The Korean Communications
in Statistics Vol. 8, No. 3, 2001
pp. 859-866

A Note On L_1 Strongly Consistent Wavelet Density Estimator for the Deconvolution Problem[1)]

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Abstract

The problem of wavelet density estimation is studied when the sample observations are contaminated with random noise. In this paper a linear wavelet estimator based on Meyer-type wavelets is shown to be L_1 strongly consistent for $f(x)$ with bounded support when Fourier transform of random noise has polynomial descent or exponential descent.

Keywords : Consistency; deconvolution; Meyer wavelet.

1. Introduction

Let X and Z be independent random variables with density functions $f(x)$ and $q(z)$, respectively, where $f(x)$ is unknown and $q(z)$ is known. One observes a sample of random variables $Y_i = X_i + Z_i$, $i=1,2,\cdots,n$. The objective is to estimate the density function $f(x)$ where $g(y)$ is the convolution of $f(x)$ and $q(z)$, $g(y) = \int_{-\infty}^{\infty} f(y-z)\, q(z)\, dz$.

The problem of measurements being contaminated with noise exists in many different fields(see, for example, Louis(1991), Zhang(1992)). The most popular approach to the problem was to estimate $f(x)$ by a kernel estimator and Fourier transform (see, for example, Carroll and Hall (1988), Taylor and Zhang(1990), Fan(1991)). Fan(1991) proved that the estimators of $f(x)$ are asymptotically optimal pointwise and globally if the Fourier transform of the kernel has bounded support.

The present paper deals with estimation of a deconvolution density using a wavelet decomposition. The underlying idea is to present $f(x)$ via a wavelet expansion and then to estimate the coefficients using a deconvolution algorithm. Wavelet methods, introduced to

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statistics by the work of Donoho and Johnstone in early 90's, show remarkable potential in nonparametric function estimation(see, for example, Donoho, Johnstone, Kerkyacharian and Picard(1995,1996)). There are several important families of wavelets(for example, Haar's wavelets, Meyer's wavelets, Franklin's wavelets, Daubechies' compactly supported wavelets). In this work we consider a wavelet decomposition based on Meyer-type wavelets rather than on wavelets with bounded support. Meyer-type wavelets allow immediate deconvolution and form a subset of the set of band-limited wavelets, that is, the Fourier transform of the wavelet has bounded support. Pensky and Vidakovic(1999) proposed the estimators based on Meyer-type wavelets to estimate $f(x)$ for two different cases in the well-known Sobolev space H^α : the case when the distribution of the error Z is supersmooth, that is, the Fourier transform $\tilde{q}$ of q has exponential descent, and the case when the distribution of the error Z is ordinary smooth, that is, $\tilde{q}$ has polynomial descent. They showed that, in the case of exponential descent, the linear wavelet estimator (2.7) in Section 2 is asymptotically optimal in the sense that the rate of convergence of the mean integrated squared error can't be improved. However, in the case of polynomial descent, the linear wavelet estimator fails to provide the optimal convergence rate when α is unknown. With the same linear estimator (2.7) Walter(1999) investigated the rates of convergence of the mean squared error under various hypotheses.

Taylor and Zhang(1990) showed the uniform and L_1 strong consistency of the estimator constructed by kernel and Fourier transform in deconvolution density estimation. In this paper the linear wavelet estimator (2.7) in Section 2 is shown to be a L_1 strongly consistent estimator for $f(x)$ with bounded support when Fourier transform $\tilde{q}(\xi)$ of $q(z)$ has polynomial descent or exponential descent. Gamma or double exponential distribution functions satisfy polynomial descent and normal or Cauchy distribution functions satisfy exponential descent.

2. Preliminaries

Throughout this paper we use the notation $\tilde{f}(\xi)$ for the Fourier transform $\int_{-\infty}^{\infty} e^{-i\xi x} f(x)\,dx$ of a function $f(x)$. We assumed that the reader is familiar with the elements of wavelet theory(see, for example, Vidakovic(1999)). Assume that $f(x)$ is square integrable and that $\tilde{q}(\xi)$ does not vanish for real ξ. If $\varphi(x)$ and $\psi(x)$, respectively, are a scaling function and a wavelet generated by an orthonormal multiresolution decomposition of $L^2(-\infty, \infty)$, then for any integer m the density function $f(x)$ allows the following representation:

$$f(x) = \sum_{k\in Z} a_{m,k}\,\varphi_{m,k}(x) + \sum_{k\in Z}\sum_{j=m}^{\infty} b_{j,k}\,\psi_{j,k}(x) \ , \tag{2.1}$$

where $\varphi_{m,k}(x) = 2^{m/2}\varphi(2^m x - k)$ and $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, and the coefficients $a_{m,k}$ and $b_{j,k}$ have the forms

$$a_{m,k} = \int_{-\infty}^{\infty} \varphi_{m,k}(x)\, f(x)\, dx \ , \quad b_{j,k} = \int_{-\infty}^{\infty} \psi_{j,k}(x)\, f(x)\, dx$$

respectively.

