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저장방식별 외부 임차공간을 고려한 창고 저장용량

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Warehouse Storage Capacity with Leased Space for Different Storage Policies

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In this paper, an approach is presented for determining the required storage capacity of a warehouse with leased public space. Storage assignment policies considered are randomized and class-based storage assignment policies. An analytic model for each of the storage policies is formulated with the objective of minimizing the cost of owned storage space and leased space while satisfying a desired service level of protection against space shortages. Cost functions used in the models are piecewise liear with fixed costs. For the models, algorithms are developed to generate optimal solutions. The approach is applied to the systems where the standard economic-order-quantity inventory model is used for all items being stored in the warehouse.

Keywords: warehouse, storage capacity, leased space, storage assignment policy, optimization

1. Introduction

In this paper, we consider a problem of determining the storage capacity of warehouse systems. Any storage requirement over the in-house service level needs to be satisfied by the storage space being leased from outside sources. The required storage capacity is then defined as the amount of storage space needed to accommodate the materials to be stored in order to meet a desired service level. The major factor that influences storage sizing is certainly the storage assignment policy which defines the way of assigning items to storage locations. In this paper, we consider two popular storage policies, randomized storage assignment (RAN) and N-class turnover-based storage (CN).

Warehouse sizing problems seeking a compromise between private warehousing and public warehousing have been substantially studied so far. Ballou (1999) presented a trial-and-error method to seek the best combination of private-public warehouse size alternatives. Dynamic natures considered in the method include the seasonality of space requirements. Hung and Fisk (1984) gave an alternative formulation to Ballou's method for static and dynamic warehouse sizing problems. They showed the problems could be formulated as linear programming models which can be solved easily by a simplex routine. Recently, Rao and Rao (1998) presented a simpler method of determining optimal solutions for the same static problem. They extended the static model to the dynamic models dealing with costs varying over time and economies of scale in fixed and / or operating costs. Cormier and Gunn (1996) considered a static problem incorporating inventory policy cost, warehouse cost, and the cost associated with leasing space from outside sources. A

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similar study (1996) without consideration of warehouse leasing was made by the same authors. Jucker *et al.* (1982) investigated a static problem to determine the capacities of a single production plant and a set of regional leased warehouses which maximize expected profit. Assuming that any regional demand exceeding warehouse capacity is lost, an efficient algorithm based on the Kuhn-Tucker conditions is developed. Dynamic problems with a single warehouse were studied by White and Francis (1971) and by Lowe *et al.* (1979). Additional review of storage capacity models is given in Cormier and Gunn (1992). Recently, Cho and Bozer (2001) presented a simple algorithm to estimate the storage capacity for automated storage / retrieval systems under stochastic demand. The design criteria considered in their study are maximum permissible overflow probability and maximum allowable storage/retrieval machine utilization. The Other related research on storage sizing includes those of Bafna (1983) and Mullens (1981) where general practical procedures are presented.

A perusal of the literature shows that no research has analytically investigated the effect of the storage policies on the storage capacity while leasing of external storage space is allowed. Recently, Lee (1998, 1999) presented the approaches to optimally determining storage capacities for different storage policies. However, the approaches are based on the simple linear cost curve without any fixed costs. Therefore, they might not be well applied in real situations in which storage space costs are not linear due to the piecewise nature with fixed costs. In this paper, we extend the approach for the RAN and CN policies presented in Lee (1999) to consider more general cost functions of privately owned and public leased storage spaces. An application of the models to systems operating under the economic-order-quantity (EOQ) inventory model is presented and the effects of demand distribution on the storage capacity and system throughput are investigated.

2. Models for Determining Storage Capacity

In general, there are two choices for storage warehousing: operating owned warehouse space and leasing space such as from a public warehouse. Depending on the costs associated with each choice of warehousing, the most economical combination of the two warehousing choices needs to be selected. Initial capital expenditures required for owned storage space are substantially higher than those of leased one and future expansion is very limited. Therefore, there is a need to meet owned storage space requirements such that a high utilization is realized and to take advantage of leased space where possible.

In order to determine the required storage capacity for storage systems, Francis *et al.* (1992) presented two different approaches, a service-level approach and a cost-based approach. In the former approach, the total amount of storage space is minimized without exceeding a given probability, $\alpha (0 < \alpha < 1)$, of a space shortage occurring (hereafter, we call it the shortage probability). The service level of the system is then equal to $1 - \alpha$. If the storage requirement is greater than the storage capacity, a space shortage occurs. Under such conditions, the excess space requirement is assumed to be met via leased storage space. In the cost-based approach, the storage capacity is determined by minimizing the sum of the cost of owning space and contracting space incurred by space shortage without consideration of the service level. In this paper, we combine the two approaches into a modified cost-based approach by which the storage capacity is determined to minimize the total cost while satisfying the service level required.

