \mu \)-essentially separably valued, i.e., there exists \( E \in \Sigma \) with \( \mu(E) = 0 \) and such that \( f(\Omega \setminus E) \) is a (norm) separable subset of \( X \), and

(ii) \( f \) is weakly \( \mu \)-measurable.

Definition 2.3. A \( \mu \)-measurable function \( f : \Omega \to X \) is called \textit{Bochner integrable} if there exists a sequence of simple functions \( \{f_n\} \) such that

\[
\lim_n \int_{\Omega} \|f_n - f\| = 0.
\]

In this case, \( \int_{E} f d\mu \) is defined for each \( E \in \Sigma \) by

\[
\int_{E} f d\mu = \lim_n \int_{E} f_n d\mu,
\]

where \( \int_{E} f_n d\mu \) is defined in the obvious way.

Definition 2.4. If \( f \) is a weakly \( \mu \)-measurable \( X \)-valued function on \( \Omega \) such that \( x^* f \in L_1(\mu) \) for all \( x^* \in X^* \), then \( f \) is called \textit{Dunford integrable}. The Dunford integral over \( E \in \Sigma \) is defined by the element \( x^*_E \) of \( X^{**} \) such that \( x^*_E(x^*) = \int_{E} x^* f d\mu \) for all \( x^* \in X^* \), and we write \( x^*_E = (D)\int_{E} f d\mu \).

In the case that \( (D)\int_{E} f d\mu \in X \) for each \( E \in \Sigma \), then \( f \) is called \textit{Pettis integrable} and we write \( (P)\int_{E} f d\mu \) instead of \( (D)\int_{E} f d\mu \) to denote the Pettis integral of \( f \) over \( E \in \Sigma \).

The above definitions and theorem are given in Diestel and Uhl [1].

Definition 2.5 (Gordon [7]). Let \( \delta(\cdot) \) be a positive function defined on the interval \([0, 1]\). A tagged interval \((x, [a, b])\) consists of an interval \([a, b] \subset [0, 1]\) and a point \( x \) in
[0, 1]. This z may not be a point in [a, b]. The tagged interval \((x, [a, b])\) is subordinate to \(\delta\) if \([a, b] \subset (x - \delta(x), x + \delta(x))\). The capital letter \(P\) will be used to denote a finite collection of non-overlapping tagged intervals. Let \(P = \{(x_i, [a_i, b_i]) : 1 \leq i \leq n\}\) be such a collection in [0, 1]. We adopt the following terminology:

(a) The points \(x_i\) are called tags of \(P\) and the intervals \([a_i, b_i]\) are called intervals of \(P\).

(b) If \((x_i, [a_i, b_i])\) is subordinate to \(\delta\) for each \(i\), then we write \(P\) is sub \(\delta\).

(c) If \(P\) is subordinate to \(\delta\) and \([0, 1] = \bigcup_{i=1}^{n} [a_i, b_i]\), then \(P\) is called a tagged partition (or McShane partition) of [0, 1].

(d) If \(P\) is a tagged partition of [0, 1] and if \(P\) is sub \(\delta\), then we write \(P\) is sub \(\delta\) on [0, 1].

(e) If \(f : [0, 1] \to X\), then \(f(P) = \sum_{i=1}^{n} f(x_i)(b_i - a_i)\).

(f) If \(F\) is defined on the intervals of [0, 1], then \(F(P) = \sum_{i=1}^{n} F([a_i, b_i])\).

(g) We will write \(\mu(P)\) for \(\sum_{i=1}^{n} (b_i - a_i)\) and \(\int_{P} f\) for \(\sum_{i=1}^{n} \int_{a_i}^{b_i} f\).

**Definition 2.6** (Gordon [7]). The function \(f : [0, 1] \to X\) is McShane integrable on [0, 1] if there exists a vector \(z\) in \(X\) with the following property: for each \(\epsilon > 0\) there exists a positive function \(\delta\) on [0, 1] such that \(\|f(P) - z\| < \epsilon\) whenever \(P\) is sub \(\delta\) on [0, 1], and \(z\) is denoted by

\[
(M)\int_{0}^{1} f \text{ or } (M)\int_{0}^{1} f(x)dx.
\]

The function \(f\) is McShane integrable on the set \(E \subset [0, 1]\) if the function \(f_{|E}\) is McShane integrable on [0, 1].

We now present the definition of the McShane-Stieltjes integral for Banach-valued functions.

