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Test for the Presence of Seasonality in Time Series Models

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Abstract

Three test statistics are proposed for the presence of seasonality in multiplicative seasonal time series models. Further their common limiting distribution is derived under some assumptions.

Key Words and Phrases:likelihood ratio statistic, limiting distribution, Rao statistic, Wald statistics.

1. Introduction

The class of multiplicative seasonal time series models has been described in detail elsewhere (for example, Box and Jenkins (1976)). It is often of interest to test whether or not a seasonal component is present in such models. In practice the principles of parsimony dictate that the smallest possible number of parameters for adequate model representation be used so that better estimates of the parameters can be obtained. Our purpose here is to provide a test for the presence of a seasonal component in the multiplicative model. Because of the interactions describing the between-year and between-seasonal dependence structure, the autocovariance function for a multiplicative seasonal autoregressive process can be quite complicated.

Our approach is to focus attention on the (seasonal) parameters under test treating the remaining (nonseasonal) parameters as nuisance parameters. Specifically, the hypothesis to be tested is that the seasonal parameters are zero. Results for testing the hypothesis that the nonseasonal parameters are zero (with the seasonal parameter being nuisance parameters) are completely analogous and so will not be presented separately. Basawa, Billard and Srinivasan(1984) have shown that three

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test statistics, the likelihood ratio statistic, the Rao statistic and the Wald statistic (Wald(1943)) have the same limiting distribution in the nonseasonal ARMA model. Lee(1993) derived the limiting distribution of the test statistics in panel time series model. For a sequence of independently and identically distributed random variables, it is well known that these statistics have the same limiting distribution, under the null hypothesis as well as under contiguous alternatives. However, the most important feature of the time series models is that the observations made at different time points are statistically dependent. We will discuss the multiplicative seasonal autoregressive model in Section 2 and three test statistics for the presence of seasonality in Section 3. Finally, We derive the limiting distribution of the statistics in Section4.

2. Seasonal Autoregressive Model

Consider the general multiplicative seasonal autoregressive process of order (p,P), and let X_t be the observation at time t, t=1, ...,n, of the process with

$$\phi(\mathrm{B})\Phi(\mathrm{B}^s)\mathrm{X}_t = \mathrm{e}_t$$

where

$$\phi(\mathrm{B}) = 1 - \phi_1\mathrm{B} - \cdots - \phi_p\mathrm{B}^p \text{ and } \Phi(\mathrm{B}^s) = 1 - \Phi_1\mathrm{B}^s - \cdots - \Phi_P\mathrm{B}^{Ps},$$

and, B is the backward shift operator and s is the periodicity or, equivalently,

$$\begin{aligned} X_t = \; & \phi X_{t-1} + \cdots + \phi_p X_{t-p} + \Phi_1 X_{t-s} + \cdots + \Phi_P X_{t-Ps} \\ & -\phi_1\Phi_1 X_{t-s-1} - \cdots - \phi_p\Phi_P X_{t-Ps-p} + e_t, \quad t = 1, \ldots, n, \end{aligned} \tag{2.1}$$

where $\{e_t\}$ are independent and identically distributed normal variables with mean zero and variance σ^2. (See Box and Jenkins (1976)). It is assumed that all the roots of the characteristic equation

$$(1 - \phi_1\mathrm{B} - \cdots \; \phi_p\mathrm{B}^p)(1 - \Phi_1\mathrm{B}^s - \cdots - \Phi_P\mathrm{B}^{Ps}) = 0$$

lie outside the unit circle, that is,the respective process is stationary. From the stationary condition, X_t can be written as

$$\mathrm{X}_t = \textstyle\sum_{r=0}^{\infty} \Psi_r e_{t-r}$$

with $\sum_{r=0}^{\infty} |\Psi_r| < \infty$. The mean $\mathrm{E}(\mathrm{X}_t)$ and the autocovariance function of lag h, $\gamma(\mathrm{h})$, can be found as

$$\mathrm{E}(\mathrm{X}_t) = \mathrm{E}(\textstyle\sum_{r=0}^{\infty} \Psi_r e_{t-r}) = 0,$$

$$\gamma(\mathrm{h}) = \sigma^2 \textstyle\sum_{\mathrm{r}=0}^{\infty} \Psi_{\mathrm{r}}\Psi_{\mathrm{r+h}}, \mathrm{h} = 0, 1, \cdots. \tag{2.2}$$