Let X and X_i, $i = 1, \ldots, n$, be random variables which represent the aggregate inventory level of the overall system and the inventory level of item i, respectively. Here, we consider the case where every X_i follows a uniform distribution with a_i and b_i which are the lower and upper bounds of the random variable, respectively. One example of such a case is a system in which the standard EOQ model with a_i being zero is applied to all items. The storage capacity at the $1 - \alpha$ service level, $S(\alpha)$, satisfies $P_r(X \leq S(\alpha)) \geq 1 - \alpha$.

2.1 Storage Space Costs

In the literature, storage system costs have been modeled in various ways. Ballou (1999) presented conceptual cost functions applied to four different systems including public, leased, and private warehousing systems. In his discussion of private versus public warehousing decisions, the private warehouse cost is modeled as a sum of a monthly fixed component and a monthly variable component which is a linearly increasing function of storage space required. The leased warehouse cost is linear without any fixed cost. For the static cases of warehouse size expansion, Cormier and Gunn (1996) used the similar cost model. Meanwhile, only linear variable costs have been considered often in the literature for the sake of simplicity (Lowe *et al.*, 1979; Rao and Rao, 1998; White and Francis, 1971). On the other hand, Jucker *et* *al.* (1982) used a nonlinear cost model which takes the form K_1K^a where a is the economies-of-scale parameter and K_1, a scaling factor. However, this simple nonlinear cost model may be too rigid to represent practical warehousing costs since fixed costs are not considered in the model and only concave ($0 < a \leq 1$) or convex ($a \geq 1$) cost functions are assumed.

For automated storage / retrieval systems(AS / RS), the total space cost must be modeled as a step function with jumps due to the addition of each S / R machine to the storage system as the system size increases. For the cost of installing a storage rack, Zollinger (1982) gave a quadratic cost function obtained by a regression analysis of cost data for more than 60 AS/RS installations. In general, each regression cost model for a particular system should have its own different structure depending on the cost data used. Hence, it is almost impossible to develop an effective optimization algorithm which can be generally applied to any regression cost model. Meanwhile, most studies on the design of AS/R system assumes a linear cost model with fixed facility costs (Asayeri *et al.*, 1985; Karasawa *et al.*, 1980).

When a combination of a private warehouse and a leased public warehouse is operated, transportation of materials to the public warehouse should be occurred. Since items are typically transported on a lump-sum basis, transportation costs obviously consist of a fixed, component and a variable one.

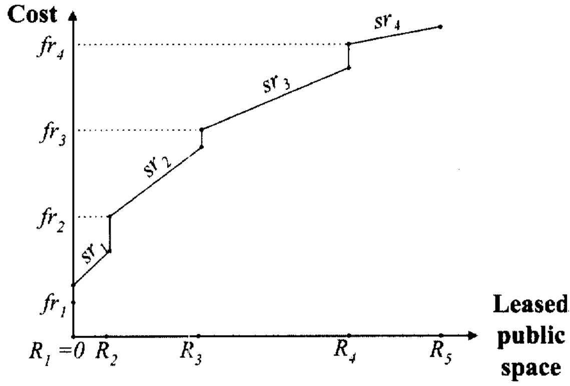
In this paper, we consider the following generalized cost models :

$$C^o(y_0) = fo_i + so_i \cdot y_0 \quad \text{if} \quad O_i \leq y_0 < O_{i+1};$$
$$C^r(y_r) = fr_j + sr_j \cdot y_r \quad \text{if} \quad R_j \leq y_r < R_{j+1}$$

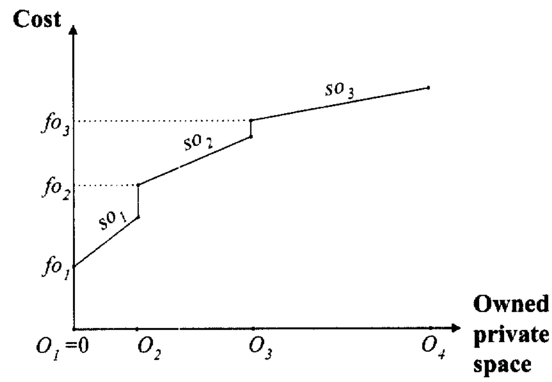
where $C^o(y_0)$ = private warehouse cost; y_0 = size of private warehouse in area units; fo_i = fixed cost for the *i-th* region of y_0, $[O_i, O_{i+1})$; so_i = variable private warehouse cost per unit area of storage for unit time; $C^r(y_r)$= leased public warehouse cost; fr_j = fixed cost for the *j-th* region of y_r, $[R_i, R_{i+1})$; sr_j = variable private warehouse cost per unit area of storage for unit time. A typical example of the cost models is shown in <Figure 1>. Note that the existing cost models (Cormier and Gunn, 1996; Lowe *et al.*, 1979; Rao and Rao, 1998; White and Francis, 1971) are special cases of our generalized cost models. In addition, the generalized cost models can represent well the transportation costs between the public and the private warehouses and any other nonlinear cost curve can be approximated flexibly due to the piecewise nature.