Let \(f : [0, 1] \to X\) and let \(\alpha\) be an increasing function on [0, 1]. Then we will use the following notation:

\[
f_{\alpha}(P) = \sum_{i=1}^{n} f(x_i)[\alpha(b_i) - \alpha(a_i)]
\]

where a tagged partition \(P = \{(x_i, [a_i, b_i]) : 1 \leq i \leq n\}\) of [0, 1] is sub \(\delta\) on [0, 1].

**Definition 2.7.** A function \(f : [0, 1] \to X\) is McShane-Stieltjes integrable with respect to \(\alpha\) on [0, 1] if there exists a vector \(z\) in \(X\) with the following property: for each \(\epsilon > 0\) there exists a positive function \(\delta\) on [0, 1] such that \(\|f_{\alpha}(P) - z\| < \epsilon\) whenever a tagged partition \(P\) is sub \(\delta\) on [0, 1].
A function \( f \) is McShane-Stieltjes integrable on a measurable set \( E \subset [0,1] \) with respect to \( \alpha \) if \( f \chi_E \) is a McShane-Stieltjes integrable function with respect to \( \alpha \) on \([0,1]\). We note that when such a number \( z \) in \( X \) exists, it is uniquely determined and is denoted by 
\[
(MS)\int_0^1 f(x) d\alpha(x) \quad \text{or} \quad (MS)\int_0^1 f d\alpha
\]
and we also say that McShane-Stieltjes integral \( (MS)\int_0^1 f d\alpha \) with respect to \( \alpha \) on \([0,1]\) exists. The function \( f \) and \( \alpha \) are referred to as the integrand function and integrator function, respectively.

And otherwise, all notions and notations used in this paper, unless mentioned, can be found in [1], [2], and [7].

3. Properties of the McShane-Stieltjes Integral

The next propositions and theorems record some of the basic computational properties of McShane-Stieltjes integral for Banach-valued functions and the proofs of these facts are virtually identical to the proofs for real-valued functions, and sometimes the concept of norm in Banach space will be required.

**Proposition 3.1.** Let \( \alpha \) be an increasing function on \( [0,1] \). A function \( f : [0,1] \to X \) is a McShane-Stieltjes integrable with respect to \( \alpha \) on \([0,1]\) if and only if for each \( \varepsilon > 0 \) there exists a positive function \( \delta \) on \([0,1]\) such that \( \|f_\alpha(P_1) - f_\alpha(P_2)\| < \varepsilon \) whenever \( P_1 \) and \( P_2 \) are sub \( \delta \) on \([0,1]\).

**Proof.** Suppose that \( f \) is a McShane-Stieltjes integrable function with respect to \( \alpha \) on \([0,1]\). Then there exists a vector \( z \) in \( X \) with the following property: for each \( \varepsilon > 0 \), there exists a positive function \( \delta \) on \([0,1]\) such that
\[
\|f_\alpha(P) - z\| < \frac{\varepsilon}{2}
\]
whenever a tagged partition \( P \) is sub \( \delta \) on \([0,1]\). If \( P_1 \) and \( P_2 \) are sub \( \delta \) on \([0,1]\), then
\[
\|f_\alpha(P_1) - z\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|f_\alpha(P_2) - z\| < \frac{\varepsilon}{2}.
\]
Hence, we get
\[
\|f_\alpha(P_1) - f_\alpha(P_2)\| = \|(f_\alpha(P_1) - z) + (z - f_\alpha(P_2))\|
\leq \|f_\alpha(P_1) - z\| + \|z - f_\alpha(P_2)\|
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Conversely, suppose that for each \( \epsilon > 0 \) there exists a positive function \( \delta \) on \([0, 1]\) such that \( \|f_\alpha(P_1) - f_\alpha(P_2)\| < \epsilon \) whenever \( P_1 \) and \( P_2 \) are sub \( \delta \) on \([0, 1]\). Then for each positive integer \( n \), there exists a positive function \( \delta'_n \) on \([0, 1]\) such that
\[
\|f_\alpha(P') - f_\alpha(P'')\| < \frac{1}{n}
\]
whenever each tagged partition \( P' \) and \( P'' \) are sub \( \delta'_n \) on \([0, 1]\).

