SCORE SEQUENCES OF HYPERTOURNAMENT MATRICES

YOUNGMEE KOH AND SANGWOOK REE

ABSTRACT. A \( k \)-hypertournament is a complete \( k \)-hypergraph with all \( k \)-edges endowed with orientations, i.e., orderings of the vertices in the edges. The incidence matrix associated with a \( k \)-hypertournament is called a \( k \)-hypertournament matrix, where each row stands for a vertex of the hypertournament. Some properties of the hypertournament matrices are investigated.

The sequences of the numbers of \( 1 \)'s and \( -1 \)'s of rows of a \( k \)-hypertournament matrix are respectively called the score sequence (resp. losing score sequence) of the matrix and so of the corresponding hypertournament. A necessary and sufficient condition for a sequence to be the score sequence (resp. the losing score sequence) of a \( k \)-hypertournament is proved.

1. INTRODUCTION

A tournament is a complete directed graph. It is well known (cf. Reid [6]) that every tournament contains a hamilton path and that a tournament is strongly connected if and only if every vertex is contained in cycles of all possible lengths. Together with these facts, one of the fundamental and well-known facts on tournaments is Landau's theorem. When we call the outdegree of a vertex of a tournament the score of the vertex, Landau's theorem shows a condition for the existence of a tournament whose vertices have the scores as given ones.

Hypergraphs are a generalization of graphs. While edges of a graph are pairs of vertices of the graph, edges of a hypergraph are subsets of the vertex set, consisting of at least two vertices. We call hypergraphs only having edges consisting of \( k \) vertices \( k \)-hypergraphs. A \( k \)-hypertournament is a complete oriented \( k \)-hypergraph.

Instead of scores of vertices in a tournament, Zhou, Yao and Zhang [7] considered scores and losing scores of vertices in a \( k \)-hypertournament, and derived a result

\footnotesize
Received by the editors September 26, 2001, and in revised form November 27, 2001.
2000 Mathematics Subject Classification. 05C50.
Key words and phrases. hypertournament, score sequence, hypertournament matrices.
This work was supported by grant No. 2001-1-10200-002-2 from the Basic Research Program of the Korea Science and Engineering Foundation.

analogous to Landau’s theorem. Landau’s theorem has attracted quite a bit of attention providing a dozen of different proofs. Included among these proofs are Fulkerson’s, Ryser’s, Brauer, Gentry and Shaw’s, Landau’s, and Bang-Sharp’s. The proof of Bang and Sharp [2] might be more noteworthy than others, as Erdős called it the proof in the Book.

In this paper, we prove the result of Zhou, Yau and Zhang [7] in the same way as Bang and Sharp’s proof. Also, we define k-hypertournament matrices as the incidence matrices of k-hypertournaments and obtain some basic properties of such matrices.

2. HYPERTOURNAMENT MATRICES

Let $V$ be the set of $n$ vertices $v_1, v_2, \ldots, v_n$. For fixed $k$, $2 \leq k \leq n - 1$, a subset of $k$ vertices is called a $k$-edge. The set of the vertices with $k$-edges defines a $k$-hypergraph (cf. Berge [3]). Especially, the $k$-hypergraph containing all of the $k$-edges on $V$ is said to be complete.

A $k$-edge endowed with an orientation is called a $k$-arc. A $k$-hypertournament $H$ is defined as the pair of the vertex set $V$ and a set $A_k$ of $k$-arcs on $V$, where there is exactly one $k$-arc for every possible $k$-edge, i.e., $H = (V, A_k)$. So, a $k$-hypertournament is the complete $k$-hypergraph with all $k$-edges given orientations, i.e., orderings of $k$ vertices in $k$-edges. Since a $k$-arc consists of $k$ vertices, the number of $k$-arcs in $A_k$ is $\binom{n}{k}$, and so it is written as $A_k = \{e_1, e_2, \ldots, e_{\binom{n}{k}}\}$. Now we define an $n \times \binom{n}{k}$ matrix $M = (m_{ij})$ by

$$m_{ij} = \begin{cases} 
1 & \text{if } v_i \text{ is in } e_j \text{ and } v_i \text{ is not the last element of } e_j, \\
-1 & \text{if } v_i \text{ is in } e_j \text{ and } v_i \text{ is the last element of } e_j, \\
0 & \text{if } v_i \text{ is not in } e_j.
\end{cases}$$

The matrix $M$ is the incidence matrix of a $k$-hypertournament $H$, and is called a $k$-hypertournament matrix. Each row and column of $M$ corresponds to a vertex and a $k$-arc of $H$, respectively. Note that our $k$-hypertournament matrices are defined in transposed form of the incidence matrices introduced in Bang and Sharp [4].

