EXISTENCE, UNIQUENESS AND NORM ESTIMATE
OF SOLUTIONS FOR THE NONLINEAR DELAY
INTEGRO-DIFFERENTIAL SYSTEM

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ABSTRACT. In this paper, we study the existence, uniqueness and norm estimate of solutions for the nonlinear delay integro-differential system.

1. INTRODUCTION

The existence for solutions of evolution equation with the nonlocal conditions in Banach space has been studied first by Byszewski [1].

In this paper, we study the existence, uniqueness and norm estimate of solutions for the following nonlinear delay integro-differential system with nonlocal initial condition:

\[
\begin{align*}
\frac{dx(t)}{dt} &= Ax(t) + f(t, x_t, \int_0^t k(t, s, x_s)ds), \quad t \in (0, T]; \\
x(t) + g(x_{t_1}, \ldots, x_{t_p})(t) &= \phi(t), \quad t \in [-h, 0],
\end{align*}
\]

where \(0 < t_1 < \cdots < t_p \leq T\) (\(p \in \mathbb{N}\)), bounded linear operator \(A\) is the infinitesimal generator of \(C_0\) semigroup on a Banach space.

\(C([-h, 0] : X)\) is a Banach space of all continuous functions from an interval \([-h, 0]\) to \(X\) with the norm defined by supremum,

\[
\begin{align*}
f : [0, T] \times C([[-h, 0] : X]) \times X &\to X, \\
g : [C([[-h, 0] : X])^p \to C([-h, 0] : X), \\
k : [0, T] \times [0, T] \times C([-h, 0] : X) &\to X
\end{align*}
\]

are given nonlinear functions and \(\phi\) is a initial function. If a function \(x\) is continuous from \([-h, 0] \cup [0, T]\) to \(X\), then \(x_t\) is an element in \(C([-h, 0] : X)\) which has pointwise
definition:
\[ x_t(\theta) = x(t + \theta) \text{ for } \theta \in [-h, 0], \ t \in [0, T]. \]

2. Existence, Uniqueness and Norm Estimate Results

We consider the following integral equation
\[
\begin{aligned}
x(t) &= S(t)\{\phi(0) - g(x_{t_1}, \cdots, x_{t_p})(0)\} \\
&\quad + \int_0^t S(t-s)f(s, x_s, \int_0^s k(s, \tau, x_\tau)d\tau)ds, \quad t \in [0, T], \\
x(t) + g(x_{t_1}, \cdots, x_{t_p})(t) &= \phi(t), \quad t \in [-h, 0].
\end{aligned}
\] (2.1)

Then the continuous solution \( x(t) \in C([-h, T] : X) \) of (2.1) is called the mild solution of (1.1).

Let \( X \) be a Banach space with the norm \( \| \cdot \| \). We shall make the following assumptions of \( f, k, g \) and a \( C_0 \) semigroup \( S(t) \).

(H1) There exists a constant \( L \) such that
\[ \| f(t, \varphi_1, \psi_1) - f(t, \varphi_2, \psi_2) \| \leq L(\| \varphi_1 - \varphi_2 \|_{C([-h, T] : X)} + \| \psi_1 - \psi_2 \|_{X}), \]
for \( \varphi_1, \varphi_2 \in C([-h, 0] : X), \psi_1, \psi_2 \in X, t \in [0, T] \) and \( f(0, 0, 0) \equiv 0 \).

(H2) The nonlinear function
\[ k : [0, T] \times [0, T] \times C([-h, 0] : X) \rightarrow X \]
satisfies a Lipschitz condition
\[ \| k(t, s, \varphi_1) - k(t, s, \varphi_2) \| \leq L_1 \| \varphi_1 - \varphi_2 \|_{C([-h, T] : X)}, \]
where \( \varphi_1, \varphi_2 \in C([-h, 0] : X), t, s \in [0, T], L_1 \) is constant and \( k(t, s, 0) \equiv 0 \).