2.2 Storage Capacity under RAN

We first consider the RAN policy where incoming



(a) Privately owned storage space



(b) Public leased storage space

Figure 1. Cost curves for owned and public storage systems.

items are equally likely to be stored in any of storage locations in the warehouse. When no shortage is allowed, the required storage capacity of the warehouse is then equal to the maximum of aggregate inventory level for all items. In real situations, due to the dynamic nature of the replenishment process and retrieval operation of items, it is extremely difficult to exactly predict the aggregate inventory level.

In this paper, for the convenience of mathematical description, we adopt most of the terminologies used in Lee (1999). By definition,

$$X = \sum_{i=1}^{n} X_i.$$

Let $Z = (X - \hat{\mu}) / \hat{\sigma}$ where $\hat{\mu} = \sum_{i=1}^{n}(a_i + b_i)/2$ and $\hat{\sigma} = \left(\sum_{i=1}^{n}(b_i - a_i)^2/12\right)^{1/2}$. Then, if n is sufficiently large, Z follows approximately the standard normal distribution, $N(0, 1)$. Thus, for a given shortage probability α_0, the storage capacity under the RAN policy can be represented by a function of the unknown variable, α :

$$S_{\text{RAN}}(\alpha) = \hat{\mu} + z_\alpha \hat{\sigma}$$

where $\alpha \leq \alpha_0$ and z_α is determined by $\Pr(Z > z_\alpha) = \alpha$ for $0 \leq \alpha \leq 1$.

As shown in (1999), the expected amount of space shortage per unit time is given by

$$E_{\text{RAN}}(\alpha) = \alpha \cdot E(X - (\hat{\mu} + z_\alpha \hat{\sigma}) | X \geq \hat{\mu} + z_\alpha \hat{\sigma}) + (1-\alpha) \cdot 0 = \hat{\sigma}[\exp(-z_\alpha^2/2)/\sqrt{2\pi} - \alpha z_\alpha]$$

<Figure 1> shows that each warehouse cost is jumping at a set of discrete points of storage space. They are $\{O_1, O_2, \cdots, O_S\}$ for the privately owned space and $\{R_1, R_2, \cdots R_T,\}$ for the public leased space. A value of α associated with each of these points can be found from the relationships :

$$S_{\text{RAN}}(\alpha_i^o) = \hat{\mu} + z_{\alpha_i^o}\hat{\sigma} = O_i,\ \ i = 1, 2, \cdots, S \text{ and}$$
$$E_{\text{RAN}}(\alpha_j^r) = \hat{\sigma}[\exp(-z_{\alpha_j^r}^2/2)/\sqrt{2\pi} - \alpha z_{\alpha_j^r}] = R_j,\ \ j = 1, 2, \cdots, T.$$

Since $S_{\text{RAN}}(\alpha)$ and $E_{\text{RAN}}(\alpha)$ are monotonically decreasing and increasing, respectively as α increases, we can easily find the associated α_i^o and α_j^r values using a simple search technique as the bisectional search method.

Now let $\mathbf{O} = \{\alpha_i^o,\ i = 1, 2, \cdots, S\}$, $\boldsymbol{R} = \{\alpha_j^r,\ j = 1, 2, \cdots, T\}$, and $\mathbf{S} = \mathbf{O} \cup \mathbf{R}$. We first sequence all elements in $\mathbf{S}$ in increasing order. Since the public storage cost curve is not defined for $\alpha > \alpha_j^r$ and also, α should be not greater than the desired service level, α_0, we delete every α_i^o from $\mathbf{S}$ which is greater than $\alpha_{\text{up}} = \min(\alpha_j^r, \alpha_0)$. Then, we define a set of mutually exclusive feasible regions, Φ_k, $k = 1, 2, \cdots, K$, of α as $\Phi_k = [\phi_k, \phi_{k+1})$ where $\phi_k =$ the *k-th* ranked value of $\alpha \in \mathbf{S}$ and $\phi_K = \alpha_{\text{up}}$.