If we take \( \delta_n = \min\{\delta'_1, \delta'_2, \ldots, \delta'_n\} \) for \( n = 1, 2, 3, \ldots \), then \( \delta_n \) is a positive function on \([0, 1]\) for \( n = 1, 2, 3, \ldots \) and
\[
\|f_\alpha(P') - f_\alpha(P'')\| < \frac{1}{n}
\]
whenever each \( P' \) and \( P'' \) are sub \( \delta_n \) on \([0, 1]\).

For each positive integer \( n \), choose a tagged partition \( P_n \) which is sub \( \delta_n \) on \([0, 1]\). For each \( \epsilon > 0 \) there exists a positive integer \( N \in \mathbb{N} \) with \( \frac{1}{N} < \epsilon \). If \( m, n > N \), then
\[
\|f_\alpha(P_m) - f_\alpha(P_n)\| < \frac{1}{N}
\]
since \( P_m \) and \( P_n \) are sub \( \delta_N \) on \([0, 1]\). Thus, a sequence \( (f_\alpha(P_n)) \) is Cauchy sequence in \( X \). Since \( X \) is a Banach space, the sequence \( (f_\alpha(P_n)) \) converges to any one vector in \( X \). Let \( z \) be the limit of this sequence and let \( \epsilon > 0 \). Then there exists a positive integer \( N_1 \) with \( \frac{1}{N_1} < \frac{\epsilon}{2} \) such that if \( N > N_1 \), then
\[
\|f_\alpha(P_N) - z\| < \frac{1}{N_1} < \frac{\epsilon}{2}.
\]

Choose a positive integer \( N \in \mathbb{N} \) such that \( N > N_1 \) and \( \frac{1}{N} < \frac{\epsilon}{2} \). Then
\[
\|f_\alpha(P) - z\| \leq \|f_\alpha(P) - f_\alpha(P_N)\| + \|f_\alpha(P_N) - z\| < \frac{1}{N} + \frac{\epsilon}{2} < \epsilon
\]
where \( P \) and \( P_N \) are sub \( \delta_N \) on \([0, 1]\). Therefore \( f \) is a McShane-Stieltjes integrable with respect to \( \alpha \) on \([0, 1]\).

\begin{proof}
Let \( [a, b] \) be any subinterval of \([0, 1]\) and let \( \epsilon > 0 \). Then, there exists a positive function \( \delta_1 \) such that \( \|f_\alpha(P_1) - f_\alpha(P_2)\| < \epsilon \) whenever \( P_1 \) and \( P_2 \) are sub \( \delta_1 \) on \([0, 1]\) by Proposition 3.1.

Let \( \delta \) be the restriction of \( \delta_1 \) to subinterval \([a, b]\) of \([0, 1]\). Then \( \delta \) is a positive function on \([a, b]\). Also, let \( P_1 \) and \( P_2 \) be any tagged partitions of \([a, b]\) which are
\end{proof}

**Proposition 3.2.** Let \( \alpha \) be an increasing function on \([0, 1]\). If \( f : [0, 1] \to X \) is a McShane-Stieltjes integrable with respect to \( \alpha \) on \([0, 1]\), then \( f \) is a McShane-Stieltjes integrable with respect to \( \alpha \) on every subinterval of \([0, 1]\).

\begin{proof}
Let \( [a, b] \) be any subinterval of \([0, 1]\) and let \( \epsilon > 0 \). Then, there exists a positive function \( \delta_1 \) such that \( \|f_\alpha(P_1) - f_\alpha(P_2)\| < \epsilon \) whenever \( P_1 \) and \( P_2 \) are sub \( \delta_1 \) on \([0, 1]\) by Proposition 3.1.

Let \( \delta \) be the restriction of \( \delta_1 \) to subinterval \([a, b]\) of \([0, 1]\). Then \( \delta \) is a positive function on \([a, b]\). Also, let \( P_1 \) and \( P_2 \) be any tagged partitions of \([a, b]\) which are
\end{proof}
sub $\delta$ on $[a, b]$. Choose a tagged partition $P_a$ of $[0, a]$ which is sub $\delta_1$, and choose a tagged partition $P_b$ of $[b, 1]$ which is sub $\delta_1$. Then two tagged partitions

$$P' = P_a \cup P_1 \cup P_b \text{ and } P'' = P_a \cup P_2 \cup P_b$$

are sub $\delta_1$ on $[0, 1]$. Hence,

$$\|f_\alpha(P_1) - f_\alpha(P_2)\| = \|f_\alpha(P') - f_\alpha(P'')\| < \epsilon.$$

By Proposition 3.1, $f$ is McShane-Stieltjes integrable with respect to $\alpha$ on $[0, 1]$. That is, $f$ is McShane-Stieltjes integrable with respect to $\alpha$ on every subinterval of $[0, 1]$.

