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Multivariate CUSUM Charts with Correlated Observations [1]

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Abstract

In this article we establish multivariate cumulative sum (CUSUM) control charts based on residual vector with correlated observations. We first find the residual vector and its expectation and variance-covariance matrix and then evaluate the average run length (ARL) of the control charts.

Key Words and Phrases : Multivariate CUSUM Control Charts, Residual Vector, ARL

1. Introduction

In statistical process control, it is usually assumed that the process outputs at different times are i.i.d.. However, for many processes the observations are correlated and control charts for monitoring these processes have recently paid much attention. Many authors suggested fitting on appropriate time series model to the process and using residuals as control statistics on the control charts (Abraham and Kartha(1979), Alwan and Roberts(1988), Harris and Ross(1991) and Longnecker and Ryan(1992)). There are many situations in which the simultaneous control of two or more related quality characteristics is necessary. So, many authors suggested multivariate control charts (Hui(1980), Lowry et al. (1992), Prabhu and Runger(1997)). Our objective is to evaluate the properties of multivariate CUSUM charts when the residual vectors are used to monitor an autocorrelated process. A simple vector time series model is used here to represent the observations from an autocorrelated process.

2. Modeling and Process Level Change

2.1 AR(1) process with measured error vector

In this section, we present a multivariate extension of the correlated structure used by Van-Brackle(1991). Let

$$\underline{x}_t = \underline{\mu}_t + \underline{\varepsilon}_t, \tag{2.1}$$

where $\underline{\chi}_t$ is the observation vector at time t $(t = 1, 2, \cdots)$, $\underline{\mu}_t$ is the process mean vector at time t, and $\underline{\varepsilon}_t$ is the measurement error vector at time t. It is assumed that $\underline{\varepsilon}_t$ is multivariate

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normally distributed with mean vector $\underset{\sim}{0}$ and variance-covariance matrix $\Sigma_{\underset{\sim}{\varepsilon}}$ and independent of the measurement error at any other time. It is also assumed that $\underset{\sim}{\mu}_t$ can be described as a vector first order autoregressive (vector AR(1)) process with process mean vector $\underset{\sim}{\xi}$, that is (Box & Jenkins(1976)),

$$\underset{\sim}{\mu}_t = (I - \Phi)\underset{\sim}{\xi} + \Phi\underset{\sim}{\mu}_{t-1} + \underset{\sim}{\delta}_t, \tag{2.2}$$

where $\underset{\sim}{\delta}_t$, the random shock vector at time t, is assumed to be multivariate normally distributed mean vector $\underset{\sim}{0}$ and variance-covanance matrix $\Sigma_{\underset{\sim}{\delta}}$, independent of measurement error vector, and independent of the random shock vector at any other time and Φ is a matrix of the autoregressive parameter.

The vector AR(1) process observed with measurement error is equivalent to a vector first-order antoregressive - first order moving average (vector ARMA(1,1)) process (Box and Jenkins(1976)). The ARMA(1, 1) process can be written as

$$(I - \Phi B)\underset{\sim}{x}_t = (I - \Phi)\underset{\sim}{\xi} + (I - \Theta B)\underset{\sim}{a}_t, \tag{2.3}$$

where B is a back shift operator such that $B\underset{\sim}{x}_t = \underset{\sim}{x}_{t-1}$, and $\underset{\sim}{a}_t$ is uncorrelated and multivariate normally distributed with mean vector $\underset{\sim}{0}$ and variance - covariance matrix $\Sigma_{\underset{\sim}{a}}$ and Θ is a matrix of the moving average parameter.