Let $i(k)$ and $j(k)$ be the indices of the space regions which satisfy

$$O_{i(k)} \leq S_{\text{RAN}}(\phi_{k+1}) < S_{\text{RAN}}(\phi_k) \leq O_{i(k)+1}$$

and

$$R_{j(k)} \leq E_{\text{RAN}}(\phi_k) < E_{\text{RAN}}(\phi_{k+1}) \leq R_{j(k)+1},$$

respectively. Then the slopes, φ_k^o and φ_k^r of the two cost curves in the *k-th* region will be

$$\varphi_k^o = so_{i(k)} \text{ and } \varphi_k^r = sr_{j(k)}.$$

Using the slopes, the total cost per unit time in the region can be expressed as a function of α :

$$TC_{\text{RAN}k}(\alpha) = \varphi_k^o S_{\text{RAN}}(\alpha) + \varphi_k^r E_{\text{RAN}}(\alpha) + FC_k = \varphi_k^o(\hat{\mu} + z_\alpha \hat{\sigma}) + \varphi_k^r \hat{\sigma} r(\alpha) + FC_k$$

where $\phi_k \leq \alpha < \phi_{k+1}$, $FC_k = fo_{i(k)} - \varphi_k^o O_{i(k)} + fr_{j(k)} - \varphi_k^r R_{j(k)}$ and $r(\alpha) = \exp(-z_\alpha^2/2)/\sqrt{2\pi} - \alpha z_\alpha$.

The optimal storage capacity under RAN can be obtained by searching for α which minimizes the total cost for all feasible regions. In this regard, for each of the region of $\alpha \leq \alpha_0$, the following sub-problem needs to be solved:

(P1_k) Minimize $\varphi_k^o(\hat{\mu} + z_\alpha \cdot \hat{\sigma}) + \varphi_k^r \hat{\sigma} \cdot r(\alpha) + FC_k$
subject to

$$\phi_k \leq \alpha < \phi_{k+1} \quad (1)$$

Once we solve all the sub-problems, (P1_k), $k = 1, 2, \cdots, K$, the optimal solution is then determined by choosing the region which yields the minimum total cost. Note that (P1_k) is a single variable optimization problem with the following property for the variable terms in the cost function (Lee, 1999) :

Property 1. z_α and $r(\alpha)$ are convex over $0 \leq \alpha \leq 0.5$.

Since FC_k is constant, it follows from Property 1 that $TC_{\text{RAN}k}(\alpha)$ is convex. Thus, ignoring the constraint (1), the unconstrained optimal solution of (P1_k) will be obtained from the following condition:

$$\frac{dTC_{\text{RAN}k}(\alpha)}{d\alpha} = \varphi_k^o \hat{\sigma} \cdot z_\alpha' + \varphi_k^r \hat{\sigma} \cdot r'(\alpha) = \hat{\sigma}(\varphi_k^r \alpha - \varphi_k^o)/\exp(-z_\alpha^2/2)/\sqrt{2\pi} = 0$$

Solving the above equation for α results in $\alpha_k^* = \varphi_k^o / \varphi_k^r$. If $\phi_k \leq \alpha_k^* < \phi_{k+1}$, the optimal solution of the sub-problem is obviously $\alpha_{\text{RAN}}^*(k) = \alpha_k^*$. Otherwise, it is enough to compare only the two boundary points due to the convexity of the cost function. This gives

$$\alpha_{\text{RAN}}^*(R) = \begin{cases} \phi_k, & \text{if } TC_{\text{RAN}k}(\phi_k) \leq TC_{\text{RAN}k}(\phi_{k+1}); \\ \phi_{k+1}, & \text{otherwise.} \end{cases}$$

In this way, we can solve every sub-problem for each region. Then the global optimal solution over all regions will be $\alpha_{\text{RAN}}^* = \alpha_{\text{RAN}}^*(k^*)$ where $k^* = \text{argmin}_k\ TC_{\text{RAN}k}(\alpha_{\text{RAN}}^*(k))$. The corresponding optimal storage capacities are

$$S_{\text{RAN}}(\alpha_{\text{RAN}}^*) = \hat{\mu} + z\alpha_{\text{RAN}}^* \hat{\sigma} \text{ and}$$
$$E_{\text{RAN}}(\alpha_{\text{RAN}}^*) = \hat{\sigma} \cdot r(\alpha_{\text{RAN}}^*)$$

with the minimum warehouse cost per unit time $= TC_{\text{RAN}k^*}(\alpha_{\text{RAN}}^*)$.

2.3 Storage Capacity under the CN Policy

The class-based storage policy seems to be more practical than the RAN policy in that groups of frequent items are stored at storage locations which are closer to the input/output point in the warehouse. Under this policy, items and storage locations are jointly partitioned into a small number of classes based on item turnover distributions and travel times of vehicles, respectively. Within any class, RAN is assumed to be applied to assign items to storage locations.

