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Asymptotic Distribution of a Nonparametric Multivariate Test Statistic for Independence

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Abstract

A multivariate statistic based on interdirection is proposed for detecting dependence among many vectors. The asymptotic distribution of the proposed statistic is derived under the null hypothesis of independence. Also we find the asymptotic distribution under the alternatives contiguous to the null hypothesis, which is needed for later use of computing relative efficiencies.

Key Words and Phrases: Asymptotic Distribution, Multivariate, Nonparametric

1. Introduction

The nonparametric sign statistic based on interdirection, called the interdirection quadrant statistic, was introduced by Geiser and Randles(1997) for testing whether the two vectors are independent. In this paper, we extend the case of two vectors to the case of many vectors. Let $X_1, ..., X_n$ be independent and identically distributed vectors with $X_i = ({X_i}^{(1)'}, {X_i}^{(2)'}, ..., {X_i}^{(c)'})'$. Here ${X_i}^{(\alpha)}$ is a $\gamma_\alpha \times 1$ vector, so X_i is a $(\gamma \equiv \sum_{\alpha=1}^{c} \gamma_a)$ vector. We assume X has a continuous distribution with density function $f_X(x^{(1)}, x^{(2)}, ..., x^{(c)})$. The vector ${X_i}^{(\alpha)}$ has a marginal density $f_\alpha(x^{(\alpha)})$, $\alpha = 1, 2, ..., c$. Here $f_\alpha(x^{(\alpha)})$ represents the density of an elliptically symmetric distribution centered at the $\gamma_\alpha \times 1$ vector θ_α. The objective is to test $H_0 : f_X(x^{(1)}, x^{(2)}, ..., x^{(c)}) = f_1(x^{(1)}) f_2(x^{(2)}) f_c(x^{(c)})$ against $H_1 : f_X(x^{(1)}, x^{(2)}, ..., x^{(c)}) \neq f_1(x^{(1)}) f_2(x^{(2)}) f_c(x^{(c)})$. We propose the test statistic $\hat{Q_n}^T = \hat{Q_n}^{1,2} + \hat{Q_n}^{1,3} + ... + \hat{Q_n}^{c-1,c}$, where $\hat{Q_n}^{\alpha,\beta}$ is the interdirection quadrant, proposed by Gieser and Randles (1997), computed between $X^{(\alpha)}$ and $X^{(\beta)}$, for each $1 \leq \alpha \leq \beta \leq c$. That is, the proposed statistic is the sum of $\frac{(c-1)c}{2}$ interdirection quadrant statistics. Let $\hat{\theta_\alpha}$ be an affine equivariant estimator of θ_α based

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on $X_1{}^{(\alpha)}, ..., X_n{}^{(\alpha)}$ that satisfies $(\hat{\theta_\alpha} - \theta_\alpha) = O_p(n^{-1/2})$. Then our extension of the interdirection quadrant statistic is given by
$\hat{Q_n}^T(\hat{\theta_1}, \hat{\theta_2}, ..., \hat{\theta_c}) =$
$$\sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} [\frac{r_\alpha r_\beta}{n} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} cos(\pi \hat{p_\alpha}(X_{i_1}^{(\alpha)}, X_{i_2}^{(\alpha)}); \hat{\theta_\alpha}) \; cos(\pi \hat{p_\beta}(X_{i_1}^{(\beta)}, X_{i_2}^{(\beta)}); \hat{\theta_\beta})]$$
where

$$\hat{p_\alpha}(X_{i_1}{}^{(\alpha)}, X_{i_2}{}^{(\alpha)}; \hat{\theta_\alpha}) = \begin{pmatrix} (C_{i_1,i_2}^{(\alpha)} + d_n) / \binom{n}{p-1} & \text{if } \; i_1 \neq i_2 \\ 0 & \text{if } \; i_1 = i_2 \end{pmatrix},$$