In a $k$-hypertournament on the vertices $v_1, \ldots, v_n$, the score $s_i$ of a vertex $v_i$ is defined to be the number of $k$-arcs which contain $v_i$ not as the last element and the losing score $r_i$ of $v_i$ to be the number of $k$-arcs containing $v_i$ as the last element. The incidence matrix of a $k$-hypertournament defined in this way, though
not telling the orientation of $k$-arcs, distinguishes the last element of each arc and provides the information on the score and the losing score of every vertex. Note that a $k$-hypertournament matrix corresponds to $\binom{n}{k}(k-1)!$ $k$-hypertournaments. The following are some properties of $k$-hypertournament matrices.

Let $H$ be a $k$-hypertournament on vertices $v_1, \cdots, v_n$ and $M$ the corresponding $k$-hypertournament matrix.

1. The matrix $M$ is a $(1,0,-1)$-matrix, and each row and column respectively corresponds to each vertex and $k$-arc of $H$.
2. Since each column of $M$ stands for a $k$-arc, it contains exactly $k-1$ 1's, one $-1$, and $n-k$ 0's.
3. Let $s_i$ be the score of vertex $v_i$ of $H$. Then $s_i$ is the number of 1's in row $i$ of $M$. Similarly, let $r_i$ be the losing score of $v_i$. Then it is the number of $-1$'s in the $i$th row of $M$. Also, for each $i \in \{1, \cdots, n\}$, $s_i + r_i = \binom{n-1}{k-1}$ is the number of $k$-arcs containing $v_i$, that is, the number of nonzeros in the $i$th row.
4. $\sum_{i=1}^{n} r_i$ is the sum of the losing scores of all vertices. This is equal to the total number of $-1$'s in the matrix $M$, which is equal to the number of columns $\binom{n}{k}$ of $M$ since each column contains one and only one $-1$.
5. The score sum $\sum_{i=1}^{n} s_i$ is the total number of 1's in the matrix $M$, which is $(k-1)$ times the number of columns of $M$, and so $(k-1)\binom{n}{k}$. And $\sum_{i=1}^{n} r_i + \sum_{i=1}^{n} s_i = k \binom{n}{k}$ standing for the number of nonzero entries of $M$.
6. The column sum vector of $M$ is given as the transpose of

\[(1, 1, \cdots, 1)M = (k-2, k-2, \cdots, k-2),\]

i.e., $M^T\mathbf{1} = (k-2)\mathbf{1}$, where $\mathbf{1} = (1, 1, \cdots, 1)^T$.

(7) $M\mathbf{1} = \begin{bmatrix} s_1 - r_1 \\ s_2 - r_2 \\ \vdots \\ s_n - r_n \end{bmatrix}$ is the row sum vector of $M$.

3. Score Sequences

In this section, we employ the idea of Bang and Sharp [2] in their proof of Landau's theorem to prove a necessary and sufficient condition for a nonnegative sequence to
be a score or a losing score sequence for a $k$-hypertournament matrix, and hence for a $k$-hypertournament.

**Proposition 1** (Landau’s Theorem [6]). Given a sequence of nonnegative integers, $0 \leq s_1 \leq s_2 \leq \cdots \leq s_n$, there exists a tournament matrix $M$ such that $s = (s_1, s_2, \cdots, s_n)^T = M1$ if and only if

$$\sum_{i=1}^{l} s_i \geq \binom{l}{2} \quad \text{for } l = 1, \cdots, n,$$

and the equality holds when $l = n$.

Bang and Sharp [2] used Hall’s theorem about systems of distinct representatives for a collection of sets. Given a collection of sets $A_1, \cdots, A_r$, a system of distinct representatives of the collection is defined as a system of distinct elements $a_1, \cdots, a_r \in \bigcup_{i=1}^{r} A_i$ such that $a_i \in A_i$ for all $i = 1, \cdots, r$.