2.2 Step change in process level

When the process is in control, i.e. mean vector $\underset{\sim}{\xi} = \underset{\sim}{\xi}_0$, the mean square error forecast vector made at time $t-1$ for t is

$$\hat{\underset{\sim}{x}}_{t-1}(1) = \underset{\sim}{\xi}_0 + \Phi(\underset{\sim}{x}_{t-1} - \underset{\sim}{\xi}_0) - \Theta\underset{\sim}{a}_{t-1}, \tag{2.4}$$

and

$$\begin{aligned} \underset{\sim}{e}_t &= \underset{\sim}{x}_t - \hat{\underset{\sim}{x}}_{t-1}(1) \\ &= \underset{\sim}{x}_t - \underset{\sim}{\xi}_0 - \Phi(\underset{\sim}{x}_{t-1} - \underset{\sim}{\xi}_0) + \Theta\underset{\sim}{a}_{t-1} \end{aligned} \tag{2.5}$$

is the residual vector at time t. Note that $\Phi\Sigma_{\underset{\sim}{\varepsilon}} = \Theta\Sigma_{\underset{\sim}{a}}$. Suppose that there is a step change from the process target mean vector $\underset{\sim}{\xi}_0$ to the process shifted mean vector $\underset{\sim}{\xi}_1$ in the process level between time $t = k-1$ and k. Then the process can be written as

$$\underset{\sim}{x}_t = \begin{cases} \underset{\sim}{\xi}_0 - \Phi\underset{\sim}{\xi}_0 + \Phi\underset{\sim}{x}_{t-1} + \underset{\sim}{a}_t - \Theta\underset{\sim}{a}_{t-1}, & t = k-1, k-2, \cdots \\ \underset{\sim}{\xi}_1 - \Phi\underset{\sim}{\xi}_0 + \Phi\underset{\sim}{x}_{t-1} + \underset{\sim}{a}_t - \Theta\underset{\sim}{a}_{t-1}, & t = k \\ \underset{\sim}{\xi}_1 - \Phi\underset{\sim}{\xi}_1 + \Phi\underset{\sim}{x}_{t-1} + \underset{\sim}{a}_t - \Theta\underset{\sim}{a}_{t-1}, & t = k+1, k+2, \cdots \end{cases} \tag{2.6}$$

and the expectation of the observation vector $\underset{\sim}{x}_t$ is

$$E(\underset{\sim}{x}_t) = \begin{cases} \underset{\sim}{\xi}_0, & t = k-1, k-2, \cdots \\ \underset{\sim}{\xi}_1, & t = k, k+1, k+2, \cdots \end{cases} \tag{2.7}$$

Therefore the residual vectors are

$$\underline{e}_t = \begin{cases} \underline{a}_t, & t = k-1, k-2, \cdots \\ (\underline{\xi}_1 - \underline{\xi}_0) + \underline{a}_t, & t = k \\ [(I + \Theta + \Theta^2 + \cdots + \Theta^{l-1})(I - \Phi) + \Theta^l](\underline{\xi}_1 - \underline{\xi}_0) + \underline{a}_t, & t = k + l, l = 1, 2, 3, \cdots \end{cases} \quad (2.8)$$

Since $\underline{a}_t$ is white noise, we have

$$E(\underline{e}_t) = \begin{cases} \underline{0}, & t = k-1, k-2, \cdots \\ \underline{\xi}_1 - \underline{\xi}_0, & t = k \\ [(I + \Theta + \Theta^2 + \cdots + \Theta^{l-1})(I - \Phi) + \Theta^l](\underline{\xi}_1 - \underline{\xi}_0), & t = k+1, k+2, \cdots \end{cases} \quad (2.9)$$

3. Multivariate CUSUM chart of Residual Vector

3.1 Multivariate CUSUM chart

Consider first the chart which reduces the multivariate data to Z_t^2, and then forms a multivariate CUSUM control statistic is

$$Y_t = \max(Y_{t-1}, 0) + (Z_t^2 - k), \quad (3.1)$$

where $Z_t^2 = \underline{e}_t' \Sigma_{\underline{a}}^{-1} \underline{e}_t$, $t = 1, 2, \cdots$, $Y_0 = 0$ and k is the reference value. The multivariate CUSUM chart based on the control statistic Y_t signals whenever $Y_t \geq h$. The properties of the multivariate CUSUM chart will be evaluated by Markov chain method or simulation.