A special class of wavelets are band-limited wavelets, the Fourier transform of which have bounded support. In this paper, we shall use a particular type of band-limited wavelet, a Meyer-type wavelet (see Walter(1994, 1999)). Let P be a probability measure with support in $[-\pi/3, \pi/3]$. Define the scaling function $\varphi(x)$ and the wavelet function $\psi(x)$ as the functions whose Fourier transforms are

$$\tilde{\varphi}(\omega) = \left[\int_{\omega-\pi}^{\omega+\pi} dP \right]^{1/2}, \qquad \tilde{\psi}(\omega) = e^{-iw/2}\left[\int_{|\omega|/2-\pi}^{|\omega|-\pi} dP \right]^{1/2},$$

the nonnegative square roots of the integrals. Then $\tilde{\varphi}(\omega)$ and $\tilde{\psi}(\omega)$ both have bounded support: supp $\tilde{\varphi} \subset [-4\pi/3, 4\pi/3]$ and supp $\tilde{\psi} \subset \Omega_1 \cup \Omega_2$ with

$$\Omega_1 = [-8\pi/3, -2\pi/3], \qquad \Omega_2 = [2\pi/3, 8\pi/3].$$

Moreover, $\tilde{\varphi}(\omega) = 1$ if $|\omega| < 2\pi/3$. In this paper we need to ensure that $\varphi(x)$ and $\psi(x)$ have sufficient rate of descent as $|x| \to \infty$. Hence we choose P to be smooth, so that the function $\tilde{\varphi}(\omega)$ and $\tilde{\psi}(\omega)$ are $s \geq 2$ times continuously differentiable on $(-\infty, \infty)$. Since $\tilde{\varphi}(\omega)$ and $\tilde{\psi}(\omega)$ have bounded support, this implies that

$$C_\varphi = sup_x\,[\,|\varphi(x)|(\ |x|^s + 1\ )] < \infty \ , \quad C_\psi = sup_x\,[\,|\psi(x)|(\ |x|^s + 1\ )] < \infty \ .$$

We also assume that

$$\int_{-\infty}^{\infty} \varphi(x)\, dx = 1 \quad \text{and} \quad \sum_{k\in Z} \varphi(x-k) \equiv 1. \tag{2.2}$$

The coefficients $a_{m,k}$ and $b_{j,k}$ can be viewed as mathematical expectations of the functions $u_{m,k}$ and $v_{j,k}$

$$a_{m,k} = \int_{-\infty}^{\infty} u_{m,k}(y)\, g(y)\, dy \ , \quad b_{m,k} = \int_{-\infty}^{\infty} v_{m,k}(y)\, g(y)\, dy, \tag{2.3}$$

provided that $u_{m,k}(y)$ and $v_{m,k}(y)$ are solutions of the following equations:

$$\int_{-\infty}^{\infty} q\,(y-x)\ u_{m,k}(y)\, dy = \varphi_{m,k}(x) \ , \quad \int_{-\infty}^{\infty} q\,(y-x)\ v_{m,k}(y)\, dy = \psi_{m,k}(x). \tag{2.4}$$

Taking the Fourier transform of both sides in (2.4), we obtain $u_{m,k}(x) = 2^{m/2}\, U_m(2^m x - k)$, $v_{j,k}(x) = 2^{j/2}\, V_j(2^j x - k)$, where $U_m(\cdot)$ and $V_j(\cdot)$ are the inverse Fourier transforms of the functions

$$\tilde{U}_m(\omega) = \tilde{\varphi}(\omega)/\ \tilde{q}(-2^m\omega)\ , \qquad \tilde{V}_j(\omega) = \tilde{\psi}(\omega)/\ \tilde{q}(-2^j\omega)\ , \tag{2.5}$$

respectively. Therefore, estimating $a_{m,k}$ and $b_{j,k}$ by

$$\hat{a}_{m,k} = n^{-1}\sum_{i=1}^{n} 2^{m/2}\, U_m(2^m Y_i - k)\ ,\quad \hat{b}_{j,k} = n^{-1}\sum_{i=1}^{n} 2^{j/2}\, V_j(2^j Y_i - k) \tag{2.6}$$

and truncating the series (2.1), we obtain a linear wavelet estimator

$$\hat{f}_n(x) = \sum_{k\in Z} \hat{a}_{m,k}\, \varphi_{m,k}(x)\ . \tag{2.7}$$

In this paper the estimator (2.7) will be shown to be L_1 strongly consistent estimator for $f(x)$ with bounded support function, $supp\ f \subset [-b, b]$.