Suppose that n items are divided into N classes such that class j consists of items $k_{j-1}+1, k_{j-1}+2, \ldots, k_j$, $j=1, \ldots, N$ where $k_0=0$, $k_N=n$, and $N \le n$. Let $\boldsymbol{\alpha}=(\alpha_1, \alpha_2, \cdots, \alpha_N)$ be a vector of class shortage probabilities which are decision variables. Then, the storage capacity for each class can be determined using the same technique discussed under the RAN policy. The system storage capacity for the CN policy is

$$S_{\mathrm{CN}}(\boldsymbol{\alpha}) = \sum_{j=1}^{N} S_{\mathrm{CN}}^{j}(\alpha_j) = \hat{\mu} + \sum_{j=1}^{N} z_{\alpha j}\hat{\sigma}_j$$

where $S_{\mathrm{CN}}^{j}(\alpha_j)=$ the storage capacity of class j when its shortage probability is given by α_j and $\hat{\sigma}_j = \left(\sum_{i=k_{j-1}+1}^{k_j} w_i^2/12\right)^{1/2}$. The space shortage for the CN policy is expressed as follows:

$$E_{\mathrm{CN}}(\boldsymbol{\alpha}) = \sum_{j=1}^{N} \hat{\sigma}_j(\exp(-z_{\alpha j}^2/2)/\sqrt{2\pi} - \alpha_j z_{\alpha j}).$$

Then, we can formulate the storage capacity problem for CN as a nonlinear integer program:

$$\text{(P2) Minimize } S_{\mathrm{CN}}(\boldsymbol{\alpha}) \sum_{s=1}^{S} so_s Y_s + E_{\mathrm{CN}}(\boldsymbol{\alpha}) \sum_{t=1}^{T} sr_t y_t + \sum_{s=1}^{S}(fo_s - so_t O_s) y_s + \sum_{t=1}^{T}(fr_t - sr_t O_t) y_t$$

subject to

$$\prod_{j=1}^{N}(1-\alpha_j) \ge 1-\alpha_0 \qquad (2)$$

$$O_s \le S_{\mathrm{CN}}(\boldsymbol{\alpha}) y_s + M(1-y_s) \qquad (3)$$

$$S_{\mathrm{CN}}(\boldsymbol{\alpha}) y_s < O_{s+1} \qquad (4)$$

$$\sum_{s=1}^{S} y_s = 1 \qquad (5)$$

$$y_s = 0 \text{ or } 1 \qquad \forall s \qquad (6)$$

$$R_t \le E_{\mathrm{CN}}(\boldsymbol{\alpha}) y_t + M(1-y_t) \qquad (7)$$

$$E_{\mathrm{CN}}(\boldsymbol{\alpha}) y_t < R_{t+1} \qquad (8)$$

$$\sum_{t=1}^{T} y_t = 1 \qquad (9)$$

$$y_t = 0 \text{ or } 1 \qquad \forall t \qquad (10)$$

$$0 \le \alpha_j \le u_\alpha \qquad \forall j \qquad (11)$$

where u_α is an upper bound of the shortage probabilities which satisfies the inequality :

$$(1-u_\alpha)^n \ge 1-\alpha_0$$

Constraint (2) describes the system shortage-probability requirement. Constraints (3)~(6) assure that the calculated owned space is included in a single space region which is defined by a pair of discrete points of space, $[O_s, O_{s+1})$, $s=1, \cdots, S$. The same condition for the public space requirement is ensured by constraints (7)~(10). Finally, a feasible range of each shortage probability is given by constraint (11).

Now, the total cost for (s, t) space region combination becomes

$$\begin{aligned} TC_{\mathrm{CN}}^{st}(\boldsymbol{\alpha}) &= \lambda_1 S_{\mathrm{CN}}(\boldsymbol{\alpha}) + \lambda_2 E_{\mathrm{CN}}(\boldsymbol{\alpha}) + FC_{st} \\ &= \lambda_1(\hat{\mu} + \sum_{j=1}^{N} z_{\alpha j}\hat{\sigma}_j) + \lambda_2 \sum_{j=1}^{N} \hat{\sigma}_j(\exp(-z_{\alpha j}^2/2) \\ &\quad /\sqrt{2\pi} - \alpha_j z_{\alpha j}) + FC_{st} \\ &= \lambda_1 \hat{\mu} + \lambda_2 \sum_{j=1}^{N} \hat{\sigma}_j[(\lambda - \alpha_j) z_{\alpha j} + \exp(-z_{\alpha j}^2/2) \\ &\quad /\sqrt{2\pi}] + FC_{st} \end{aligned} \qquad (12)$$

where $FC_{st} = fo_s - so_s O_s + fr_t - sr_t R_t$, $\lambda_1 = so_s$, $\lambda_2 = sr_t$, and $\lambda = \lambda_1/\lambda_2$. Since we want to minimize the total cost such that overall shortage probability should not exceed α, after eliminating the constant terms in (12), the sub-problem of (P2) for (s, t) space region combination can be formulated as follows:

$$\begin{aligned} (\mathrm{P2}_{st}) \text{ Minimize } & \sum_{j=1}^{N} \hat{\sigma}_j[(\lambda - \alpha_j) z_{\alpha j} + \exp(-z_{\alpha j}^2/2)/\sqrt{2\pi}] \\ \text{subject to } & \prod_{j=1}^{N}(1-\alpha_j) \ge 1-\alpha_0 \\ & O_s \le S_{\mathrm{CN}}(\boldsymbol{\alpha}) < O_{s+1} \\ & R_t \le E_{\mathrm{CN}}(\boldsymbol{\alpha}) < R_{t+1} \\ & 0 \le \alpha_j \le u_\alpha \quad \forall j \end{aligned}$$