3.2 Markov chain approach for ARL

The continuous-state Markov chain is discretized by dividing the interval between 0 and the control limit h into n subintervals of width $w = h/n$. Each such subinterval represents a transient state of the Markov chain. The region (h, ∞), which represents a signal region from the multivariate CUSUM chart, corresponds to an absorbing state of the Markov chain. Let S_j be the midpoint of jth interval, $j = 1, 2, \cdots, n$. The control statistic Y_t is said to be in transient state j at time t if $S_j - \frac{w}{2} < Y_t < S_j + \frac{w}{2}$. The process is considered to be in control if Y_t is in a transient state and is considered to be out of control if Y_t is in the absorbing state.

The transient probability matrix, represented in partitioned matrix form for this Markov chain, can be written as

$$P = \begin{pmatrix} R & (I - R)\underline{1} \\ 0^T & 1 \end{pmatrix}, \quad (3.2)$$

where the submatrix R is $n \times n$ matrix which contains the transient probability of going from one transient state to another. I is a $n \times n$ identity matrix, $\underline{0}$ is a column vector of zero and $\underline{1}$ is a column vector of ones.

Let p_{jk} be the transient probability of going from transient state j to transient state k. Then p_{jk} can be approximated by assuming that when the CUSUM statistic is in state j, the CUSUM statistic value Y_t is equal to S_j. The transition probability p_{jk} $(j, k = 1, 2, \cdots, n)$ for R is given by

$$p_{jk} = P(\text{going to state } k \mid \text{in state } j)$$

$$
\begin{aligned}
&= P(S_k - \frac{w}{2} < Y_k \le S_k + \frac{w}{2} \mid \text{in state} \quad j) \\
&= P(S_k - \frac{w}{2} < Y_j + Z_k^2 - k \le S_k + \frac{w}{2}) \\
&= P((k - j - \frac{1}{2})w + k < Z_k^2 \le (k - j + \frac{1}{2})w + k). \qquad (3.3)
\end{aligned}
$$

Brook and Evans (1972) has shown that the ARL vector $\underline{N}$, when the process is in control, is given by

$$\underline{N} = (I - R)^{-1}\underline{1}, \qquad (3.4)$$

where the jth element of $\underline{N}$ represents the ARL for the process starts from state j.

4. Numerical Results and Conclusions

The following control procedures will be compared with their ARL performances. In our computation, the ARL of multivariate CUSUM chart when the process is in control are fixed to be 200 and for the conveniency of calculations the sample size for each sample observation is one. The control limit h of this chart is obtained by using Markov chain method and the values are 13.462, 10.232, 8.591 with respect to k is 2.5, 3.0, 3.5 respectively, when the process is in control. The types of shifts in the parameters for comparison among the proposed control schemes when the process is out of control are stated as follows:

(1)

$$\underline{\xi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \underline{\xi}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \underline{\xi}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \underline{\xi}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

(2)

$$\Sigma_{\underline{\varepsilon}} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \quad \rho = 0.2,\ 0.5,\ 0.8$$

(3)

$$\Phi = \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{pmatrix} \quad \Phi = \begin{pmatrix} 0.3 & 0.1 \\ 0.1 & 0.3 \end{pmatrix}$$

$$\Phi = \begin{pmatrix} 0.8 & 0.5 \\ 0.5 & 0.8 \end{pmatrix} \quad \Phi = \begin{pmatrix} 0.9 & 0.8 \\ 0.8 & 0.9 \end{pmatrix}$$

Tables 1-4 are obtained with various combination of (1), (2), (3) and all the ARLs by simulation for the proposed types of shifts are obtained with 10,000 iterations.