It is easy to prove the following fact from the above two properties.

**Proposition 3.3.** Let $f : [0, 1] \to X$ and let $t \in (0, 1)$. If $f$ is a McShane-Stieltjes integrable with respect to $\alpha$ on each of the interval $[0, t]$ and $[t, 1]$, then $f$ is a McShane-Stieltjes integrable with respect to $\alpha$ on $[0, 1]$ and

$$(MS) \cdot \int_0^1 f \, d\alpha = (MS) \cdot \int_0^t f \, d\alpha + (MS) \cdot \int_t^1 f \, d\alpha.$$

Moreover, the Propositions above give, mutatis mutandis, the following remarkable property:

**Proposition 3.4.** Let $f$ and let $g$ be McShane-Stieltjes integrable with respect to $\alpha$ on $[0, 1]$ into $X$. Then for any real numbers $s$ and $t$, the function $sf + tg$ is McShane-Stieltjes integrable with respect to $\alpha$ on $[0, 1]$ and

$$(MS) \cdot \int_0^1 (sf + tg) \, d\alpha = s \left[(MS) \cdot \int_0^1 f \, d\alpha\right] + t \left[(MS) \cdot \int_0^1 g \, d\alpha\right].$$

The following theorem shows the linearity of the McShane-Stieltjes integrator functions.

**Theorem 3.5.** Let $\alpha$ and $\beta$ be increasing functions on $[0, 1]$ and let $c_1$ and $c_2$ be nonnegative real numbers. If $f : [0, 1] \to X$ is a McShane-Stieltjes integrable with respect to $\alpha$ and $\beta$ respectively, then

$$(MS) \cdot \int_0^1 f \, d(c_1 \alpha + c_2 \beta) = c_1 \left[(MS) \cdot \int_0^1 f \, d\alpha\right] + c_2 \left[(MS) \cdot \int_0^1 f \, d\beta\right].$$

**Proof.** We first show that $(MS) \cdot f_0^1 f \, d(\alpha + \beta) = (MS) \cdot f_0^1 f \, d\alpha + (MS) \cdot f_0^1 f \, d\beta$ and next show that $(MS) \cdot f_0^1 f \, d(k \alpha) = k[(MS) \cdot f_0^1 f \, d\alpha]$ for a negative real number $k.$
First, suppose that \( f : [0, 1] \to X \) is a McShane-Stieltjes integrable with respect to both \( \alpha \) and \( \beta \). Then for each \( \epsilon > 0 \), there exists a positive function \( \delta_1 \) on \([0, 1]\) such that \( \| f_\alpha(P_1) - (MS)_\int_0^1 f d\alpha \| < \frac{\epsilon}{2} \) whenever a tagged partition \( P_1 \) on \([0, 1]\) is sub \( \delta_1 \), and a positive function \( \delta_2 \) on \([0, 1]\) such that \( \| f_\beta(P_2) - (MS)_\int_0^1 f d\beta \| < \frac{\epsilon}{2} \) whenever a tagged partition \( P_2 \) of \([0, 1]\) is sub \( \delta_2 \).

Choose a tagged partition \( P = \{(x_i, [a_i, b_i]) : 1 \leq i \leq n\} \) of \([0, 1]\) consisting of elements that is the intersection of every element of \( P_1 \) and \( P_2 \).

Let \( \delta(x) = \min\{\delta_1(x), \delta_2(x)\} \) for \( x \) in \([0, 1]\). Then \( \delta \) is a positive function on \([0, 1]\) and also \( P \) is sub \( \delta \) on \([0, 1]\). Thus the following properties hold: for given \( \epsilon > 0 \),

\[
\left\| f_\alpha(P) - (MS)_\int_0^1 f d\alpha \right\| < \frac{\epsilon}{2}
\]

whenever a tagged partition \( P \) is sub \( \delta \) on \([0, 1]\) and

\[
\left\| f_\beta(P) - (MS)_\int_0^1 f d\beta \right\| < \frac{\epsilon}{2}
\]