In particular, when $\mathrm{p} = \mathrm{P} = 1$, we can show that

$$\Psi_{\mathrm{r}} = \phi^{\mathrm{h}}[\phi^{(\mathrm{u}+1)\mathrm{s}} - \Phi^{(\mathrm{u}+1)}]/(\phi^{\mathrm{s}} - \Phi)$$

where $\mathrm{r} = \mathrm{us} + \mathrm{h}, \mathrm{h} = 0, 1, \ldots, \mathrm{s} - 1$, and $\mathrm{u} = 0, 1, \ldots$, and hence the variance is given by

$$\gamma(0) = \sigma^2 \textstyle\sum_{\mathrm{r}=0}^{\infty} \Psi_{\mathrm{r}}^2 = \frac{\sigma^2(1+\phi^s\Phi)}{(1-\phi^2)(1-\Phi^2)(1-\phi^s\Phi)}.$$

Now, let us consider testing the hypothesis that our model is a pure nonseasonal autoregressive model. The null hypothesis to be tested is $\mathrm{H} : \Phi_{\mathrm{i}} = 0, \mathrm{i} = 1, \ldots, \mathrm{P}$, while the alternative hypothesis is K : at least one Φ_{i} is not zero. Let us write $\underset{\sim}{\phi} = (\phi_1, \ldots, \phi_{\mathrm{p}})$ and $\underset{\sim}{\Phi} = (\Phi_1, \ldots, \Phi_{\mathrm{P}})$.

The likelihood function based on the n observations is given by

$$\begin{aligned} L = \; & (2\pi\sigma^2)^{-n/2} \exp\{-(2\sigma^2)^{-1} \textstyle\sum_{t=1}^{n} (X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} - \Phi_1 X_{t-s} \\ & - \cdots - \Phi_P X_{t-Ps} + \phi_1 \Phi_1 X_{t-s-1} + \cdots + \phi_p \Phi_P X_{t-Ps-p})^2\}. \end{aligned} \tag{2.3}$$

Then, the solutions $(\hat{\underset{\sim}{\phi}}, \hat{\underset{\sim}{\Phi}})$ of $(\partial \ln \mathrm{L}/\partial \underset{\sim}{\phi}) = 0$ and $(\partial \ln L/\partial \underset{\sim}{\Phi}) = 0$, evaluated at $(\underset{\sim}{\phi}, \underset{\sim}{\Phi}) = (\hat{\underset{\sim}{\phi}}, \hat{\underset{\sim}{\Phi}})$, give the unrestricted maximum likelihood estimators $\hat{\underset{\sim}{\phi}}$ and $\hat{\underset{\sim}{\Phi}}$ of $\underset{\sim}{\phi}$ and $\underset{\sim}{\Phi}$, respectively. Likewise, the solution of $(\partial \ln \mathrm{L}/\partial \sigma^2) = 0$, evaluated at $\sigma^2 = \hat{\sigma}^2$, gives the maximum likelihood estimator $\hat{\sigma}^2$ of σ^2, viz.,

$$\begin{aligned} \hat{\sigma}^2 = \; & \mathrm{n}^{-1} \textstyle\sum_{\mathrm{t}=1}^{\mathrm{n}} (\mathrm{X_t} - \hat{\phi}_1 \mathrm{X_{t-1}} - \cdots - \hat{\phi}_{\mathrm{p}} \mathrm{X_{t-p}} - \hat{\Phi}_1 \mathrm{X_{t-s}} - \cdots - \hat{\Phi}_{\mathrm{P}} \mathrm{X_{t-Ps}} \\ & + \hat{\phi}_1 \hat{\Phi}_1 X_{t-s-1} + \cdots + \hat{\phi}_p \hat{\Phi}_P X_{t-Ps-p})^2. \end{aligned}$$

Under the null hypothesis, the likelihood function based on n observations is given by

$$L_H = (2\pi\sigma^2)^{-n/2} \exp\{-(2\sigma^2)^{-1} \textstyle\sum_{t=1}^{n} (X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p})^2\}. \tag{2.4}$$