From the numerical results, we conclude that the ARL values decrease as ρ and the absolute values of the difference of elements of $\underline{\xi}_1$ increase and the ARL values decrease as the absolute values of the difference of elements of $\underline{\Phi}$ decrease. The ARL values decrease as k increases.

(Table 1). ARL values for multivariate CUSUM chart based on residual vector with

$$\Phi = \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} 0.5 & -0.4 \\ -0.4 & 0.5 \end{pmatrix}$$

ξ'	ρ	k=2.5	k=3.0	k=3.5
	0.2	2.084	1.954	1.894
(1 0)	0.5	1.812	1.690	1.628
	0.8	1.189	1.102	1.705
	0.2	2.087	1.957	1.892
(0 1)	0.5	1.815	1.689	1.628
	0.8	1.175	1.102	1.077
	0.2	1.179	1.103	1.079
(-1 1)	0.5	1.017	1.006	1.004
	0.8	1.000	1.000	1.000
	0.2	1.776	1.096	1.071
(2 0)	0.5	1.018	1.006	1.004
	0.8	1.000	1.000	1.000

(Table 2). ARL values for multivariate CUSUM chart based on residual vector with

$$\Phi = \begin{pmatrix} 0.3 & 0.1 \\ 0.1 & 0.3 \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} 0.5 & -0.4 \\ -0.4 & 0.5 \end{pmatrix}$$

ξ'	ρ	k=2.5	k=3.0	k=3.5
	0.2	2.818	2.626	2.557
(1 0)	0.5	2.346	2.183	2.115
	0.8	1.689	1.548	1.488
	0.2	2.814	2.628	2.548
(0 1)	0.5	2.337	2.180	2.103
	0.8	1.688	1.543	1.472
	0.2	1.695	1.551	1.485
(-1 1)	0.5	1.372	1.253	1.208
	0.8	1.002	1.001	1.000
	0.2	1.675	1.534	1.474
(2 0)	0.5	1.363	1.252	1.208
	0.8	1.002	1.001	1.000

(Table 3). ARL values for multivariate CUSUM chart based on residual vector with

$$\Phi = \begin{pmatrix} 0.8 & 0.5 \\ 0.5 & 0.8 \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} 0.5 & -0.4 \\ -0.4 & 0.5 \end{pmatrix}$$

$\underline{\xi}'$	ρ	k=2.5	k=3.0	k=3.5
	0.2	3.487	3.260	3.174
(1 0)	0.5	2.875	2.676	2.599
	0.8	1.971	1.826	1.767
	0.2	3.476	3.260	3.174
(0 1)	0.5	2.864	2.673	2.594
	0.8	1.973	1.824	1.756
	0.2	1.980	1.830	1.767
(-1 1)	0.5	1.653	1.504	1.438
	0.8	1.051	1.022	1.015
	0.2	1.971	1.825	1.759
(2 0)	0.5	1.652	1.510	1.446
	0.8	1.052	1.024	1.016

(Table 4). ARL values for multivariate CUSUM chart based on residual vector with

$$\Phi = \begin{pmatrix} 0.9 & 0.8 \\ 0.8 & 0.9 \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} 0.5 & -0.4 \\ -0.4 & 0.5 \end{pmatrix}$$

$\underline{\xi}'$	ρ	k=2.5	k=3.0	k=3.5
	0.2	2.093	1.963	1.903
(1 0)	0.5	1.820	1.700	1.699
	0.8	1.189	1.103	1.077
	0.2	2.089	1.964	1.903
(0 1)	0.5	1.820	1.695	1.637
	0.8	1.178	1.102	1.078
	0.2	1.176	1.103	1.067
(-1 1)	0.5	1.017	1.006	1.004
	0.8	1.000	1.000	1.000
	0.2	1.188	1.101	1.074
(2 0)	0.5	1.019	1.006	1.005
	0.8	1.000	1.000	1.000

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