EQUIVALENT CONDITIONS FOR
A DIRECT INJECTIVE MODULE

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ABSTRACT. The purpose of this paper is to find the necessary and sufficient conditions for a module to be a direct injective module. Moreover, we focus on the possibility that a direct injective module can be related with arbitrary module and Hom functor like an injective module.

1. INTRODUCTION

Throughout, $R$ is a ring with unity, all modules are unitary $R$-modules and all maps are $R$-maps. A module $M$ is said to be direct injective if given a direct summand $N$ of $M$ with inclusion $i : N \rightarrow M$ and any monomorphism $f : N \rightarrow M$, there exists an endomorphism $g$ of an $R$-module $M$ such that the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{g} & M \\
\downarrow{i} & & \\
O & \xrightarrow{f} & N
\end{array}
\]

commutes, i.e., $g \circ f = i$. The concept of a direct injective module as a generalization of a quasi-injective module was introduced by Nicholson [2] in 1976. Xue [3] showed the characterizations of hereditary rings and semisimple rings by using direct projective modules and direct injective modules.

In this paper, we obtain the necessary and sufficient conditions for a module to be a direct injective module. As the results of it, we obtain equivalent conditions for a module to be a direct injective module.

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2. **Main Results**

**Theorem 2.1.** A module $M$ is direct injective if and only if for any direct summand $N$ of $M$ and a monomorphism $f : N \to A$, there exists a map $f' : A \to N$ such that $f' \circ f = I_N$, where $A$ is a submodule of $M$.

**Proof.** Assume that $M$ is a direct injective module. Let $N$ be a direct summand of $M$, $A$ be a submodule of $M$, and $g : A \to M$ be a monomorphism. Then for each monomorphism $f : N \to A$, there exists a map $h \in \text{End}(M)$ which completes the following diagram

\[
\begin{array}{ccc}
M & \overset{i}{\rightarrow} & N \overset{f}{\rightarrow} A \overset{g}{\rightarrow} M \\
\downarrow{h} & & \downarrow{h} \\
\end{array}
\]

as a commutative diagram, i.e., $h \circ g \circ f = i$.

Let $p : M \to N$ be the projection map and define a map $k : A \to M$ by $k = h \circ g$.

Let $f' = p \circ k$. Then we have

\[
f' \circ f = (p \circ k) \circ f = p \circ (h \circ g) \circ f = p \circ (h \circ g \circ f) = p \circ i = I_N,
\]

i.e.,

\[
\begin{array}{ccc}
M & \overset{i}{\rightarrow} & N \overset{f}{\leftarrow} A \overset{g}{\rightarrow} M. \\
\downarrow{h} & \overset{f'}{\leftarrow} & \end{array}
\]

Hence, $f' \circ f = I_N$.

Conversely, assume that $N$ is a direct summand of a module $M$ and $i : N \to M$ be the inclusion map and let $f' : M \to N$ be a map such that $f' \circ f = I_N$, i.e.,
\[ \begin{array}{cccc}
O & \longrightarrow & N & \xrightarrow{f} & M. \\
\uparrow & & \downarrow^{f'} & & \downarrow^{f} \\
\end{array} \]

Define a map \( h : M \longrightarrow M \) by \( h = i \circ f' \), then
\[ h \circ f = i \circ f' \circ f = i \circ I_N = i. \]

So we have the diagram
\[ \begin{array}{cccc}
M & \xleftarrow{i} & \quad & \downarrow^{h} \\
& & \downarrow & \\
O & \longrightarrow & N & \xrightarrow{f} & M \\
\end{array} \]

as a commutative diagram, i.e., \( h \circ f = i \). Hence, \( M \) is a direct injective module. This completes the proof. \( \square \)

**Theorem 2.2.** A module \( M \) is direct injective if and only if, given any direct summand \( N \) of \( M \) and any map \( g : N \longrightarrow M \), for each monomorphism \( f : N \longrightarrow M \), there exists a map \( h \in \text{End}(M) \) such that the following diagram
\[ \begin{array}{cccc}
M & \xleftarrow{g} & \quad & \downarrow^{h} \\
& & \downarrow & \\
O & \longrightarrow & N & \xrightarrow{f} & M \\
\end{array} \]
commutes, i.e., \( h \circ f = g \).