whenever a tagged partition \( P \) is sub \( \delta \) on \([0, 1]\). Hence,

\[
\left\| f_{\alpha+\beta}(P) - \left[ (MS)_\int_0^1 f d\alpha + (MS)_\int_0^1 f d\beta \right] \right\|
\]

\[
= \left\| \sum_{i=1}^n f(x_i)((\alpha + \beta)(b_i) - (\alpha + \beta)(a_i)) \right. - \left[ (MS)_\int_0^1 f d\alpha + (MS)_\int_0^1 f d\beta \right] \right\|
\]

\[
= \left\| \left[ \sum_{i=1}^n f(x_i)(\alpha(b_i) - \alpha(a_i)) + \sum_{i=1}^n f(x_i)(\beta(b_i) - \beta(a_i)) \right] \right.
\]

\[
- \left[ (MS)_\int_0^1 f d\alpha + (MS)_\int_0^1 f d\beta \right] \right\|
\]

\[
\leq \left\| f_\alpha(P) - (MS)_\int_0^1 f d\alpha \right\| + \left\| f_\beta(P) - (MS)_\int_0^1 f d\beta \right\|
\]

\[
< \epsilon
\]

whenever a tagged partition \( P = \{(x_i, [a_i, b_i]) : 1 \leq i \leq n\} \) is sub \( \delta \) on \([0, 1]\). Thus, we get

\[
(MS)_\int_0^1 f d(\alpha + \beta) = (MS)_\int_0^1 f d\alpha + (MS)_\int_0^1 f d\beta.
\]

Second, suppose that \( f \) is a McShane-Stieltjes integrable with respect to \( \alpha \) on \([0, 1]\) and \( k \) is a nonnegative real number.

Case 1: \( k = 0 \). It is trivial.
Case 2: $k > 0$. Given $\epsilon > 0$, there exists a positive function $\delta$ on $[0, 1]$ such that

$$\left\|f_\alpha(P) - (MS)\int_0^1 f\,d\alpha\right\| < \frac{\epsilon}{k}$$

whenever a tagged partition $P$ is sub $\delta$ on $[0, 1]$. Thus we obtain

$$\left\|f_{(k\alpha)}(P) - k\left[(MS)\int_0^1 f\,d\alpha\right]\right\|$$

$$= \left\|\sum_{i=1}^n f(x_i)[k\alpha(b_i) - k\alpha(a_i)] - k\left[(MS)\int_0^1 f\,d\alpha\right]\right\|$$

$$= \left\|k\sum_{i=1}^n f(x_i)[\alpha(b_i) - \alpha(a_i)] - k\left[(MS)\int_0^1 f\,d\alpha\right]\right\|$$

$$= \left\|kf_\alpha(P) - k\left[(MS)\int_0^1 f\,d\alpha\right]\right\|$$

$$< \epsilon$$

whenever a tagged partition $P$ is sub $\delta$ on $[0, 1]$. Hence,

$$(MS)\int_0^1 fd(k\alpha) = k\left[(MS)\int_0^1 f\,d\alpha\right].$$

Consequently, we have the following result as required:

$$(MS)\int_0^1 fd(c_1\alpha + c_2\beta) = c_1\left[(MS)\int_0^1 f\,d\alpha\right] + c_2\left[(MS)\int_0^1 f\,d\beta\right]. \quad \square$$

**Theorem 3.6.** If $f : [0, 1] \to X$ is a McShane-Stieltjes integrable with respect to $\alpha$ and if $T : X \to Y$ is a bounded linear operator, then the composition $T \circ f : [0, 1] \to Y$ is a McShane-Stieltjes integrable with respect to $\alpha$ and

$$T\left[(MS)\int_0^1 f\,d\alpha\right] = (MS)\int_0^1 T \circ f \, d\alpha$$

**Proof.** There exists $M \in \mathbb{R}^+$ such that $\|T\| \leq M$ since $T : X \to Y$ is a bounded linear operator. Now let $(MS)\int_0^1 f\,d\alpha = z$. Then, for given $\epsilon > 0$ there exists a positive function $\delta$ on $[0, 1]$ such that

$$\left\|z - \sum_{i=1}^n f(x_i)[\alpha(b_i) - \alpha(a_i)]\right\| < \frac{\epsilon}{M}.$$
whenever a tagged partition \( \{(x_i, [a_i, b_i]) : 1 \leq i \leq n\} \) is sub \( \delta \) on \([0, 1]\). Thus,