Then, the solution $\hat{\underset{\sim}{\phi}}_{\mathrm{H}} = (\hat{\phi}_{1\mathrm{H}}, \ldots, \hat{\phi}_{\mathrm{pH}})^{\mathrm{T}}$ of $(\partial \ln \mathrm{L_H}/\partial \underset{\sim}{\phi}) = 0$ is the resticted maximum likelihood estimator $\hat{\underset{\sim}{\phi}}_{\mathrm{H}}$ of $\underset{\sim}{\phi}$ under H. These solutions are given as

$$\hat{\underset{\sim}{\phi}}_H = D_n^{-1} \underset{\sim}{e}_n$$

where $\underset{\sim}{\mathrm{e}}_{\mathrm{n}}$ is the $(\mathrm{p} \times 1)$ vector

$$\underset{\sim}{e}_n^T = [\textstyle\sum_{t=1}^{n} X_t X_{t-1}, \sum_{t=1}^{n} X_t X_{t-2}, \ldots, \sum_{t=1}^{n} X_t X_{t-p}]$$

and $\mathrm{D_n}$ is the $(\mathrm{p} \times \mathrm{p})$ matrix

$$\mathrm{D_n} = \begin{bmatrix} \sum_{\mathrm{t}=1}^{\mathrm{n}} \mathrm{X}_{\mathrm{t}-1}^2 & \sum_{\mathrm{t}=1}^{\mathrm{n}} \mathrm{X_{t-2}X_{t-1}} & \cdots & \sum_{\mathrm{t}=1}^{\mathrm{n}} \mathrm{X_{t-p}X_{t-1}} \\ \sum_{\mathrm{t}=1}^{\mathrm{n}} \mathrm{X_{t-1}X_{t-2}} & \sum_{\mathrm{t}=1}^{\mathrm{n}} \mathrm{X}_{\mathrm{t}-2}^2 & \cdots & \sum_{\mathrm{t}=1}^{\mathrm{n}} \mathrm{X_{t-p}X_{t-2}} \\ \cdot & \cdot & \cdot & \cdot \\ \sum_{\mathrm{t}=1}^{\mathrm{n}} \mathrm{X_{t-1}X_{t-p}} & \sum_{\mathrm{t}=1}^{\mathrm{n}} \mathrm{X_{t-2}X_{t-p}} & \cdots & \sum_{\mathrm{t}=1}^{\mathrm{n}} \mathrm{X}_{\mathrm{t}-\mathrm{p}}^2 \end{bmatrix}.$$

3.Test Statistics

For a single realization, the three test statistics most frequently used in large sample tests, that is, the likelihood ratio, Rao and Wald statistics are derived as follows. Let $\underset{\sim}{\theta} = (\underset{\sim}{\phi}, \underset{\sim}{\Phi})^{\mathrm{T}}$ be the $(\mathrm{p}+\mathrm{P}) \times 1$ vector of parameters.

The Wald statistic is given by

$$Q_1 n = n(\hat{\underset{\sim}{\theta}} - \underset{\sim}{\theta})^T \{DG_n^{-1}(\hat{\underset{\sim}{\theta}})D^T\}^{-1}(\hat{\underset{\sim}{\theta}} - \underset{\sim}{\theta}) \tag{3.1}$$

where $\hat{\underset{\sim}{\theta}}$ is the unrestricted maximum likelihood estimator of $\underset{\sim}{\theta}$ and $\mathrm{D} = (\mathrm{I}, \underset{\sim}{\mathrm{O}})$ is the $\mathrm{p} \times (\mathrm{p}+\mathrm{P})$ matrix with I being the $(\mathrm{p} \times \mathrm{p})$ identity matrix, and where $\mathrm{G_n}(\hat{\underset{\sim}{\theta}})$ is the $(\mathrm{p}+\mathrm{P}) \times (\mathrm{p}+\mathrm{P})$ matrix

$$G_n(\hat{\underset{\sim}{\theta}}) = \quad -\mathrm{n}^{-1}(\partial^2 \mathrm{logL}/\partial \underset{\sim}{\theta_r} \partial \underset{\sim}{\theta_s}) \,|_{\underset{\sim}{\theta}=\hat{\underset{\sim}{\theta}}, \underset{\sim}{\Phi}=\hat{\underset{\sim}{\Phi}}}, \ \mathrm{r}, \mathrm{s} = 1, ..., \mathrm{p}+\mathrm{P}.$$