**Proof.** Assume that a module \( M \) is direct injective. Then by Theorem 2.1, there exists a map \( f' : M \longrightarrow N \) such that \( f' \circ f = I_N \). Therefore define a map \( h : M \longrightarrow M \) by \( h = g \circ f' \). Then
\[ h \circ f = g \circ f' \circ f = g \circ I_N = g. \]
Hence there exists a map \( h \in \text{Hom}(M) \) such that \( h \circ f = g \).

Conversely, suppose that given any direct summand \( N \) of \( M \) and any map \( g : N \longrightarrow M \), for each monomorphism \( f : N \longrightarrow M \), there exists a map \( h \in \text{End}(M) \)
such that \( h \circ f = g \). i.e., the diagram

\[
\begin{array}{ccc}
M & \downarrow g & \\
O & \xrightarrow{h} & N \\
& \xrightarrow{f} & M
\end{array}
\]

is commutative. If we take an inclusion map \( i : N \to M \) instead of arbitrary map \( g : N \to M \), then we have an immediate consequence from the above assumption and the definition of direct injective module.

\[\blacksquare\]

**Theorem 2.3.** A module \( M \) is direct injective if and only if, given an exact sequence

\[
O \to A \to B \to C \to O
\]

for a direct summand \( A \) of \( M \) and submodules \( B, C \) of \( M \), then

\[
O \to \text{Hom}(C, M) \xrightarrow{F_{\beta}} \text{Hom}(B, M) \xrightarrow{F_{\alpha}} \text{Hom}(A, M) \to O
\]

is exact sequence.

**Proof.** Assume that a module \( M \) is direct injective and let \( A \) be a direct summand of \( M \) and \( B, C \) be submodules of \( M \). Since \( \text{Hom}(, M) \) is a left exact contravariant functor, for an exact sequence

\[
O \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to O,
\]

to prove that

\[
O \to \text{Hom}(C, M) \xrightarrow{F_{\beta}} \text{Hom}(B, M) \xrightarrow{F_{\alpha}} \text{Hom}(A, M) \to O
\]

is exact sequence, it is enough to show that \( F_{\alpha} \) is an epimorphism. For arbitrary map \( g \in \text{Hom}(A, M) \), since a module \( M \) is direct injective, by Theorem 2.1, we have \( \alpha' : B \to A \) with \( \alpha' \circ \alpha = I_A \). Define a map \( f : B \to M \) by \( f = g \circ \alpha' \). Then there is a map \( f \in \text{Hom}(B, M) \) such that

\[F_{\alpha}(f) = f \circ \alpha = g \circ \alpha' \circ \alpha = g \in \text{Hom}(A, M)\]

The diagram
commutes. Hence $F\alpha$ is an epimorphism.

Conversely, let $A$ be a direct summand of $M$ and $B$, $C$ be submodules of $M$. Assume that for an exact sequence

$$O \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow O,$$

the sequence

$$O \longrightarrow \text{Hom}(C, M) \xrightarrow{F\beta} \text{Hom}(B, M) \xrightarrow{F\alpha} \text{Hom}(A, M) \longrightarrow O$$

is exact. If we take $M$ instead of $B$ from the above assumption, then for an arbitrary map $g : A \longrightarrow M$ and each monomorphism $\alpha : A \longrightarrow M$, there exists a map $h \in \text{End}(M)$ such that $h \circ \alpha = g$, i.e.,

$$\begin{array}{c}
O \longrightarrow A \xrightarrow{\alpha} M.
\end{array}$$

Hence by Theorem 2.2, the module $M$ is direct injective. \hfill \Box

From Theorem 2.3, we know that direct injective modules can be defined by Hom functor with some conditions.