$$d_n = \frac{1}{2}[\binom{n}{p-1} - \binom{n-2}{p-1}],$$

and the interdirection count $C_{i_1,i_2}{}^{(\alpha)}$, first defined by Randles (1989), denotes the number of hyperplanes defined by the origin and $\gamma_\alpha - 1$ observations $X_i{}^{(\alpha)} - \hat{\theta_\alpha}$ (excluding $X_{i1}{}^{(\alpha)} - \hat{\theta_\alpha}$ and $X_{i2}{}^{(\alpha)} - \hat{\theta_\alpha}$) such that $X_{i1}{}^{(\alpha)} - \hat{\theta_\alpha}$ and $X_{i2}{}^{(\alpha)} - \hat{\theta_\alpha}$ are on opposite of the hyperplane. The interdirection counts are used to measure the angular distance between the centered vectors $X_{i1}{}^{(\alpha)} - \hat{\theta_\alpha}$ and $X_{i2}{}^{(\alpha)} - \hat{\theta_\alpha}$ relative to the origin and the positions of the other observations. The term $\hat{p_\beta}(X_{i_1}{}^{(\beta)}, X_{i_2}{}^{(\beta)}; \theta(\hat{\beta}))$ is similarly defined among the $X_i{}^{(\beta)} - \hat{\theta_\beta}$ vectors. We note that $\hat{Q_n^T}$ is affine invariant since the interdirection count is affine invariant under the group $G = g((D, b) \mid g(D, b)(X) = DX + b)$, where D is the direct sum of $D_1, D_2, ..., D_c$ (D_α nonsingular $\gamma_\alpha \times \gamma_\alpha$) and b is a real number with dimension γ, as described by Muirhead(1982). In the next section, we will find asymptotic null distribution of $\hat{Q_n^T}$ when the underlying marginal distributions are elliptically symmetric.

2. Asymptotic Null Distribution of $\hat{Q_n^T}$

We shall define the class of elliptically symmetric distribution before deriving the asymptotic null distribution of $\hat{Q_n^T}$.

Definition. The $\gamma_\alpha \times 1$ random vector X^α is said to have an elliptically symmetric distribution with parameters $\theta_\alpha(\gamma_\alpha \times 1)$ and $\sum_\alpha(\gamma_\alpha \times \gamma_\alpha)$ if the density function of X^α is of the form $f_\alpha(X^{(\alpha)}) = K_\alpha |\sum_\alpha|^{-1/2} \; g_\alpha[(X^{(\alpha)} - \theta_\alpha)' {\sum_\alpha}^{-1}(X^{(\alpha)} - \theta_\alpha)]$, (1) where g_α is a nonnegative, real-valued, differentiable function, $\sum_\alpha$ is positive definite and symmetric and K_α is a positive scalar such that (1) represents a density function. An elliptically symmetric distribution with dispersion parameter $\sigma^2 \; I_\alpha$ is called a spherically symmetric distribution. Since $\hat{Q_n^T}$ is affine invariant and an elliptically symmetric distribution is transformable into a spherically symmetric distribution through a nonsingular linear transformation, we assume without loss of generality that $f_\alpha(x^{(\alpha)})$ is a spherically symmetric distribution centered at the origin.

In order to find the asymptotic distribution of $\hat{Q_n^T}$ under H_0, we construct an approximating quantity which has the same aymptotic distribution. We first consider the case where $\theta_1, \theta_2, ..., \theta_c$ are known. Under the assumed underlying distribution, we define

$$\hat{Q_n}^T(\theta_1,\theta_2,...,\theta_c) = \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c}[\frac{r_\alpha r_\beta}{n}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n} cos(\pi\hat{p_\alpha}(X_{i_1}^{(\alpha)}, X_{i_2}^{(\alpha)})) \\ cos(\pi\hat{p_\beta}(X_{i_1}^{(\beta)}, X_{i_2}^{(\beta)}))].$$