The likelihood ratio statistic is given by

$$\begin{aligned} Q_2 n \ &= -2\log\{L(\hat{\underset{\sim}{\phi}}_H)/L(\hat{\underset{\sim}{\phi}}, \hat{\underset{\sim}{\Phi}})\} \\ &= \sigma^{-2}\textstyle\sum_{t=1}^{n}(X_t - \hat{\phi}_{H1}X_{t-s} - \cdots - \hat{\phi}_{Hp}X_{t-ps})^2 \\ &\quad -\sigma^{-2}\textstyle\sum_{t=1}^{n}(X_t - \hat{\phi}_1 X_{t-1} - \cdots - \hat{\Phi}_{t-s} - \cdots + \hat{\phi}_1\hat{\Phi}_1 X_{t-s-1} \\ &\quad + \cdots + \hat{\phi}_p\hat{\Phi}_P X_{t-Ps-p})^2. \end{aligned} \tag{3.2}$$

The Rao Statistic is given by

$$Q_3 n = \Delta_n^T(\hat{\underset{\sim}{\phi}}_H)G_n^{-1}(\hat{\underset{\sim}{\phi}}_H)\Delta_n(\hat{\underset{\sim}{\phi}}_H) \tag{3.3}$$

where

$$\Delta_n(\underset{\sim}{\phi}_H) \quad = \mathrm{n}^{-1/2}(\partial \log \mathrm{L}/\partial\theta)|_{\underset{\sim}{\phi}=\hat{\underset{\sim}{\phi}}_H, \underset{\sim}{\Phi}=0},$$

and

$$G_n(\hat{\underset{\sim}{\phi}}_H) \quad = -n^{-1}(\partial^2 \log L/\partial\theta_r\partial\theta_s)|_{\underset{\sim}{\phi}=\hat{\underset{\sim}{\phi}}_H, \underset{\sim}{\Phi}=0}, \ r, s = 1, \cdots, p+P,$$

4.Limiting Distribution of the Test Statistics

In this section, we discuss the common limiting distribution of our three test statistics. We first need the following results.

The score function is given by

$$(\partial \log L/\partial \underset{\sim}{\theta}) = S_n(\underset{\sim}{\theta}). \tag{4.1}$$

Let a solution of the likelihood equation $S_n(\underset{\sim}{\theta}) = 0$ be denoted by $\hat{\underset{\sim}{\theta}}$, and suppose the true parameter is denoted by $\underset{\sim}{\theta}$. Consider the two-term Taylor expansion,

$$S_n(\hat{\underset{\sim}{\theta}}) = S_n(\underset{\sim}{\theta}) - \{W_n(\underset{\sim}{\theta}^*)\}(\hat{\underset{\sim}{\theta}} - \underset{\sim}{\theta})$$

where

$$W_n(\underset{\sim}{\theta}^*) = (-\partial^2 \log L/\partial\theta_r\partial\theta_s)|_{\theta=\theta^*}, \ r, s = 1, \cdots, p+P, \tag{4.2}$$

with $\underset{\sim}{\theta}^*$ in place of $\hat{\underset{\sim}{\theta}}$, and $\underset{\sim}{\theta}^*$ is the row evaluated at possible different points on the line segment between $\hat{\underset{\sim}{\theta}}$ and $\underset{\sim}{\theta}$. Crowder(1976) has shown that $\hat{\underset{\sim}{\theta}}$ is weakly consistent for $\underset{\sim}{\theta}$ under the condition which requires the matrix $-B_n^{-1/2} W_n(\underset{\sim}{\theta}^*)$ to tend to infinity in some sense where B_n is the Fisher information matrix $B_n = E[-W_n(\underset{\sim}{\theta})]$. For asymptotic normality of $\hat{\underset{\sim}{\theta}}$ this condition is augmented by

$$-B_n^{-1} W_n(\underset{\sim}{\theta}) \xrightarrow{p} I_{(p+P)\times(p+P)}$$

where I is the $(p+P) \times (p+P)$ unit matrix. In our model since $G_n(\theta)$ is a $(p+P) \times (p+P)$ matrix and $G_n(\underset{\sim}{\theta}) = n^{-1} W_n(\underset{\sim}{\theta})$,

$$\begin{aligned} -\{E(-W_n(\underset{\sim}{\theta}))\}^{-1} W_n(\underset{\sim}{\theta}) &= \{E(-nG_n(\underset{\sim}{\theta}))\}^{-1}\{-nG_n(\underset{\sim}{\theta})\} \\ &= G^{-1}(\underset{\sim}{\theta}) G_n(\underset{\sim}{\theta}) \xrightarrow{p} I_{(p+P)\times(p+P)}. \end{aligned}$$