Notice that if we let $\tau_j = -\ln(1-\alpha_j)$, since $z_{\alpha j} = -z_{1-\alpha j}$, the variable term in the objective function of $(\mathrm{P2}_{st})$ becomes

$$\begin{aligned} &(\lambda - \alpha_j) z_{\alpha j} + \exp(-z_{\alpha j}^2/2)/\sqrt{2\pi} \\ &= (1 - e^{-\tau j} - \lambda) z_{e^{-\tau_j}} + \exp(-z_{e^{-\tau j}}^2/2)/\sqrt{2\pi} \end{aligned}$$

This objective function has the following property (Lee, 1999):

Property 2. Let $g(\tau) = (1 - e^{-\tau} - \lambda) z_{e^{-\tau}} + \exp(-z_{e^{-\tau}}^2/2)/\sqrt{2\pi}$. Then $g(\tau)$ is convex over $0 \le \tau \le -\ln 0.5$

Following Property 2, we can rewrite the problem ($P2_{st}$) as a separable convex program:

$$\text{(P3) Minimize } \sum_{j=1}^{N} \hat{\sigma}_j g(\tau_j) \quad (13)$$

$$\text{subject to } \sum_{j=1}^{N} \tau_j \le -\ln(1-\alpha_0)$$

$$O_s \le S'_{CN}(\boldsymbol{\tau}) < O_{s+1} \quad (14)$$

$$R_t \le E'_{CN}(\boldsymbol{\tau}) < R_{t+1} \quad (15)$$

$$0 \le \tau_j \le u_\tau \quad (16)$$

where τ = an N-dimensional vector of unknowns, $S'_{CN}(\boldsymbol{\tau}) = S_{CN}(\boldsymbol{\alpha})\,|_{\alpha_j = 1-e^{-\tau_j}}$, $\forall j$, $E'_{CN}(\boldsymbol{\tau}) = E_{CN}(\boldsymbol{\alpha})\,|_{\alpha_j = 1-e^{-\tau_j}}$, $\forall j$, and $u_\tau = -\ln(1-u_\alpha)$.

The first derivative of $g(\tau_j)$ in the objective function (13) is obtained by using the chain rule :

$$\delta(\tau_j) = \frac{\partial g(\tau_j)}{\partial \tau_j} = \frac{\lambda z_\alpha + r(\alpha)}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial \tau_j}$$

$$= [(\alpha - \lambda)/y(z_\alpha)]e^{-\tau_j}$$

$$= e^{-\tau_j}(1 - e^{-\tau_j} - \lambda)/y(z_{1-e^{-\tau_j}}) \quad (17)$$

where $y(z) = \exp(-z^2/2)/\sqrt{2\pi}$. Note that when $\lambda < 1$,

$$\delta(\tau_j) \le 0 \text{ for } \tau_j \le = -\ln(1-\lambda),$$

$$\delta(\tau_j) > 0, \text{ otherwise.}$$

On the other hand, when $\lambda \ge 1$, $\delta(\tau_j) < 0 \ \forall \tau_j$. Therefore it follows from the convexity property of $g(\tau_j)$ that when $\lambda < 1$, $g(\tau_j)$ is non-increasing over $\tau_j \le -\ln(1-\lambda)$, and non-decreasing over $\tau_j > -\ln(1-\lambda)$. When $\lambda \ge 1$, $g(\tau_j)$ will be non-increasing $\forall \tau_j$. By the way, it is easily found from (17) that there is at most only a single value, τ_j^0 of τ_j which satisfies $\delta(\tau_j) = 0$, for all j:

$$\tau_j^0 = \tau_0 = -\ln(1-\lambda) \quad \text{for } \lambda < 1,$$

$$= \infty, \qquad \text{otherwise.}$$

From this analysis, we know that ignoring constraints (14) and (15) the problem (P2) is a special case of the following nonlinear program:

$$\text{(P4) Minimize } Z = \sum_{i=1}^{n} c_i f(x_i) \quad (18)$$

$$\text{subject to } \sum_{i=1}^{n} x_i \le r_1 \quad (19)$$

$$0 \le x_i \le r_2 \quad \forall i \quad (20)$$

where $c_i (c_i \ge c_j,\ i<j)$, r_1 and r_2 $(r_1 \ge r_2)$ are positive constants and $f(x_i)$ is a nonnegative convex function of x_i over $0 \le x_i \le \max(x_0, r_2)$ whose minimum lies at $x_i = x_0 > 0\ \forall i$. Also, $f'(x_i)$ is assumed to be nonpositive and concave for $0 \le x_i \le x_0$. Lee (1998) suggested a search procedure (here we call it *Algorithm* 1) which generates optimal solutions for (P4). Therefore, (P3) without constraints (14) and (15) can be solved optimally using the algorithm with the following mapping :