3. L_1 Strong Consistency

The main result of this paper is Theorem 3.1 which establishes the L_1 strong consistency of the linear wavelet estimator $\hat{f}_n(x)$ to $f(x)$ with bounded support, that is,

$$\int_{-b}^{b} |\ \hat{f}_n(x) - f(x)|\ dx \to 0,\ \ a.s.,\ \text{as}\ \ n\to\infty.$$

The following lemma is needed to prove Theorem 3.1.

Lemma 3.1 (Taylor and Zhang (1990))

(a) If $\{Y_n\}$ are independent random variables with $sup_n\ E\ Y_n^{\ 2} < \infty$, then

$$n^{-(1-\beta)} \sup_{|t|\le n^{\beta}} \left|\sum_{j=1}^{n} D_j(t,\varepsilon)\right| \to 0 \quad a.s.$$

where $D_j(t,\varepsilon) = e^{itY_j}\, I[\,|Y_j| \le j^{\frac{1+\varepsilon}{2}}\,] - E\, e^{itY_j}\, I[\,|Y_j| \le j^{\frac{1+\varepsilon}{2}}\,],\ 0<\beta<\frac{1}{2}\ ,\ \varepsilon>0.$

(b) If $\{Y_n\}$ are independent random variables with $sup_n\ E\ Y_n^{\ 2} < \infty$, then

$$\int_{F_n} |\,g_n(t)\,|\,dt \ \to 0\ \ a.s.$$

implies

$$\int_{F_n} g_n(t)\ \Big|\sum_{j=1}^{n}\big(\ e^{itY_j} - E\, e^{itY_j} - D_j(t,\varepsilon)\big)\Big|\ dt \ \to 0 \quad a.s.$$

where g_n is an integrable function and F_n is a subset of R .

Now the L_1 strong consistency of $\hat{f}_n(x)$ to $f(x)$ will be shown in the following theorem.

Theorem 3.1 Let X and Z be independent random variables and $EY^2 < \infty$, $Y = X + Z$. Assume that (i) $f(x)$ has bounded support, $supp\ f \subset [-b, b]$, (ii) $2^m \to \infty$ as $n \to \infty$, and (iii) $\tilde{q}(\xi)$ does not vanish for all real $\xi \in R$. Then,

$$\int_{-b}^{b} |\hat{f}_n(x) - f(x)|\, dx \to 0,\ a.s.,\ \text{as}\ n \to \infty\ \text{if}\ G(2^{m+2}\pi/3) = o(n^{\beta})$$

for some $0 < \beta < 1/2$ where $G(\cdot)$ is defined as $G(x) = \dfrac{x}{\inf_{|\xi| \le x} |\tilde{q}(\xi)|}$.

proof.

First, observe that from (2.5), (2.6), the Fourier inversion formula and Parseval's identity,

$$\begin{aligned}
\hat{f}_n(x) - f(x) &= \sum_{k\in Z} \hat{a}_{m,k}\, \varphi_{m,k}(x) - f(x) \\
&= \sum_{k\in Z}\left(\frac{1}{n}\sum_{j=1}^{n} 2^{m/2} U_m(2^m Y_j - k)\right)\varphi_{m,k}(x) - \sum_{k\in Z} a_{m,k}\,\varphi_{m,k}(x) - \sum_{k\in Z}\sum_{l=m}^{\infty} b_{l,k}\,\psi_{l,k}(x) \\
&= \sum_{k\in Z}\sum_{j=1}^{n} \frac{2^{m/2}}{2\pi n}\int_{-\infty}^{\infty} \frac{\tilde{\varphi}(\xi)\, e^{i\xi(2^m Y_j - k)}}{\tilde{q}(-2^m\xi)}\, d\xi \times 2^{m/2}\varphi(2^m x - k) \\
&\quad - \sum_{k\in Z}\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2^{-m/2} e^{-i\xi k/2^m}\tilde{\varphi}(\xi/2^m)\,\tilde{g}(-\xi)}{\tilde{q}(-\xi)}\, d\xi \times 2^{m/2}\varphi(2^m x - k) \\
&\quad - \sum_{k\in Z}\sum_{l=m}^{\infty} b_{l,k}\,\psi_{l,k}(x) \\
&= \sum_{k\in Z}\varphi(2^m x - k)\left\{\frac{1}{2\pi n}\int_{-\infty}^{\infty}\frac{2^m\tilde{\varphi}(\xi)\sum_{j=1}^{n}\left(e^{i\xi 2^m Y_j} - \tilde{g}(-2^m\xi)\right)}{\tilde{q}(-2^m\xi)} \times e^{-ik\xi}\, d\xi\right\} \\
&\quad - \sum_{k\in Z}\sum_{l=m}^{\infty} b_{l,k}\,\psi_{l,k}(x).
\end{aligned} \tag{3.1}$$