This $Q_n^T\ (\theta_1,\theta_2,...,\theta_c)$ is the same as the statistic $\hat{Q_n^T}\ (\hat{\theta_1},\hat{\theta_2},...,\hat{\theta_c})$ except the estimated proportion is replaced with $p_\alpha({X_{i_1}}^{(\alpha)}, {X_{i_2}}^{(\alpha)}) =$ $E_{H_0}[\hat{p_\alpha}({X_{i_1}}^{(\alpha)}, {X_{i_2}}^{(\alpha)}) \mid {X_{i_1}}^{(\alpha)}, {X_{i_2}}^{(\alpha)}]$ = (radian measure of the angle between ${X_{i_1}}^{(\alpha)}$ and ${X_{i_2}}^{(\alpha)}$) $/\pi$. Now let ${X_i}^{(\alpha)} = R_i^{(\alpha)}{U_i}^{(\alpha)}$ where $R_i^{(\alpha)} = [{X_i}^{(\alpha)\prime}{X_i}^{(\alpha)}]^{1/2}$. Note that $U^{(\alpha)}$ is distributed uniformly on the γ_α dimensional unit hypersphere and is independent of positive quantity $R^{(\alpha)}$. The estimated proportion of hyperplanes, $\hat{p_\alpha}$, and p_α only require the directions of ${U_{i_1}}^{(\alpha)}$ and ${U_{i_2}}^{(\alpha)}$, not their lengths so that $cos(\pi p_\alpha({X_{i_1}}^{(\alpha)}, {X_{i_2}}^{(\alpha)}))$ = cos(angle between ${U_{i_1}}^{(\alpha)}$ and ${U_{i_2}}^{(\alpha)}$) $= {U_{i_1}}^{(\alpha)\prime}$ and ${U_{i_2}}^{(\alpha)}$. Thus have

$$\begin{aligned} &{Q_n}^T(\theta_1,\theta_2,...,\theta_c) \\ &= \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c}[\frac{r_\alpha r_\beta}{n}\sum_{i_\alpha=1}^{n}\sum_{i_\beta=1}^{n} {U_{i_\alpha}}^{(\alpha)\prime}{U_{i_\beta}}^{(\alpha)}{U_{i_\alpha}}^{(\beta)\prime}{U_{i_\beta}}^{(\beta)}] \\ &= \sum_{\alpha=1}^{c-1}\sum_{\beta=\alpha+1}^{c}[\frac{1}{n}\sum_{s_\alpha=1}^{n}\sum_{s_\beta=1}^{n}(\sum_{j=1}^{n}\sqrt{\gamma_\alpha\gamma_\beta}U_{s_\alpha j}^{(\alpha)}U_{s_\beta j}^{(\beta)})^2], \end{aligned}$$

where $U_{s_\alpha j}^{(\alpha)}$ $(U_{s_\beta j}^{(\beta)})$ is the s_α th (s_β th) component of $U_j^{(\alpha)}$ $(U_j^{(\beta)})$. The following theorem results from this approximation.

Theorem 1 . Under H_0, $Q_n^T(\theta_1,\theta_2,...,\theta_c) \xrightarrow{d} \chi^2_{\gamma_1\gamma_2+\gamma_1\gamma_3+...+\gamma_{c-1}\gamma_c}$ where the notation $\xrightarrow{p}$ means the convergence in distribution.

Proof Let $B = B_{1,2} \bigoplus B_{1,3} \bigoplus \cdots \bigoplus B_{c-1,c}$ Where

$B_{1,2} = (b_{s_1 s_2})_{\gamma_1\times\gamma_2}, B_{1,3} = (b_{s_1 s_2})_{\gamma_1\times\gamma_3}, \cdots, B_{c-1,c} = (b_{s_{c-1}s_c})_{\gamma_{c-1}\times\gamma_c}$ are arbitrary nonzero but fixed matrices. Define $Z = Z_{1,2} \bigoplus Z_{1,3} \bigoplus \cdots \bigoplus Z_{c-1,c}$ where $Z_{1,2} = (\sum_{j=1}^{n}\sqrt{\gamma_1\gamma_2}U_{s_1 j}^{(1)}U_{s_2 j}^{(2)})_{\gamma_1\times\gamma_2}, Z_{1,3} = (\sum_{j=1}^{n}\sqrt{\gamma_1\gamma_3}U_{s_1 j}^{(1)}U_{s_3 j}^{(3)})_{\gamma_1\times\gamma_3}, ..., Z_{c-1,c} =$ $(\sum_{j=1}^{n}\sqrt{\gamma_{c-1}\gamma_c}U_{s_{c-1}j}^{(c-1)}U_{s_c j}^{(c)})_{\gamma_{c-1}\times\gamma_c}$.
then

$$vec(B)'vec(Z) \quad = \quad vec(B_{1,2})'vec(Z_{1,2}) + vec(B_{1,3})'vec(Z_{1,3}) + ... + vec(B_{c-1,c})'vec(Z_{c-1,c})$$

$$= \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} \sum_{s_\alpha=1}^{\gamma_\alpha} \sum_{s_\beta=1}^{\gamma_\beta} b_{s_\alpha \sigma_\beta} \; Z_{s_\alpha s_\beta}$$