(H3) The nonlinear function
\[ g : C([-h, 0] : X)^P \rightarrow C([-h, 0] : X) \]
satisfies a Lipschitz condition
\[ \| g(x_{t_1}, \cdots, x_{t_p})(s) - g(\tilde{x}_{t_1}, \cdots, \tilde{x}_{t_p})(s) \| \leq K \| x_t - \tilde{x}_t \|_{C([-h, 0] : X)}, \]
where \( x_t, \tilde{x}_t \in C([-h, 0] : X), s \in [-h, 0] \) and \( K > 0 \) is constant.

(H4) \( \phi \in C([-h, 0] : X) \).

(H5) \( \| S(t) \| \leq M \).

Now, we will prove the following theorem.
Theorem 2.1. Assume that the hypotheses (H1)-(H5) are satisfied and
\[ M \{ K + LT(1 + L_1 T) \} < 1, \]
then nonlocal Cauchy problem (1.1) has unique mild solution.

Proof. Define the operator \( F : C([-h, T] : X) \to C([-h, T] : X) \) by
\[
(Fx_t)(0) = \begin{cases} 
\phi(t) - g(x_{t_1}, \cdots, x_{t_p})(t), & t \in [-h, 0], \\
S(t) \{ \phi(0) - g(x_{t_1}, \cdots, x_{t_p})(0) \} \\
+ \int_0^t S(t - s) f(s, x_s, \int_0^s k(s, \tau, x_{t_\tau}) d\tau) ds, & t \in [0, T],
\end{cases}
\]
for \( x_t \in C([-h, T] : X) \).

We will prove that \( F \) is contractive mapping defined by \( (Fx_t)(\theta) = (Fx_{t+\theta})(0) \).

If \( x_t, y_t \in C([-h, T] : X) \) and \( t + \theta \in [-h, 0] \),
\[
(Fx_t)(\theta) - (Fy_t)(\theta) = (Fx_{t+\theta})(0) - (Fy_{t+\theta})(0) \\
= g(x_{t_1}, \cdots, x_{t_p})(t + \theta) - g(y_{t_1}, \cdots, y_{t_p})(t + \theta).
\]
(2.2)

And if \( x_t, y_t \in C([-h, T] : X) \) and \( t + \theta \in [0, T] \), then
\[
(Fx_t)(\theta) - (Fy_t)(\theta) \\
= (Fx_{t+\theta})(0) - (Fy_{t+\theta})(0) \\
= S(t + \theta) \{ g(x_{t_1}, \cdots, x_{t_p})(0) - g(y_{t_1}, \cdots, y_{t_p})(0) \} \\
+ \int_0^{t+\theta} S(t + \theta - s) \left\{ f(s, x_s, \int_0^s k(s, \tau, x_{t_\tau}) d\tau) - f(s, y_s, \int_0^s k(s, \tau, y_{t_\tau}) d\tau) \right\} ds.
\]
(2.3)

From (2.2) and (H3),
\[
\|(Fx_t)(\theta) - (Fy_t)(\theta)\| \leq K\|x_t - y_t\|_{C([-h, T] : X)}.
\]

Hence
\[
\|(Fx_t) - (Fy_t)\|_{C([-h, T] : X)} = \sup_{-h \leq \theta \leq 0} \|(Fx_t)(\theta) - (Fy_t)(\theta)\| \\
\leq K\|x_t - y_t\|_{C([-h, T] : X)}.
\]
From 2.3 and (H1)-(H5),
\[
\|(F x_t)(\theta) - (F y_t)(\theta)\|
\leq \|S(t + \theta)\| \|g(x_{t_1}, \cdots, x_{t_p})(0) - g(y_{t_1}, \cdots, y_{t_p})(0)\|
+ \int_0^{t+\theta} \|S(t + \theta - s)\| \left\| f(s, x_s, \int_0^s k(s, \tau, x_{\tau}) d\tau) - f(s, y_s, \int_0^s k(s, \tau, y_{\tau}) d\tau) \right\| ds
\leq MK\|x_t - y_t\|_{C([-h, T] \setminus X)}
+ M \int_0^{t+\theta} L \left\{ \|x_s - y_s\|_{C([-h, T] \setminus X)} + \int_0^s L_1 |x_{\tau} - y_{\tau}|_{C([-h, T] \setminus X)} d\tau \right\} ds
\leq MK\|x_t - y_t\|_{C([-h, T] \setminus X)} + ML \int_0^{t+\theta} \left\{ 1 + L_1(t + \theta) \right\} \|x_s - y_s\|_{C([-h, T] \setminus X)} ds
\leq MK\|x_t - y_t\|_{C([-h, T] \setminus X)} + ML \left\{ 1 + L_1(t + \theta) \right\} (t + \theta) \|x_t - y_t\|_{C([-h, T] \setminus X)} \int_0^{t+\theta} ds
\leq M \{ K + LT(1 + L_1 T) \} \|x_t - y_t\|_{C([-h, T] \setminus X)}.
\]
Thus
\[
\|(F x_t)(\theta) - (F y_t)(\theta)\| = \sup_{-h < \theta < 0} \|(F x_t)(\theta) - (F y_t)(\theta)\|
\leq M \{ K + LT(1 + L_1 T) \} \|x_t - y_t\|_{C([-h, T] \setminus X)}.
\]
Since \(M \{ K + LT(1 + L_1 T) \} < 1\), \(F\) is a contractive mapping on \(C([-h, T] : X)\).
Consequently, an unique fixed point of \(F\) on \(C([-h, T] : X)\) is a unique mild solution of (1.1).
\[
\Box
\]
Next theorem is characteristic for the continuous dependence of the nonlinear functional integro-differential system (1.1) with the classical initial condition.