Since our model satisfies the above condition given by Crowder(1976), we have

$$\{W_n(\underset{\sim}{\theta}^*) - W_n(\underset{\sim}{\theta})\} \xrightarrow{p} 0 \quad as \quad n \to \infty.$$

Hence, we obtain the relationship

$$\sqrt{n}(\hat{\underset{\sim}{\theta}} - \underset{\sim}{\theta}) \cong \{n^{-1} W_n(\underset{\sim}{\theta})\}^{-1}\{n^{-1/2} S_n(\underset{\sim}{\theta})\} = G_n^{-1}(\underset{\sim}{\theta})(n^{-1/2} S_n(\underset{\sim}{\theta})). \tag{4.3}$$

Lemma 4.1.

Let $\{X_t\}, t = 1, \ldots, n$, be a stationary time series satisfying (2.1). Then,

$$G_n(\underset{\sim}{\theta}) \xrightarrow{p} G(\underset{\sim}{\theta}) \quad as \quad n \to \infty \tag{4.4}$$

where $G_n(\underset{\sim}{\theta})$ is defined in (3.1) and $G(\underset{\sim}{\theta})$ is a nonrandom $(p+P) \times (p+P)$ matrix.

Proof

Let

$$X_{t,m} = \textstyle\sum_{r=0}^{m} \Psi_r e_{t-r}. \tag{4.5}$$

Then, $\{X_{t,m}\}, t = 1, 2, \ldots$, is an m-dependent process.

By the Law of Large Numbers for m-dependent process (of Hoeffding and Robbins,1948), for each $h = 0, 1, \ldots, n^{-1}\sum_{t=1}^{n} X_t X_{t+h} \xrightarrow{P} \gamma(h)$ as $n \to \infty$ where $\gamma(h), h = 0, 1, \ldots$, are defined as in (2.2). Thus, $n^{-1}W_n(\underset{\sim}{\theta}) = G_n(\underset{\sim}{\theta}) \xrightarrow{P} G(\underset{\sim}{\theta})$ as $n \to \infty$ where $G_n(\underset{\sim}{\theta})$ and $G(\underset{\sim}{\theta})$ are defined as before.

Lemma 4.2.
Let $\{X_t\}, t = 1, \ldots, n$, be a stationary time series satisfying (2.1). Then,

$$n^{-1/2}S_n(\underset{\sim}{\theta}) \xrightarrow{d} N_{p+P}(0, G(\underset{\sim}{\theta})) \quad as \quad n \to \infty \tag{4.6}$$

where $S_n(\underset{\sim}{\theta})$ and $G_n(\underset{\sim}{\theta})$ are defined as before.

Proof
Let $X_{t,m}$ be defined as in (4.5) and let

$$\begin{aligned} Y^i_{t,m} &= X_{t-i,m} - \Phi_1 X_{t-s-i,m} - \cdots - \Phi_P X_{t-Ps-i,m} \quad i = 1, \ldots, p, \\ Z^j_{t,m} &= X_{t-js,m} - \phi_1 X_{t-js-1,m} - \cdots - \phi_p X_{t-js-p,m} \quad j = 1, \ldots, P. \end{aligned}$$

Then, $\{Y^i_{t,m}e_t\}$ and $\{Z^j_{t,m}e_t\}$ are (m+1)-dependent processes for $i = 1, \ldots, p$, and $j = 1, \ldots, P$, respectively.

Since $E(e_{t-r}) = 0$ for $r = 0, 1, \ldots, k$, $E(Y^i_{t,m}e_t) = 0$ for $i = 1, \ldots, p$, and $E(Z^j_{t,m}e_t) = 0$ for $j = 1, \ldots, P$, we can show that

$$\begin{aligned} \lim_{m\to\infty} \mathrm{Var}(Y^i_{t,m}e_t) &= \sigma^2 \lim_{m\to\infty} E(X_{t-i,m} - \Phi_1 X_{t-s-i,m} - \ \ldots \ - \Phi_P X_{t-Ps-i,m})^2 \\ &= \sigma^2\{[1 + \sum_{k=1}^{P} \Phi_k^2]\gamma(0) - 2\sum_{k=1}^{P} \Phi_k \gamma(ks) - 2\sum_{k\,<}^{P}\sum_{l}^{P} \Phi_k \Phi_l \gamma((k-l)s)\} \\ &= \sigma^2 g(0) \end{aligned}$$

where $g(i), i = 0, 1, \ldots, p$, is provided by the stationary condition.