$$= \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} \sum_{j=1}^{n} [\sum_{s_\alpha=1}^{\gamma_\alpha} \sum_{s_\beta=1}^{\gamma_\beta} \sqrt{\gamma_\alpha \gamma_\beta} \; b_{s_\alpha s_\beta} U_{s_\alpha j}^{(\alpha)} U_{s_\beta j}^{(\beta)}$$

$$= \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} \sum_{j=1}^{n} U_j{}^{(\alpha)'} (\sqrt{\gamma_\alpha \gamma_\beta} B_{\alpha,\beta}) U_j{}^{(\beta)}$$

$$= \sum_{j=1}^{n} \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} U_j{}^{(\alpha)'} (\sqrt{\gamma_\alpha \gamma_\beta} B_{\alpha,\beta}) U_j{}^{(\beta)}$$

where $vec(A) = (a_1{}', ..., a_t)'$, a_k is a column vector. The summands in the last formula are the sum of iid random variables with mean zero and variance $vec(B_{\alpha,\beta})' vec(B_{\alpha,\beta})$, and the covariance between any two terms are zero. Thus by the usual central limit theorem, $n^{-1/2} vec(B)' vec(Z) \sim AN(0, vec(B)' vec(B))$. It follows that $n^{-1} vec(Z)' vec(Z) \xrightarrow{d} \chi^2_{\gamma_1\gamma_2 + \gamma_1\gamma_3 + ... + \gamma_{c-1}\gamma_c}$.

Since $n^{-1} vec(Z)' vec(Z)$
$= n^{-1} \sum_{s_1=1}^{\gamma_1} \sum_{s_2=1}^{\gamma_2} Z_{1,2}^2(s_1, s_2) + \sum_{s_1=1}^{\gamma_1} \sum_{s_3=1}^{\gamma_3} Z_{1,3}^2(s_1, s_3) + ... + \sum_{s_{c-1}=1}^{\gamma_{c-1}} \sum_{s_c=1}^{\gamma_c} Z_{c-1,c}^2(s_{c-1}, s_c))$
$= Q_n^T(\theta_1, \theta_2, ..., \theta_c)$
where $Z_{s_\alpha, s_\beta}(s_\alpha, s_\beta)$ is the element in the s_α th row and s_β th column of $z_{\alpha,\beta}$, the result is proved.

Now using the result $\hat{Q_n}^{\alpha,\beta}(\theta_\alpha, \theta_\beta) = Q_n^{\alpha,\beta}(\theta_\alpha, \theta_\beta) + o_p(1)$, in Gieser and Randles (1997), we are able to find the asymptotic null distribution of $\hat{Q_n^T}(\theta_1, \theta_2, ..., \theta_c)$.

Theorem 2. Under H_0, $\hat{Q_n^T}(\theta_1, \theta_2, ..., \theta_c) \xrightarrow{d} \chi^2_{\gamma_1\gamma_2 + \gamma_1\gamma_3 + ... + \gamma_{c-1}\gamma_c}$

proof

$$\begin{aligned} \hat{Q_n}^T - Q_n{}^T &= \hat{Q_n}^{1,2} + \hat{Q_n}^{1,3} + ... + \hat{Q_n}^{c-1,c} - Q_n{}^{1,2} - Q_n{}^{1,3} - ... - Q_n{}^{c-1,c} \\ &= (\hat{Q_n}^{1,2} - Q_n{}^{1,2}) + (\hat{Q_n}^{1,3} - Q_n{}^{1,3}) + ... + (\hat{Q_n}^{c-1,c} - Q_n{}^{c-1,c}) \\ &= o_p(1). \end{aligned}$$

Next we consider the case in which $\theta_1, \theta_2, ..., \theta_c$ are unknown and hence estimated. The asymptotic null distribution is easily obtained using the result, $\hat{Q_n}^{\alpha,\beta}(\hat{\theta}_\alpha, \hat{\theta}_\beta) = \hat{Q_n}^{\alpha,\beta}(\theta_\alpha, \theta_\beta) + o_p(1)$, by Gieser and Randles and following the same procedure as in Theorem 2 above. Thus the asymptotic null distribution of $\hat{Q_n}^T$ remains the same whether the symmetric center is known or not, provided the center is estimated using $\hat{\theta}_\alpha$ with the stated properties.