For the second part of (3.1) we have , by Theorem 8.4 of Wojtaszczyk(1997),

$$\int_{-b}^{b} \left|\sum_{k\in Z}\sum_{l=m}^{\infty} b_{l,k}\,\psi_{l,k}(x)\right| dx \to 0,\ \text{as}\ m \to \infty.$$

Next, for the first part of (3.1)

$$\begin{aligned}
&\int_{-b}^{b}\left|\sum_{k\in Z}\varphi(2^m x - k)\cdot\frac{2^m}{2\pi n}\int_{-\infty}^{\infty}\frac{\tilde{\varphi}(\xi)\, e^{-ik\xi}\sum_{j=1}^{n}\left(e^{i2^m\xi Y_j} - \tilde{g}(-2^m\xi)\right)}{\tilde{q}(-2^m\xi)}\, d\xi\right| dx \\
&\le \frac{2^m}{2\pi n}\int_{-\infty}^{\infty}\left|\frac{\tilde{\varphi}(\xi)\sum_{j=1}^{n}\left(e^{i2^m\xi Y_j} - \tilde{g}(-2^m\xi)\right)}{\tilde{q}(-2^m\xi)}\right| d\xi \times \int_{-b}^{b}\sum_{k\in Z}|\varphi(2^m x - k)|\, dx.
\end{aligned} \tag{3.2}$$

For the first part of (3.2), notice that $|\text{supp}(\tilde{\varphi}(\xi))| \le \frac{4}{3}\pi$. Then

$$\frac{2^m}{2\pi n}\int_{-\frac{4}{3}\pi}^{\frac{4}{3}\pi}\left|\frac{\tilde{\varphi}(\xi)\sum_{j=1}^{n}(e^{i2^m\xi Y_j}-\tilde{g}(-2^m\xi)-D_j(2^m\xi,\varepsilon)+D_j(2^m\xi,\varepsilon))}{\tilde{q}(-2^m\xi)}\right|d\xi$$

$$(\text{ where } D_j(\xi,\varepsilon) = e^{i\xi Y_j}I[\,|Y_j|\le j^{\frac{1+\varepsilon}{2}}] - E\,e^{i\xi Y_j}I[\,|Y_j|\le j^{\frac{1+\varepsilon}{2}}])$$

$$\le\frac{2^m}{2\pi n}\int_{-\frac{4}{3}\pi}^{\frac{4}{3}\pi}\frac{\left|\sum_{j=1}^{n}(e^{i2^m\xi Y_j}-\tilde{g}(-2^m\xi)-D_j(2^m\xi,\varepsilon))\right|}{|\tilde{q}(-2^m\xi)|}d\xi$$

$$+\frac{2^m}{2\pi n}\int_{-\frac{4}{3}\pi}^{\frac{4}{3}\pi}\frac{\left|\sum_{j=1}^{n}D_j(2^m\xi,\varepsilon)\right|}{|\tilde{q}(-2^m\xi)|}d\xi$$

$$=\frac{1}{2\pi}\int_{-\frac{2^{m+2}}{3}\pi}^{\frac{2^{m+2}}{3}\pi}\frac{1}{n|\tilde{q}(-\xi)|}\left|\sum_{j=1}^{n}(e^{i\xi Y_j}-Ee^{i\xi Y_j}-D_j(\xi,\varepsilon))\right|d\xi$$

$$+\frac{2^m}{2\pi n}\int_{-\frac{4}{3}\pi}^{\frac{4}{3}\pi}\frac{\left|\sum_{j=1}^{n}D_j(2^m\xi,\varepsilon)\right|}{|\tilde{q}(-2^m\xi)|}d\xi. \tag{3.3}$$

Since, by assumption, $G(2^{m+2}\pi/3) = o(n^\beta)$ for some $0<\beta<1/2$,

$$\int_{-\frac{2^{m+2}}{3}\pi}^{\frac{2^{m+2}}{3}\pi}\frac{1}{n|\tilde{q}(-\xi)|}d\xi \le o(n^\beta)\times n^{-1}.$$