Thus, by the Central Limit Theorem for m-dependent processes, we have that

$$n^{-1/2}S_n(\underset{\sim}{\theta}) \xrightarrow{d} N_{p+P}(0, G(\underset{\sim}{\theta})) \quad \text{as} \quad n \to \infty.$$

Theorem 4.1.
Let $\{X_t\}, t = 1, \ldots, n$, be a stationary time series satisfying (2.1). Then,

$$\sqrt{n}(\hat{\underset{\sim}{\theta}} - \underset{\sim}{\theta}) \xrightarrow{d} N_{p+P}(0, \sigma^2 G^{-1}(\underset{\sim}{\theta})) \quad as \quad n \to \infty. \tag{4.7}$$

Proof

From Lemma 4.1 and Lemma 4.2, and using Slutsky's Theorem, we obtain directly that $\sqrt{n}(\hat{\underset{\sim}{\theta}} - \underset{\sim}{\theta}) \xrightarrow{d} N_{p+P}(0, \sigma^2 G^{-1}(\underset{\sim}{\theta}))$ as $n \to \infty$.The proof is completed.

We are now in a position to derive the limiting distribution of our three test statistics. Basawa, Billard and Srinivasan (1984) have shown that the three test statistics have the same limiting distribution under the regularity conditions when the obervations made at different time points are dependent. First, we derive the limiting distribution of the Wald statistic under the null hypothesis H using the unrestricted maximum likelihood estimates only. Consequently, we then describe the common limiting distribution of the three statistics under both the null hypothesis and the alternative hypothesis.

Theorem 4.2.

Let $\{X_t\}, t = 1, \ldots, n$, be a stationary time series with (2.1). Then under the null hypothesis H, the Wald statistics Q_{1n} converges in distribution to a Chi-square distribution with P degrees of freedom.

Proof

Let $\eta_t = \Phi_t$ and let $\hat{\eta}_t = \hat{\Phi}_t$ be the maximum likelihood estimators of η_t and Φ_t, respectively, for $t = 1, \ldots, P$. Then the null hypothesis H becomes $H^* : \eta_t = 0, \quad t = 1, \ldots, P$. Under H^*, we have from theorem 4.1

$$\sqrt{n}(\hat{\underset{\sim}{\eta}} - \underset{\sim}{\eta}) \xrightarrow{d} N_P(0, J(\hat{\underset{\sim}{\theta}})) \quad as \quad n \to \infty \tag{4.8}$$

where $J(\hat{\underset{\sim}{\theta}}) = \sigma^2 H G^{-1}(\hat{\underset{\sim}{\theta}}) H^T$ and H is a $P \times (P + p)$ matrix, $H = (\partial \underset{\sim}{\eta} / \partial \underset{\sim}{\theta})^T = [I, O]$ with I being the $(P \times P)$ identity matrix, and $G(\underset{\sim}{\theta})$ is defined as before.

Thus, by the definition of a Wald statistic, we have, under H^*,

$$Q_{1n} = n(\hat{\underset{\sim}{\eta}} - \underset{\sim}{\eta})^T J^{-1}(\hat{\underset{\sim}{\theta}})(\hat{\underset{\sim}{\eta}} - \underset{\sim}{\eta}) = n\hat{\underset{\sim}{\eta}} J^{-1}(\hat{\underset{\sim}{\theta}})\hat{\underset{\sim}{\eta}} \xrightarrow{d} \chi^2_{(P)} \quad \text{as} \quad n \to \infty,$$

and the proof is completed.

Remarks

The limiting distribution of the three test statistics, $Q_{in}, i = 1, 2, 3$, that is, the likelihood ratio statistic, Rao statistic and Wald statistic are identical (Basawa et al(1984)). The commom limiting distribution under the null hypothesis H is a Chi-square distribution with P degrees of freedom, and under the alternative hypothesis K is a noncentral Chi-square distribution with P degrees of freedom and noncentrality parameter $\delta^2 = h^T J^{-1}(\theta_H) h$ with $h^T = (h_1, \ldots, h_P)^T$ and h_j being real numbers.

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