3. Asymptotic Distribution of $\hat{Q}_n^T$ under Contiguous Alternatives

We will find the asymptotic distribution of $\hat{Q}_n^T$ under a sequence of alternatives approaching the null hypothesis. In doing so, we first will propose a model that shows a dependence among the components of the observations. In this model we express the dependence as a function of a non-negative real-valued parameter Δ . The sequence of alternatives defined by the model converges to the null hypothesis as $\Delta \to 0$ in such a way that the alternatives are contiguous to the null hypothesis. We will use the model obtained by generalizing the one which Konijin (1956) studied. The general multivariate version of this model is given by (2) $Y^{(1)}, Y^{(2)}, ..., Y^{(c)}$ where are independent random vectors with dimensions of respectively and

$M_{1,2}, M_{1,3}, ..., M_{c,c-1}$ are fixed non-zero matrices of dimensions $\gamma_1 \times \gamma_2, \gamma_1 \times \gamma_3, ..., \gamma_c \times \gamma_c - 1$, respectively, and $0 \leq \Delta \leq 1/c$. Assmume A_Δ is nonsingular for nonsingular for Δ in a neighborhood of 0. Note that $\Delta = 0$ corresponds to the null hypothesis of independence. We assume that each $Y^{(\alpha)}$ is elliptically symmetric with zero mean vector and variance covariance matrix $\sum_\alpha, \alpha = 1, ..., c$. In other words,
$f_\alpha(y^{(\alpha)}) = K_\alpha \; g_\alpha((y^{(\alpha)} - \theta_\alpha)'\Sigma_\alpha{}^{-1}(y^{(\alpha)} - \theta_\alpha))$ where $g_\alpha()$ does not depend on θ_k or Σ_k . With these underlying distributions, we will apply LeCam's three lemmas as described in Hajek and Sidak (1967, pp. 201-214) to find the asymptotic distribution. The sequence of alternatives $H_1 \; : \; \Delta_n = n^{-1/2}\Delta_0$, where $\Delta_0 > 0$, can be shown to be contiguous to the null hypothesis by following the same arguments as in Gieser and Randles (1997). This contiguity helps us find the asymptotic distribution

$$
\begin{aligned}
X &= \begin{pmatrix} X^{(1)} \\ X^{(2)} \\ \vdots \\ X^{(c)} \end{pmatrix} \\
&= \begin{pmatrix} (1-(c-1)\Delta)Y^{(1)} & +\Delta M_{1,2}Y^{(2)} & +\cdots & +\Delta M_{1,c}Y^{(c)} \\ \Delta M_{2,1}Y^{(1)} & +(1-(c-1)\Delta)Y^{(2)} & +\cdots & +\Delta M_{2,c}Y^{(c)} \\ & \vdots & & \\ \Delta M_{c,1}Y^{(1)} & +\Delta M_{c,2}Y^{(2)} & +\cdots & +(1-(c-1)\Delta)Y^{(c)} \end{pmatrix} \\
&= \begin{pmatrix} (1-(c-1)\Delta)I_{\gamma 1} & \Delta M_{1,2} & \cdots & \Delta M_{1,2} \\ \Delta M_{2,1} & (1-(c-1)\Delta)I_{\gamma 2} & \cdots & \Delta M_{2,c} \\ \vdots & \vdots & & \vdots \\ \Delta M_{c,1} & \Delta M_{c,2} & \cdots & (1-(c-1)\Delta)I_{\gamma_c} \end{pmatrix} \begin{pmatrix} Y^{(1)} \\ Y^{(2)} \\ \vdots \\ Y^{(c)} \end{pmatrix} \\
&= A_\Delta \begin{pmatrix} Y^{(1)} \\ Y^{(2)} \\ \vdots \\ Y^{(c)} \end{pmatrix} = A_\Delta Y, \qquad (2)
\end{aligned}
$$