$$n = N,\ i = j,\ c_i = \hat{\sigma}_j,\ \ x_i = \tau_j,\ \ f(x_i) = g(\tau_j),$$

$$r_1 = -\ln(1-\alpha_0),\ r_2 = u_\tau, \text{ and } x_0 = \tau_0. \quad (21)$$

Consequently, considering (16) the feasible solution for a given μ will be

$$\tau_j(\mu) = \min(\max(0,\ \tau_j),\ u_\tau) \text{ and}$$

$$\alpha_j = 1 - \exp[-\min(\max(0,\ \tau_j),\ u_\tau)].$$

Due to the monotonic nature of $S'_{CN}(\boldsymbol{\tau})$ and $E'_{CN}(\boldsymbol{\tau})$ according to the variation of μ, constraints (14) and (15) reduce to corresponding boundary constraints of the dual variable μ. As a result, (P3) and its special version of ($P2_{st}$) can be solved exactly by the following procedure called *Algorithm* 2:

Step 0: Set $s = t = 1$; $Z_{\min} = M$.

Step 1: Set up ($P2_{st}$), (P3), and (P4). Let u be the dual variable associated with the constraint (19). Find $u = u^*$ by solving (P4) using *Algorithm* 1 with $\lambda = so_s / sr_t$ and (21).

Step 2: Determine the boundary values of u corresponding to the constraints (14) and (15), u_1^{lb}, u_1^{ub}, u_2^{lb} and u_2^{ub}, respectively. Solve (P3) with them and compute the corresponding optimal solution of ($P2_{st}$). If ($P2_{st}$) is infeasible, set $TC_{CN}^{st}(\boldsymbol{\alpha}) = M$.

Step 3: If $Z_{\min} > TC_{CN}^{st}(\boldsymbol{\alpha})$, then let $Z_{\min} = TC_{CN}^{st}(\boldsymbol{\alpha})$ and reset the optimal value of $\boldsymbol{\alpha}$, $\alpha_i^* = \alpha_i^{st}$ $\forall i$. Repeat Step 1 to Step 3 for every combination of ($s \le S$, $t \le T$).

3. Application to the EOQ Model

For the application of the developed storage capacity models, we consider a unit-load AS/R system. In the system all items are assumed to be ordered based on the standard EOQ inventory model. As the previous studies (Hausman, Schwarz and Graves, 1976; Lee, 1998; Lee, 1999), we approximate the demand rate by a discrete geometric probability distribution which is given by:

$$d_i = D_0 f_d(i) = D_0 p(1-p)^{i-1} / (1-(1-p)^n),$$

$i=1, \cdots, n$

where D_0 and p are the total demand per period for all items measured in full pallet loads and the skewness parameter of the distribution, respectively. In this case, the inventory level of item i, X_i, can be considered to follow the uniform distribution, $U(0, b_i)$ where $b_i = [2\xi D_0 p(1-p)^{i-1}/(1-(1-p)^n)]^{1/2}$.

We solved example problems using the following hypothetical data:

$n = 100$, $D_0 = 10000$, $\xi = 1$, $u_a = 0.05$, $a_0 = 0.1$,
$p = 0.0075$, 0.0448, 0.1088, 0.1391, $S = 9$, $T = 10$,
fo_i, $i = 1, \cdots, 9$ = 400, 1600, 2200, 2640, 3040, 3400, 3720, 4000, 4240,
O_i, $i = 1, \cdots, 10$ = 0, 400, 600, 800, 1000, 1200, 1400, 1600, 1800, 10000,
so_i, $i = 1, \cdots, 9$ = 2, 1.5, 1.2, 1, 0.8, 0.6, 0.4, 0.2, 0.1,
fr_i, $i = 1, \cdots, 10$ = 0, 25, 45, 60, 72.5, 82.5, 90, 97, 100, 107,
R_i, $i = 1, \cdots, 11$ = 0, 2, 4, 6, 8, 10, 12, 14, 16, 20, 40,
sr_i, $i = 1, \cdots, 10$ = 10, 7.5, 5, 3.75, 2.5, 1.25, 1, 0.8, 0.5, 0.3.

For the purpose of comparison, we consider the rule of thumb which can be used in practice is to set the capacity equal to 85% of that required for the FULL policy with $a_0 = 0$ (Cormier and Gunn, 1996). The numbers of different items included in each class for the CN policy are set to be equal in the example problems.