\[
\begin{align*}
\| &T[(MS)\int_0^1 f\,d\alpha] - (MS)\int_0^1 (T \circ f)\,d\alpha\| \\
= &\|Tz - \sum_{i=1}^n (T \circ f)(x_i)[\alpha(b_i) - \alpha(a_i)]\| \\
= &\|T \cdot \left[ z - \sum_{i=1}^n f(x_i)[\alpha(b_i) - \alpha(a_i)] \right]\| \\
= &\|T\| \cdot \left\| z - \sum_{i=1}^n f(x_i)[\alpha(b_i) - \alpha(a_i)] \right\| \\
< &\ M \cdot \frac{\epsilon}{M} = \epsilon
\end{align*}
\]

whenever also \( \{(x_i, [a_i, b_i]) : 1 \leq i \leq n\} \) is sub \( \delta \) on \([0, 1]\). Therefore, \(Tof : [0, 1] \rightarrow Y\) is a McShane-Stieltjes integrable operator with respect to \( \alpha \) and

\[
(MS)\int_0^1 Tof\,d\alpha = T(z) = T\left[(MS)\int_0^1 f\,d\alpha\right].
\]

\( \square \)

**Corollary 3.7.** If \( f \) is a McShane-Stieltjes integrable with respect to \( \alpha \) on \([0, 1]\) into \( X \) and for each \( t \) in \([0, 1]\), then \( x^*f \) is a McShane-Stieltjes integrable with respect to \( \alpha \) on \([0, 1]\) and for each \( x^* \) in \( X^* \)

\[
(MS)\int_0^1 x^*f\,d\alpha = x^*\left[(MS)\int_0^1 f\,d\alpha\right].
\]

**Proof.** If \( x^* = 0 \), then the result follows immediately. Now we consider the case that \( x^* \) is not zero. Since \( f \) is a McShane-Stieltjes integrable with respect to \( \alpha \) on \([0, 1]\), there exists a positive function \( \delta \) on \([0, 1]\) such that for each \( \epsilon > 0 \)

\[
\left\| f\alpha(P) - (MS)\int_0^1 f\,d\alpha \right\| < \frac{\epsilon}{\|x^*\|},
\]

whenever a tagged partition \( P \) is sub \( \delta \) on \([0, 1]\). And then,

\[
\begin{align*}
\left\| (x^*f)\alpha(P) - x^*\left[(MS)\int_0^1 f\,d\alpha\right] \right\| \\
= &\left\| \sum_{i=1}^n (x^*f)(x_i)[\alpha(b_i) - \alpha(a_i)] - x^*\left[(MS)\int_0^1 f\,d\alpha\right] \right\|
\end{align*}
\]
\[ \|x^*\left(\sum_{i=1}^{n} f(x_i)[\alpha(b_i) - \alpha(a_i)] - (MS)\int_0^1 f\,d\alpha\right)\| = \|x^*\| \cdot \left|\int_0^1 (MS) - (MS)\int_0^1 f\,d\alpha\right| = \|x^*\| \cdot \frac{\epsilon}{\|x^*\|} = \epsilon \]

whenever also \( P = \{(x_i, [a_i, b_i]): 1 \leq i \leq n\} \) is sub \( \delta \) on \([0, 1]\). Hence, \( x^*f \) is a McShane-Stieltjes integrable with respect to \( \alpha \) on \([0, 1]\) and for each \( x^* \in X^* \),

\[ (MS)\int_0^1 (x^*f)\,d\alpha = x^*\left[(MS)\int_0^1 f\,d\alpha\right]. \]

Moreover, for each \( t \in [0, 1] \), \( f \) is a McShane-Stieltjes integrable with respect to \( \alpha \) on \([0, t]\) by Proposition 3.2.

Considering the above argument carefully, \( x^*f \) is a McShane-Stieltjes integrable with respect to \( \alpha \) on \([0, t]\) and

\[ (MS)\int_0^t x^*f\,d\alpha = x^*\left[(MS)\int_0^t f\,d\alpha\right] \]

for \( x^* \in X^* \).

\[ \square \]

4. COMPARISON WITH THE PETTIS INTEGRAL

We now proceed to prove that every measurable and Pettis integrable function is McShane-Stieltjes integrable.

**Theorem 4.1.** Let \( f: [0, 1] \to X \) be McShane-Stieltjes integrable with respect to \( \alpha \) on \([0, 1]\). If \( f = g \) almost everywhere on \([0, 1]\), then \( g \) is McShane-Stieltjes integrable with respect to \( \alpha \) on \([0, 1]\) and \((MS)\int_0^1 f\,d\alpha = (MS)\int_0^1 g\,d\alpha\).