**Theorem 2.2.** Suppose that the hypotheses (H1)-(H5) are holds and
\[
M \{ K + LT(1 + L_1 T) \} < 1.
\]
Then for each \(\phi_1, \phi_2 \in C([-h, 0] : X)\) and mild solution \(x_1^t, x_2^t\) of the equations
\[
\begin{align*}
\frac{dx(t)}{dt} &= Ax(t) + f(t, x_t, \int_0^t k(t, s, x_s) ds), \quad t \in [0, T] \\
x_i(t) + g(x_{i_1}, \cdots, x_{i_p})(t) &= \phi_i(t), \quad t \in [-h, 0], \ i = 1, 2,
\end{align*}
\]
the following inequality is established
\[
\|(x_1 - x_2)_{C([-h, T] : X)} \leq \frac{M}{1 - MK} \exp \frac{ML(1 + L_1 T)T}{1 - MK} \|\phi_1 - \phi_2\|_{C([-h, 0] : X)}.
\]
Proof. Let \( \phi_i \in C([-h, 0] : X) \) \((i = 1, 2)\) and \( x^1_i \) \((i = 1, 2)\) be the mild solution of (2.4). For \( t + \theta \in [0, T] \),

\[
\begin{aligned}
x^1_t(\theta) - x^2_t(\theta) & = S(t + \theta)\{\phi_1(0) - \phi_2(0)\} - S(t + \theta)\{g(x^1_i, \ldots, x^1_{p_i})(0) - g(x^2_i, \ldots, x^2_{p_i})(0)\} \\
& \quad + \int_0^t S(t + \theta - s)\left\{f(s, x^1_s, \int_0^s k(s, \tau, x^1_\tau) d\tau) - f(s, x^2_s, \int_0^s k(s, \tau, x^2_\tau) d\tau)\right\} ds
\end{aligned}
\]

and for \( t + \theta \in [-h, 0] \),

\[
\begin{aligned}
x^1_t(\theta) - x^2_t(\theta) & = \phi_1(t + \theta) - \phi_2(t + \theta) + g(x^1_i, \ldots, x^1_{p_i})(t + \theta) - g(x^2_i, \ldots, x^1_{p_i})(t + \theta).
\end{aligned}
\]

From 2.6 and (H1)–(H5),

\[
\begin{aligned}
\|x^1_t(\theta) - x^2_t(\theta)\| & \leq M\|\phi_1 - \phi_2\|_{C([-h,0]:X)} + MK\|x^1_0 - x^2_0\|_{C([-h,T]:X)} \\
& \quad + ML\int_0^{t+\theta} L\left(\|x^1_\tau - x^2_\tau\|_{C([-h,T]:X)}\right) d\tau ds \\
& \leq M\|\phi_1 - \phi_2\|_{C([-h,0]:X)} + MK\|x^1_0 - x^2_0\|_{C([-h,T]:X)} \\
& \quad + ML(1 + L_1(t + \theta))\int_0^{t+\theta}\|x^1_\tau - x^2_\tau\|_{C([-h,T]:X)} d\tau ds.
\end{aligned}
\]