The trade-off between warehouse size and system throughput needs to be analyzed in the earlier stage of system design. Once the optimal storage capacity is found, we can compute the expected travel time taken by a stacker crane to store or retrieve a unit load in the system by the slightest modification of the existing sttistical techniques (Bozer and White, 1984; Hausman *et al.*, 1976). Then, the throughput of the system is determined by the reciprocal of the expected travel time. Here, we present expected travel time figures for the square-in time AS/R system (Bozer and White, 1984).

The overall results including those obtained by the rule of thumb (here, denoted by RAN1) are summarized in <Tables 1>, <Table 2>. In <Table 1>, α^*_{RAN} denotes the optimal value of α for the RAN policy and α^*_j, $j=1, \cdots, N$, that for class j, and CJ, the J-class based storage policy. All the figures in <Table 2> are expressed as a relative ratio to the corresponding value of the RAN policy. Usually, the number of classes is not greater than 3 in most real applications. Here, we consider warehouses with up to 5 classes due to our

Table 1. Optimal owned storage capacities and shortage probabilities for different values of p

p	Storage Capacity			Optimal Shortage Probability					
	RAN	RAN1	C5	α^*_{RAN}	α^*_1	α^*_2	α^*_3	α^*_4	α^*_5
0.0075	1,759.46	2,672.25	2,044.38	0.02	0.01	0.01	0.01	0.01	0.01
0.0448	1,494.43	2,268.86	1,691.67	0.04	0.02	0.02	0.02	0.02	0.02
0.1088	1,045.8	1,579.21	1,150.03	0.1	0.05	0.036	0.013	0.004	0.001
0.1391	933.7	1,388.63	1,012.69	0.1	0.05	0.04	0.01	0.002	0

Table 2. Owned storage capacity required and expected travel time for different combinations of storage policy and skewness parameter

p	RAN	RAN1	C2	C3	C4	C5
0.0075	1.0 1.0	1.519 1.232	1.063 1.004	1.101 1.016	1.134 1.028	1.162 1.04
0.0448	1.0 1.0	1.518 1.232	1.027 0.933	1.058 0.908	1.108 0.912	1.132 0.912
0.1088	1.0 1.0	1.510 1.229	1.044 0.994	1.061 0.961	1.077 0.936	1.112 0.919
0.1391	1.0 1.0	1.487 1.220	1.039 1.008	1.052 0.984	1.066 0.961	1.085 0.942

* Figures in the upper lines for each combination of 2 and p denote relative ratios of storage capacities to the corresponding value of RAN and figures in the lower, those of expected travel times.

interest in the behavior of the CN policy.

The following observations are made from <Table 1>:

1) The storage capacity for RAN1 is much larger than that for RAN, which indicates that warehouses designed based on the rule of thumb are significantly larger than actually required.
2) In case of C4 and C5, higher shortage probabilities are preferentially assigned to highly frequent classes due to the nonincreasing function of $\hat{\sigma}_j$ in (13).
3) As the value of p increases, so does α^*_{RAN}.

From <Table 2>, we observe:

1) The storage capacities obtained for the seven policies always meet the relationship,

$$S_{\mathrm{RAN}}(\alpha) < S_{C2}(\alpha) < \cdots < S_{C5}(\alpha) < S_{\mathrm{RAN1}}(\alpha)$$

which verifies the intuition in the beginning.

2) The storage capacities required for the class-based policies are not greatly larger than those for RAN. The largest % increment from even C5 over the RAN policy is only 16.2% which is obtained when $p = 0.0075$. However, in case of RAN1, the % increment increases up to 51.9%.
3) The % increment in storage capacity for class-based policies over RAN seems to decrease as the skewness of demand distribution increases. Particularly for C5, the % increment decreases from 16.2% to 8.5%.
4) Any improvement in travel time from class-based policies over RAN may not be expected when the skewness of demand curve is low, i.e., $p = 0.0075$. However, except those cases dramatic throughput improvements are obtained as the number of class becomes larger. These observations may give the system designer a motivation to use the class-based policies unless the increment in storage space outweigh the throughput improvement.

4. Conclusions

In this paper, we consider the storage sizing problem for warehouses under the randomized and class-based storage policies. The total cost of the problem includes the cost incurred from owning the storage space for the warehouse and that from contracting space outside of the company for shortage space. Therefore, solutions of the problems show a trade-off analysis between privately owned space and public leased space. The problem for each of the storage policies has been formulated as a nonlinear optimization model. Exact optimal solutions of the model for the RAN policy can be easily obtained by taking advantage of the convexity property for the objective function. An efficient iterative search procedure is developed to generate optimal solutions for the CN policy based on an existing search procedure. The methodology presented in this paper may contribute to determining optimal warehouse sizes when outsourcing of storage space is needed for installing a reasonable supply chain system of inventories.

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