**Proof.** It is sufficient to prove that if \( f = \theta \) (the zero of \( X \)) almost everywhere on \([0, 1]\) then \( f \) is McShane-Stieltjes integrable with respect to \( \alpha \) on \([0, 1]\) and \((MS)\int_0^1 f\,d\alpha = \theta\). Since \( \|f\| = 0 \) a.e. on \([0, 1]\), the function \( \|f\| \) is McShane-Stieltjes integrable with respect to \( \alpha \) on \([0, 1]\) and it is Lebesgue integrable since \( \int_0^1 \|f\| = 0 \). Let \( \epsilon > 0 \) and choose a positive function \( \delta \) on \([0, 1]\) such that \( \|f\|_\alpha(P) < \epsilon \) whenever \( P \) is sub \( \delta \) on \([0, 1]\). Let \( P \) be sub \( \delta \) on \([0, 1]\) and compute \( \|f_\alpha(P) - \theta\| = \|f_\alpha(P)\| \leq \|f\|_\alpha(P) < \epsilon \). This shows that \( f \) is McShane-Stieltjes integrable with respect to \( \alpha \) on \([0, 1]\) and \((MS)\int_0^1 f\,d\alpha = \theta\). \[ \square \]
The next definition and the proof of the theorems can be found in Gordon [7]. We shall have necessarily any modifications about them.

**Definition 4.2.** Let \( \{f_n\} \) be a collection of McShane-Stieltjes integrable functions with respect to \( \alpha \) on \([0,1]\). The collection \( \{f_n\} \) is uniformly McShane-Stieltjes integrable with respect to \( \alpha \) on \([0,1]\) if there exists a set \( E \in [0,1] \) such that \( \mu(E) = 1 - 0 = 1 \) and for each \( \epsilon > 0 \) there exists a positive function \( \delta \) on \([0,1]\) such that

\[
\left\| (f_n)_{X_E}(P) - (MS) \int_0^1 f_n d\alpha \right\| < \epsilon
\]

for all \( n \) and whenever \( P \) is sub \( \delta \) on \([0,1]\).

**Theorem 4.3.** Let \( f_n : [0,1] \to X \) be a McShane-Stieltjes integrable function with respect to \( \alpha \) on \([0,1]\) for each positive integer \( n \). If \( f_n \to f \) uniformly on \([0,1]\), then \( f \) is McShane-Stieltjes integrable with respect to \( \alpha \) on \([0,1]\) and

\[
(MS) \int_0^1 f d\alpha = \lim_{n \to \infty} (MS) \int_0^1 f_n d\alpha.
\]

**Theorem 4.4.** Let \( \{E_n\} \) be a sequence of disjoint measurable sets in \([0,1]\), let \( \{x_n\} \) be a sequence in \( X \), and let \( f : [0,1] \to X \) be defined by \( f(t) = \sum_n x_n 1_{E_n}(t) \).

If the series \( \sum_n \mu(E_n)x_n \) is unconditionally convergent, then the function \( f \) is McShane-Stieltjes integrable with respect to \( \alpha \) on \([0,1]\) and

\[
(MS) \int_0^1 f d\alpha = \sum_n \mu(E_n)x_n[\alpha(b_i) - \alpha(a_i)].
\]

Now we are ready to verify the following two theorems that will be used to prove Theorem 4.9.

**Theorem 4.5.** If \( f : [0,1] \to X \) is Bochner integrable on \([0,1]\), then \( f \) is McShane-Stieltjes integrable with respect to \( \alpha \) on \([0,1]\).

**Proof.** Since \( f \) is measurable, there exist \( E \subset [0,1] \) with \( \mu(E) = 1 - 0 = 1 \) and a sequence \( \{f_n\} \) of countably-valued functions such that for each \( n \) the inequality \( \|f_n(t) - f(t)\| \leq \frac{1}{n} \) holds for all \( t \) in \([0,1]\). It is clear that each \( f_n \) is Bochner integrable on \([0,1]\). For each \( n \), let \( f_n = \sum_k x_k^n 1_{E_k^n} \) where the sets \( \{E_k^n : k \geq 1\} \) are disjoint and measurable. The series \( \sum_k \mu(E_k^n)x_k^n \) is absolutely convergent and hence unconditionally convergent for each \( n \). By Theorem 4.4, each of the functions \( f_n \) is McShane-Stieltjes integrable with respect to \( \alpha \) on \([0,1]\).
Since $f_{\chi_E}$ is the uniform limit of $\{f_n\}$ on $[0,1]$, the function $f_{\chi_E}$ is McShane-Stieltjes integrable with respect to $\alpha$ on $[0,1]$ by Theorem 4.3. And by Theorem 4.1 the function $f$ is McShane-Stieltjes integrable with respect to $\alpha$ on $[0,1]$.