**Theorem 2.4.** A module $M$ is direct injective if and only if for each monomorphism $f : N \longrightarrow M$, a direct summand $N$ and a module $K$, any map $g : N \longrightarrow K$ can be extended to a map $h : M \longrightarrow K$ which completes the diagram

$$\begin{array}{c}
K \\
\text{or} \\
O \longrightarrow N \xrightarrow{f} M
\end{array}$$

commutes, i.e., $h \circ f = g$.\hfill \Box
Proof. ($\Rightarrow$) Suppose that $M$ is a direct injective module. Let $N$ be a direct summand of $M$. Then by Theorem 2.1, for each monomorphism $f : N \rightarrow M$, there is a map $f' : M \rightarrow N$ such that

$$f' \circ f = I_N.$$ 

For an arbitrary module $K$ and any map $g : N \rightarrow K$, define a map $h : M \rightarrow K$ by

$$h = g \circ f'.$$

Then

$$h \circ f = g \circ f' \circ f = g \circ I_N = g.$$ 

Therefore, there is a map $h : M \rightarrow K$ such that

$$h \circ f = g.$$ 

Hence, we have the diagram

\[
\begin{array}{ccc}
K & \overset{h}{\longrightarrow} & M \\
\downarrow{g} & & \downarrow{f'} \\
O & \longrightarrow & N & \overset{f}{\longrightarrow} & M
\end{array}
\]

as a commutative diagram.

($\Leftarrow$) The converse proof is trivial by taking inclusion map $i$ instead of arbitrary map $g$. \hfill \square

**Theorem 2.5.** For a module $M$, let $A$ be a direct summand of $M$ and $B$, $C$ be submodules of $M$. $M$ is a direct injective module if and only if, given an exact sequence

\[
O \longrightarrow A \overset{\alpha}{\longrightarrow} B \overset{\beta}{\longrightarrow} C \longrightarrow O
\]

we have

\[
O \longrightarrow \text{Hom}(C, K) \overset{F\beta}{\longrightarrow} \text{Hom}(B, K) \overset{F\alpha}{\longrightarrow} \text{Hom}(A, K) \longrightarrow O
\]

as an exact sequence, for any module $K$.

Proof. Assume that $M$ is a direct injective module. Let $A$ be a direct summand of $M$ and $B$, $C$ be submodules of $M$. Since $\text{Hom}(\cdot, K)$ is a left exact contravariant functor, in order to prove that
\[ \begin{array}{cccccc}
O & \longrightarrow & \text{Hom}(C, K) & F_\beta \longrightarrow & \text{Hom}(B, K) & F_\alpha \longrightarrow & \text{Hom}(A, K) & \longrightarrow & O \\
& & \downarrow F_\beta & \downarrow F_\alpha & & \downarrow F_\alpha & \downarrow F_\alpha & & \\
& & \text{Hom}(B, K) & \longrightarrow & \text{Hom}(A, K) & \longrightarrow & O \\
\end{array} \]

is an exact sequence for an exact sequence
\[ \begin{array}{cccccc}
O & \longrightarrow & A & \alpha \longrightarrow & B & \beta \longrightarrow & C & \longrightarrow & O \\
& & \downarrow \alpha & & \downarrow \beta & & \downarrow \beta & & \\
& & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & O \\
\end{array} \]

and an arbitrary module \( K \), we will show that \( F_\alpha \) is an epimorphism, i.e., we want to complete the diagram

\[ \begin{array}{ccc}
K & \longrightarrow & O \\
& \downarrow g & \downarrow \alpha \\
O & \longrightarrow & A \\
& \downarrow \alpha & \downarrow \alpha \\
& \longrightarrow & B \\
\end{array} \]

for an arbitrary map \( g \in \text{Hom}(A, K) \). Since \( M \) is a direct injective module, we have \( \alpha' : B \longrightarrow A \) such that \( \alpha' \circ \alpha = I_A \). Define a map \( f : B \longrightarrow K \) by
\[ f = g \circ \alpha'. \]

Then
\[ F_\alpha(f) = f \circ \alpha = g \circ \alpha' \circ \alpha = g \in \text{Hom}(A, K). \]

Hence, we have the diagram

\[ \begin{array}{ccc}
K & \longrightarrow & O \\
& \downarrow g & \downarrow \alpha \\
O & \longrightarrow & A \\
& \downarrow \alpha & \downarrow \alpha' \\
& \longrightarrow & B \\
\end{array} \]

as a commutative diagram. Therefore, \( F_\alpha \) is an epimorphism.