**Lemma 2** (Hall’s Theorem [6]). The set $A_1, \cdots, A_r$ possess a system of distinct representatives if and only if, for each $m \leq r$,

$$|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_m}| \geq m$$

for any $\{i_1, i_2, \cdots, i_m\} \subset \{1, 2, \cdots, r\}$.

**Theorem 3.** A nondecreasing sequence of nonnegative integers $0 \leq r_1 \leq r_2 \leq \cdots \leq r_n$ is the losing score sequence of a $k$-hypertournament $H$ if and only if it satisfies

$$\sum_{i=1}^{l} r_i \geq \binom{l}{k} \quad \text{for } l = 1, 2, \cdots, n,$$

and the equality holds when $l = n$.

**Proof.** We follow the proof of Bang and Sharp [2] (see also Ree and Koh [6]) for score sequences of tournaments.

For the necessity, if $l < k$ then clearly $\sum_{i=1}^{l} r_i \geq \binom{l}{k} = 0$. Assume that $l \geq k$. Let $v_1, v_2, \cdots, v_l$ be the vertices with losing scores $r_1, r_2, \cdots, r_l$. Then the induced $k$-hypertournament $H_l$ on the $l$ vertices $v_1, v_2, \cdots, v_l$ is contained in $H$ whose vertex set is $\{v_1, \cdots, v_n\}$. Each vertex $v_i$ for $1 \leq i \leq l$ is possibly the loser, or the last element, of some arcs containing some of the vertices $v_j$’s for $l + 1 \leq j \leq n$. That is, for $1 \leq i \leq l$, the losing score of $v_i$ in $H_l$ is less than or equal to $r_i$. Since the losing score sum of $H_l$ is $\binom{l}{k}$ by property 4, $\sum_{i=1}^{l} r_i \geq \binom{l}{k}$ and the losing score sum of $H$ is $\sum_{i=1}^{n} r_i = \binom{n}{k}$. 

For the sufficiency, let \( X_1, X_2, \ldots, X_n \) be pairwise disjoint sets with \( |X_i| = r_i \) for \( 1 \leq i \leq n \). Consider these \( n \) sets as the vertices of a complete \( k \)-hypergraph on \( n \) vertices.

Define the orientation of each \( k \)-edge \( \{X_{i_1}, X_{i_2}, \ldots, X_{i_k} \} \) as follows. Form the \( \binom{n}{k} \) set
\[
F = \{ X_{i_1} \cup \cdots \cup X_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n \}.
\]
For \( 1 \leq l \leq \binom{n}{k} \), consider the union of any \( l \) members of \( F \). Let \( I \) denote the set of distinct subscripts of the \( X_i \)’s that make up these \( l \) members of \( F \). Note that the index set \( I \) can be made from at most \( \binom{|I|}{k} \) distinct members of \( F \), i.e., \( l \leq \binom{|I|}{k} \).

From the assumption that \( r_i \)’s are nondecreasing and \( \sum_{i=1}^{|I|} r_i \geq \binom{|I|}{k} \), the following inequalities are derived.
\[
\left| \bigcup_{i \in I} X_i \right| = \sum_{i \in I} |X_i| = \sum_{i \in I} r_i \geq \sum_{i=1}^{|I|} r_i \geq \binom{|I|}{k} \geq l.
\]
So by Hall’s Theorem, \( F \) has \( \binom{n}{k} \) distinct representatives from the union of the members of \( F \) so that each set of \( F \) contains one of the representatives.

Orient a \( k \)-edge \( \{X_{i_1}, \cdots, X_{i_k} \} \) to form a \( k \)-arc so that \( X_{i_j} \) is the last element in this arc if and only if the representative of the member \( X_{i_1} \cup \cdots \cup X_{i_k} \) in \( F \) is in \( X_{i_j} \). Since both of the number of representatives and \( |X_1 \cup \cdots \cup X_n| = \sum_{i=1}^n |X_i| = \sum_{i=1}^n r_i \) are \( \binom{n}{k} \), each element of \( X_1 \cup \cdots \cup X_n \) appears exactly once as a representative, i.e., the losing score of \( X_i \) is \( |X_i| = r_i \). Hence, we obtain a \( k \)-hypertournament with losing score sequence \( 0 \leq r_1 \leq r_2 \leq \cdots \leq r_n \).