using the log-likelihood function. Finding the asymptotic distribution under the contiguous alternatives involves establishing the asymptotic bivariate normality of the appropriately defined statistic and the log-likelihood function Λ_n under H_0 where $\Lambda_n = \sum_{i=1}^{n} logL(X_i;\Delta_n)$, and $L(x;\Delta_n) = f_x(x;\Delta_n)/f_x(x;0)$. Under H_0 the log-likelihood function is approximated by $T_n = \Delta_n \sum_{i=1}^{n} L^*(X_i;0)$ where $L^*(x;0) \equiv \frac{\delta}{\delta\Delta} logL(x;0) \mid_{\Delta}= 0$. LeCam's third lemma states that if, under H_0, $\begin{pmatrix} S_n \\ \Lambda_n \end{pmatrix} \sim AN\left(\begin{pmatrix} \mu_1 \\ -\sigma_2^2/2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}\right)$, then $S_n \sim AN(\mu_1 + \sigma_{12}, \sigma_1^2)$ under a continuous sequence of alternatives. Thus, obtaining the asymptotic bivariate normality of (S_n, T_n) under H_0 yields the asymptotic normality of S_n under the contiguous alternatives. Now we use the reation $X^{(\alpha)} - \theta_\alpha = R^{(\alpha)}U^{(\alpha)}$ and make a transformation by multiplying by a nonsingular matrix D_α, where $\sum_\alpha = D_\alpha \acute{D}_\alpha$ in order to get the representation of the elliptically symmetric distribution from the spherically symmetric distribution. Using the same assumption and arguments as in Gieser and Randles (1997), the asymptotic distribution of $\hat{Q_n^T}$ and Δ_n is derived. Since $\hat{Q_n^T}$ is affine invariant, we put $\sum_\alpha = I_\alpha$ without loss of generality.

Theorem 3. Under Δ_n, $\hat{Q_n^T} \xrightarrow{d} \chi_{\gamma_1\gamma_2+\gamma_1\gamma_3+\ldots+\gamma_{c-1}\gamma_c}(\lambda_1)$,
where the noncentrality paramater(denoted by λ_1) is given by
$\lambda_1 = \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} \frac{4\Delta_0^2}{\gamma_\alpha\gamma_3} vec(E_{H_0}[H^{\alpha,\beta}])' vec(E_{H_0}[H^{\alpha,\beta}])$,
$H^{\alpha,\beta} = R^{(\alpha)}R^{(\beta)}\acute{D}_\alpha(\phi_\alpha((R^{(\alpha)})^2)\sum_\alpha^{-1} M_{\alpha,\beta} + \phi_\beta((R^{(\beta)})^2)M'_{\beta,\alpha}\sum_\beta^{-1})D_\beta$ and
$\phi_k(t) = g'_k(t)/g_k(t)$.

proof. Let $a = (a_1, a_2)'$ be an arbitrary vector of constants none of which are zero.
Let $S_n = \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} S_n^{\alpha,\beta}$,where

$$S_n^{\alpha,\beta} = n^{-1/2} \sum_{i=1}^{n} U_i^{(\alpha)'}(\sqrt{\gamma_\alpha\gamma_\beta} B_{\alpha,\beta})U_i^{(\beta)}$$

Then

$$\acute{a}\begin{pmatrix} S_n \\ T_n \end{pmatrix} =$$

$$\sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} [n^{-1/2} \sum_{i=1}^{n} 2\Delta_0 a_2((R_i^{(\alpha)})^2\phi_1((R_i^{(\alpha)})^2) + \frac{\gamma_\alpha}{2}) + a_2((R_i^{(\beta)})^2\phi_2((R_i^{(\beta)})^2) + \frac{\gamma_\beta}{2})$$
$$+U_i^{(\alpha)'}(a_2 H_i^{\alpha,\beta} + a_1 \frac{\sqrt{\gamma_\alpha\gamma_\beta}}{2\Delta_0} B_{\alpha,\beta})U_i^{(\beta)}].$$

Since $E_{H_0}[S_n^{\alpha,\beta}] = 0$ and $E_{H_0}[T_n^{\alpha,\beta}] = 0$, it follows that $E_{H_0}[\acute{a}(S_n, T_n)'] = 0$
Also

$$V_{H_0}\acute{a}(S_n, T_n) = V_{H_0} a_1(S_n^{1,2} + S_n^{1,3} + \ldots + S_n^{c-1,c}) + a_2 T_n$$
$$= a_1^2 \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} V_{H_0}(S_n^{\alpha,\beta}) + a_2^2 V_{H_0}(T_n)$$