Thus, the first part in (3.3) goes to 0 a.s. by applying Lemma 3.1(b). For the second part of (3.3), similarly as the first part,

$$\frac{1}{2\pi n}\int_{-\frac{2^{m+2}}{3}\pi}^{\frac{2^{m+2}}{3}\pi}\frac{\left|\sum_{j=1}^{n}D_j(\xi,\varepsilon)\right|}{|\tilde{q}(-\xi)|}d\xi$$

$$\le\frac{1}{\pi}\,\frac{1}{n^\beta}\,\frac{2^{m+2}\pi/3}{\inf_{|\xi|\le\frac{2^{m+2}}{3}\pi}|\tilde{q}(\xi)|}\,\frac{1}{n^{(1-\beta)}}\sup_{|\xi|\le n^\beta}\left|\sum_{j=1}^{n}D_j(\xi,\varepsilon)\right|. \tag{3.4}$$

Notice that $|\xi|\le\frac{2^{m+2}}{3}\pi\le n^\beta$. Thus (3.4) goes to 0 a.s. by applying Lemma 3.1(a) and $G(2^{m+2}\pi/3) = o(n^\beta)$ for some $0<\beta<1/2$.

For the second part of (3.2), since $\sum_{k\in Z}|\varphi(2^m x-k)| \equiv 1$,

$$\int_{-b}^{b}\sum_{k\in Z}|\varphi(2^m x-k)|\,dx = 2b < \infty.$$

Therefore, (3.2) goes to 0 a.s. and hence Theorem 3.1 is proved.

Remark. When $\tilde{q}(\xi)$ has polynomial descent or exponential descent. the assumption in the theorem, $G(2^{m+2}\pi/3) = o(n^{\beta})$, can also play the role of the condition, $2^m \to \infty$ as $n \to \infty$. When $\tilde{q}(\xi)$ has polynomial descent or exponential descent, we can assume that $\tilde{q}(\xi)$ satisfies the following inequality,

$$|\tilde{q}(\xi)| \geq A(1+\xi^2)^{-\gamma/2} e^{-B|\xi|^{\nu}}, \quad A>0,\ B\geq 0,\ \gamma\geq 0,\ \nu>0.$$

For the polynomial descent, $|\tilde{q}(\xi)| \geq A(1+\xi^2)^{-\gamma/2}$ and hence the assumption in the theorem $G(2^{m+2}\pi/3) = o(n^{\beta})$ implies $2^m = o(n^{\beta/(1+\gamma)})$ where m is the wavelet resolution and n is the sample size. Thus we can take $2^m = O(n^{\delta})$ for some $0 < \delta < \beta/(1+\gamma)$, so that $2^m \to \infty$ as $n \to \infty$. For the exponential descent, $2^m = O((\log n)^{1/\nu})$ and hence $2^m \to \infty$ as $n \to \infty$.

We conclude this section with two examples of Theorem 3.1. One is the case when $\tilde{q}$ has polynomial descent and the other is the case when $\tilde{q}$ has exponential descent.

Example 1. Let $q(x) = 0.5\,\alpha e^{-\alpha|x|}$, the probability density function(p.d.f.) of a double exponential distribution. Then $\tilde{q}(\xi) = (\alpha^2\xi^2+1)^{-1}$ and hence $\tilde{q}$ has polynomial descent. Suppose that $supp\ f \subset [-b, b]$. Then Theorem 3.1 yields

$$\int_{-b}^{b} |\hat{f}_n(x) - f(x)|\,dx \to 0 \quad a.s. \text{ if } 2^m = o(n^{\beta/3}) \text{ for some } 0<\beta<1/2.$$

Example 2. Let $q(x) = (\sqrt{2\pi})^{-1} e^{-x^2/2}$ be the standard normal p.d.f. Then $\tilde{q}(\xi) = e^{-0.5\xi^2}$ and hence $\tilde{q}$ has exponential descent. Suppose that $supp\ f \subset [-b, b]$. Then Theorem 3.1 yields $\int_{-b}^{b} |\hat{f}_n(x) - f(x)|\,dx \to 0 \quad a.s.$ if $2^m = O(\sqrt{\log n})$.

4. Concluding Remarks

In the wavelet framework 2^{-m} plays the role of usual window h_n and hence the wavelet scale m is very important. As the above examples indicate, the choice of the wavelet scale m depends only on the known noise distribution, that is, in the case of polynomial descent $2^m = o(n^{\beta/(1+\gamma)})$ and in the case of exponential descent $2^m = O((\log n)^{1/\nu})$. Thus the smoothness of f does not affect the wavelet resolution m in obtaining the L_1 strong consistency of $\hat{f}_n$ to f when $f(x)$ has bounded support.

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