**Corollary 4.** A nonincreasing sequence of nonnegative integers \( s_1 \geq s_2 \geq \cdots \geq s_n \geq 0 \) is a score sequence of a \( k \)-hypertournament \( H \) if and only if it satisfies
\[
\sum_{i=1}^l s_i \leq l \binom{n-1}{k-1} - \binom{l}{k} \quad \text{for } l = 1, 2, \cdots, n,
\]
and the equality holds when \( l = n \).

If we arrange the score sequence in nondecreasing order, then we obtain the same inequalities for \( s_i \) as in Zhou, Yao and Zhang [7]:
\[
\sum_{i=1}^l s_i \geq l \binom{n-1}{k-1} + \binom{n-l}{k} - \binom{n}{k} \quad \text{for } l = 1, 2, \cdots, n,
\]
and the equality holds when \( l = n \).
Corollary 5. Sequences $0 \leq s_1 \leq s_2 \leq \cdots \leq s_n$ and $r_1 \geq r_2 \geq \cdots \geq r_n \geq 0$ are the score and losing score sequences of a $k$-hypertournament if and only if they satisfy

$$s_i + r_i = \binom{n-1}{k-1}, \quad \sum_{i=1}^{l} s_i \geq l \binom{n-1}{k-1} + \binom{n-l}{k} - \binom{n}{k}\text{ and } \sum_{i=1}^{l} r_i \leq \binom{n}{k} - \binom{n-l}{k}$$

for $1 \leq l \leq n$, and the equalities hold when $l = n$.

Proof. Let $\tilde{s}_j = s_{n-j+1}$ and $\tilde{r}_j = r_{n-j+1}$. Then the sequences $\tilde{s}_j$ and $\tilde{r}_j$ satisfy the conditions in Theorem 3 and Corollary 4. So using $k \binom{n}{k} = n \binom{n-1}{k-1}$, we have

$$\sum_{j=1}^{l} \tilde{s}_j = \sum_{j=1}^{l} \tilde{s}_{n-j+1} = (k-1) \binom{n}{k} - \sum_{i=1}^{n-l} \tilde{s}_i \geq n \binom{n-1}{k-1} - \binom{n}{k} - (n-l) \binom{n-1}{k-1} + \binom{n-l}{k} = l \binom{n-1}{k-1} + \binom{n-l}{k} - \binom{n}{k}$$

and

$$\sum_{j=1}^{l} \tilde{r}_j = \sum_{j=1}^{l} \tilde{r}_{n-j+1} = \binom{n}{k} - \sum_{i=1}^{n-l} \tilde{r}_i \leq \binom{n}{k} - \binom{n-l}{k},$$

where $1 \leq l \leq n$, and equalities hold if $l = n$. \hfill \Box

Given a nonincreasing sequence of integers, we also get a condition for the existence of a $k$-hypertournament matrix having the given sequence as the elements of its row sum vector.

Corollary 6. A nonincreasing sequence of integers \( \{t_i | i = 1, 2, \cdots, n\} \) is the elements of the row sum vector of a $k$-hypertournament matrix $M$ on $n$ vertices if and only if $t_i$ has the same parity as that of $\binom{n-1}{k-1}$ for all $i = 1, 2, \cdots, n$, and

$$\sum_{i=1}^{l} t_i \leq l \binom{n-1}{k-1} - 2 \binom{l}{k} \text{ for } l = 1, 2, \cdots, n,$$

and the equality holds when $l = n$, i.e.,
\[ \sum_{i=1}^{n} t_i = n \binom{n-1}{k-1} - 2 \binom{n}{k} = (k-2) \binom{n}{k}. \]

REFERENCES


(Y. Koh) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SUWON, SUWON P. O. BOX 77, GYEONGGI-DO, 440-600, KOREA

E-mail address: ynkoh@mail.suwon.ac.kr

(S. Ree) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SUWON, SUWON P. O. BOX 77, GYEONGGI-DO, 440-600, KOREA

E-mail address: swee@mail.suwon.ac.kr