Therefore

\[
\begin{aligned}
\|x^1_t - x^2_t\|_{C([-h,T]:X)} & = \sup_{-h \leq \theta \leq 0} \|x^1_t(\theta) - x^2_t(\theta)\| \\
& \leq M\|\phi_1 - \phi_2\|_{C([-h,0]:X)} + MK\|x^1_0 - x^2_0\|_{C([-h,T]:X)} \\
& \quad + ML(1 + L_1T)\int_0^T\|x^1_\tau - x^2_\tau\|_{C([-h,T]:X)} d\tau ds.
\end{aligned}
\]

From 2.7 and (H3)–(H4),

\[
\|x^1_t(\theta) - x^2_t(\theta)\| \leq \|\phi_1 - \phi_2\|_{C([-h,0]:X)} + K\|x^1_0 - x^2_0\|_{C([-h,T]:X)}.
\]

Thus

\[
\begin{aligned}
\|x^1_t - x^2_t\|_{C([-h,T]:X)} & = \sup_{-h \leq \theta \leq 0} \|x^1_t(\theta) - x^2_t(\theta)\| \\
& \leq \|\phi_1 - \phi_2\|_{C([-h,0]:X)} + K\|x^1_0 - x^2_0\|_{C([-h,T]:X)}.
\end{aligned}
\]
Since $M \geq 1$ and $MK < 1$, then 2.8 and 2.9 imply that
\[ \|x_t^1 - x_t^2\|_{C([-h,T]:X)} \leq \frac{M}{1 - MK} \|\phi_1 - \phi_2\|_{C([-h,0]:X)} + \frac{ML(1 + L_1 T)}{1 - MK} \int_0^T \|x_s^1 - x_s^2\|_{C([-h,T]:X)} ds. \]

By Gronwall's inequality,
\[ \|x_t^1 - x_t^2\|_{C([-h,T]:X)} \leq \frac{M}{1 - MK} \|\phi_1 - \phi_2\|_{C([-h,0]:X)} \exp \left( \frac{ML(1 + L_1 T)T}{1 - MK} \right). \quad \square \]

**Remark 2.1.** Let $0 < t_1 < \cdots < t_p \leq T$ ($p \in N$), Theorems 2.1 and 2.2 can be employed the following $g$ defined by
\[ g(x_{t_1}, \cdots, x_{t_p})(s) = \sum_{k=1}^p c_k x(t_k + s), \]
where $x \in C([-h,T]:X)$, $s \in [-h,0]$ and $c_k$ ($k = 1, \cdots, p$) is constant satisfying
\[ (2.10) \quad M \left\{ \sum_{k=1}^p |c_k| + LT(1 + L_1 T) \right\} < 1. \]

**Remark 2.2.** Let $0 < t_1 < \cdots < t_p$ and $\epsilon_k$ ($k = 1, \cdots, p$) is constant such that $0 < t_1 - \epsilon_1$ and $t_{k-1} < t_k - \epsilon_k$ ($k = 2, \cdots, p$). If the nonlinear function $g$ is defined by
\[ g(x_{t_1}, \cdots, x_{t_p})(s) = \sum_{k=1}^p \frac{c_k}{\epsilon_k} \int_{t_k - \epsilon_k}^{t_k} x(\tau + s) d\tau, \]
where $x \in C([-h,T]:X)$, $s \in [-h,0]$ and $c_k$ ($k = 1, \cdots, p$) is constant satisfying 2.10. For $s \in [-h,0]$, since
\[ \|g(x_{t_1}, \cdots, x_{t_p})(s) - g(y_{t_1}, \cdots, y_{t_p})(s)\| = \left\| \sum_{k=1}^p \frac{c_k}{\epsilon_k} \int_{t_k - \epsilon_k}^{t_k} (x(\tau + s) - y(\tau + s)) d\tau \right\| \leq \left( \sum_{k=1}^p |c_k| \right) \|x_t - y_t\|_{C([-h,T]:X)}, \]
Theorems 2.1 and 2.2 can be employed the function $g$.

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REFERENCES


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