$$+ 2a_1a_2 \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} Cov_{H_0}[S_n^{\alpha,\beta}, T_n]$$
$$= a_1^2(vec(B_{1,2})'vec(B_{1,2})vec(B_{1,3})'vec(B_{1,3}) + ... + vec(B_{c-1,c})'vec(B_{c-1,c}))$$
$$+ a_2^2\sigma^2$$
$$+ 2a_1a_2 \sum_{\alpha=1}^{c-1} \sum_{\beta=\alpha+1}^{c} 2\Delta_0\sqrt{\gamma_\alpha\gamma_\beta}E_{H_0}[U^{(\acute{\alpha})}H^{\alpha,\beta}U^{(\beta)}U^{(\acute{\alpha})}B_{\alpha,\beta}U^{(\beta)}]$$

where $\sigma_2 = V(T_n)$. The second equality follows from the fact that $Cov_{H_0}[S_n^{\alpha,\beta}] = 0$ when one or more of the superscripts is unique. Also since

$$\begin{aligned} E_{H_0}[U^{(\acute{\alpha})}H^{\alpha,\beta}U^{(\beta)}U^{(\acute{\alpha})}B_{\alpha,\beta}U^{(\beta)}] &= \frac{1}{\gamma\beta}E_{H_0}[U^{(\acute{\alpha})}H^{\alpha,\beta}B_{\alpha,\beta}U^{(\alpha)}] \\ &= \frac{1}{\gamma_\alpha\gamma_\beta}vec(B_{\alpha,\beta})'vec(E_{H_0}[(H^{\alpha,\beta})]), \end{aligned}$$

we have $\sigma_1^2 = vec(\acute{B})vec(B)$,$\sigma_2^2 = \sigma^2$, and

$$\begin{aligned} \sigma_{12} &= \begin{pmatrix} vec(B_{1,2}) \\ vec(B_{1,3}) \\ \vdots \\ vec(B_{c-1,c}) \end{pmatrix} \begin{pmatrix} \frac{2\Delta_0}{\sqrt{\gamma_1\gamma_2}} vec(E_{H_0}[H^{1,2}]) \\ \frac{2\Delta_0}{\sqrt{\gamma_1\gamma_3}} vec(E_{H_0}[H^{1,3}] \\ \vdots \\ \frac{2\Delta_0}{\sqrt{\gamma_{c-1}\gamma_c}} vec(E_{H_0}[H^{c-1,c}] \end{pmatrix} \\ &= vec(\acute{B}) \begin{pmatrix} \frac{2\Delta_0}{\sqrt{\gamma_1\gamma_2}} vec(E_{H_0}[H^{1,2}]) \\ \frac{2\Delta_0}{\sqrt{\gamma_1\gamma_3}} vec(E_{H_0}[H^{1,3}] \\ \vdots \\ \frac{2\Delta_0}{\sqrt{\gamma_{c-1}\gamma_c}} vec(E_{H_0}[H^{c-1,c}] \end{pmatrix} \end{aligned}$$

Then the asymptotic normality of $\acute{a}(S_nT_n)'$ follows by applying the usual central limit theorem , and hence, under H_0 $\begin{pmatrix} S_n \\ T_n \end{pmatrix} \sim AN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}\right)$. Thus under Δ_n, $S_n \sim AN(\sigma_{12}, vec(\acute{B})vec(B))$, and $Q_n^T \longrightarrow \chi^2_{\gamma_1\gamma_2+\gamma_1\gamma_3+...+\gamma_{c-1}\gamma_c}(\lambda_1)$. We note also that Q_n^T and $\hat{Q}_n^T$ have the same asymptotic distribution under Δ_n as well as under H_0 because $|{Q_n}^T - \hat{Q_n}^T| = o_p(1)$ under Δ_n. Therefore the result follows.

References

1. Geiser, P.W. and Randles, R. H. (1997)' Nonparametric test of independence between two vectors, *Journal of American Statistical Association*, 92, 561-567.

2. Hajek, J.and Sidak, Z. (1967). Theory of Rank Tests, *Academic Press*, New York.

3. Konijin, H.S.(1956), On the power of certain tests for independence in bivariate population, *Annals of Mathematical Statistics, 27, 300-323.*

4. *Muirhead, R.J.(1982), Aspects of Multivariate Satistical Theory, Wiley, New York.*

5. *Randles, R.H.(1989), A distribution-free multivariate sign test based on interdirections, Journal of the American Statistical Association*, 84, 1045-1050.