Conversely, let \( A \) be a direct summand of \( M \) and \( C \) be a submodule of \( M \). Assume that for an exact sequence
\[ \begin{array}{cccccc}
O & \longrightarrow & A & \alpha \longrightarrow & M & \beta \longrightarrow & C & \longrightarrow & O \\
& & \downarrow \alpha & & \downarrow \beta & & \downarrow \beta & & \\
& & A & \longrightarrow & M & \longrightarrow & C & \longrightarrow & O \\
\end{array} \]

and an arbitrary module \( K \), the sequence
\[ \begin{array}{cccccc}
O & \longrightarrow & \text{Hom}(C, K) & F_\beta \longrightarrow & \text{Hom}(M, K) & F_\alpha \longrightarrow & \text{Hom}(A, K) & \longrightarrow & O \\
& & \downarrow F_\beta & \downarrow F_\alpha & \downarrow F_\alpha & \downarrow F_\alpha & \downarrow F_\alpha & \downarrow F_\alpha & \\
& & \text{Hom}(M, K) & \longrightarrow & \text{Hom}(A, K) & \longrightarrow & O \\
\end{array} \]

is an exact sequence, i.e., we have the diagram
commutative. Then for an arbitrary map \( g : A \rightarrow K \) and each monomorphism \( \alpha : A \rightarrow M \), there exists a map \( h : M \rightarrow K \) which completes the following diagram

\[
\begin{array}{ccc}
K & \xrightarrow{h} & M \\
\xrightarrow{g} & \downarrow & \downarrow \\
O & \xrightarrow{\alpha} & A \\
\xrightarrow{} & & \xrightarrow{\alpha'}
\end{array}
\]

as a commutative diagram, i.e., \( h \circ \alpha = g \). This implies that \( M \) is a direct injective module. This completes the proof. \( \square \)

**Theorem 2.6.** \( M \) is a direct injective module if and only if for every pair of direct summands \( A, B \) of \( M \), the injection \( i : A \rightarrow M \), and every monomorphism \( f : A \rightarrow B \), there exists a map \( g : B \rightarrow M \) which completes the following diagram

\[
\begin{array}{ccc}
M & \xrightarrow{g} & O \\
\xleftarrow{i} & \downarrow & \downarrow \circ \alpha \\
A & \xrightarrow{f} & B
\end{array}
\]

as a commutative diagram, i.e., \( g \circ f = i \).

**Proof.** (\( \Rightarrow \)) Suppose that \( M \) is a direct injective module. Then for the injection maps \( i : A \rightarrow M \), \( i' : B \rightarrow M \) and each monomorphism \( f : A \rightarrow B \), there exists a map \( h \in \text{End}(M) \) which completes the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{h} & A \\
\xleftarrow{g} & \downarrow & \downarrow \circ \alpha \\
O & \xrightarrow{i} & B \\
\xrightarrow{} & & \xrightarrow{i'}
\end{array}
\]

commutes, i.e., \( h \circ i' \circ f = i \). Let

\[ g = h \circ i' \]
then we have $g \circ f = i$. Therefore, there exists a map $g : B \rightarrow M$ such that

$$g \circ f = i.$$ 

This completes the proof of “$\Rightarrow$” part.

$(\Leftarrow)$ The converse case is omitted since it is the same as the “$\Rightarrow$” part by replacing $B$ with $M$. \hfill \Box

Through the above long proofs, we know that from Theorem 2.1 to Theorem 2.6, they are equivalent. We want to focus on the possibility that a direct injective module can be related with arbitrary module and Hom functor like an injective module.

REFERENCES


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