**Theorem 4.6.** Let $f : [0,1] \to X$ be measurable. If $f$ is Pettis integrable on $[0,1]$, then $f$ is McShane-Stieltjes integrable with respect to $\alpha$ on $[0,1]$.

*Proof.* Since $f$ is measurable, there exist $E \subset [0,1]$ with $\mu(E) = 1 - 0 = 0$ and a countably-valued function $g : [0,1] \to X$ such that $\|g(t) - f_{\chi_E}(t)\| \leq 1$ for all $t$ in $[0,1]$. It is easy to see that $g - f_{\chi_E}$ is Bochner integrable on $[0,1]$ and that $g$ is Pettis integrable on $[0,1]$. By Theorem 4.5 the function $g - f_{\chi_E}$ is McShane-Stieltjes integrable with respect to $\alpha$ on $[0,1]$. Let $g = \sum_n x_n \chi_{E_n}$ where the $E_n$'s are disjoint, measurable sets in $[0,1]$. Since $g$ is Pettis integrable on $[0,1]$, every subseries of $\sum_n \mu(E_n)x_n$ is weakly convergent. By a theorem of Orlicz and Pettis (cf. Diestel and Uhl [1, p. 22]), the series $\sum_n \mu(E_n)x_n$ is unconditionally convergent. By Theorem 4.4, the function $g$ is McShane-Stieltjes integrable with respect to $\alpha$ on $[0,1]$, and it follows that $f_{\chi_E} = g - (g - f_{\chi_E})$ is McShane-Stieltjes integrable with respect to $\alpha$ on $[0,1]$. By Theorem 4.1, the function $f$ is McShane-Stieltjes integrable with respect to $\alpha$ on $[0,1]$.

**Theorem 4.7.** Suppose that $X$ contains no copy of $c_0$ and let $f : [0,1] \to X$ be Dunford integrable on $[0,1]$. If $\int_I f \in X$ for every interval $I \subset [0,1]$, then $f$ is Pettis integrable on $[0,1]$.

*Proof.* The proof is a consequence of the Bessaga-Pelczynski characterization of Banach spaces that do not contain a copy of $c_0$ (cf. Diestel and Uhl [1, p. 22]).

From the fact that every McShane integrable function is Dunford integrable and $X$-valued on intervals, we obtain the corollary below:

**Corollary 4.8.** Suppose that $X$ contains no copy of $c_0$. If $f : [0,1] \to X$ is McShane-Stieltjes integrable with respect to $\alpha$ on $[0,1]$, then $f$ is Pettis integrable on $[0,1]$.

Combining Theorem 4.6 and Corollary 4.8, we have the important following result:

**Theorem 4.9.** Suppose that $X$ is separable and contains no copy of $c_0$. A function $f : [0,1] \to X$ is McShane-Stieltjes integrable with respect to $\alpha$ on $[0,1]$ if and only if $f$ is Pettis integrable on $[0,1]$. 

Proof. Suppose that $X$ is separable and contains no copy of $c_0$. If $f : [0, 1] \to X$ is McShane-Stieltjes integrable with respect to $\alpha$ on $[0, 1]$, then $f$ is Pettis integrable on $[0, 1]$ by Corollary 4.8.

Conversely, if $f : [0, 1] \to X$ is Pettis integrable on $[0, 1]$, then $f$ is measurable by Theorem 2.2. Therefore $f$ is McShane-Stieltjes integrable with respect to $\alpha$ on $[0, 1]$ by Theorem 4.6.

\[ \square \]

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DEPARTMENT OF MATHEMATICS, KYONGGI UNIVERSITY, SAN 94-6 IUI-DONG, Paldal-gu, Suwon, Gyeonggi-do 442-760, Korea

E-mail address: biseung@kuic.kyonggi.ac.kr