# GAUSS SUMS OVER GALOIS RINGS OF CHARACTERISTIC 4 

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#### Abstract

In this paper, we define and study Gauss sums over Galois rings of characteristic 4. In particular, we give the absolute value of Gauss sum over Galois rings of characteristic 4.


## 1. Introduction

Let $G F(2)$ be the prime field of characteristic 2 and $G F\left(2^{r}\right)$ an extension field of degree $r$. Then $G F\left(2^{r}\right)$ is a simple algebraic extension over $G F(2)$. That is, if $\Theta$ is a primitive element of $G F\left(2^{r}\right)$, then

$$
\begin{equation*}
G F\left(2^{r}\right)=G F(2)[\Theta] \cong G F(2)[x] /(F(x)) \tag{1.1}
\end{equation*}
$$

where $F(x)$ is a monic irreducible polynomial in $G F(2)[x]$ of degree $r$ having $\Theta$ as a root.

Let $\mathbb{Z} / 4 \mathbb{Z}$ denote the ring of integers modulo 4 . It is a finite local commutative ring with the unique maximal ideal $m=2(\mathbb{Z} / 4 \mathbb{Z}) \cong$ $\mathbb{Z} / 2 \mathbb{Z}$. Let $\mu_{1}: \mathbb{Z} / 4 \mathbb{Z} \rightarrow(\mathbb{Z} / 4 \mathbb{Z}) / m \cong G F(2)$ denote reduction modulo 2. We can extend $\mu_{1}$ to $(\mathbb{Z} / 4 \mathbb{Z})[x]$ in the natural way.

In (1.1), since $\Theta$ is a simple zero of $F(x)$, if $f \in(\mathbb{Z} / 4 \mathbb{Z})[x]$ is a preimage of $F$ under the homomorphism $\mu_{1}$, then by Theorem 1.1 below, there is precisely one element $\theta$ such that $\mu_{1}(\theta)=\Theta$ and $f(\theta)=0$.

Theorem 1.1 [3, Lemma (XV.1)]. Let $f \in(\mathbb{Z} / 4 \mathbb{Z})[x]$ be a regular polynomial (i.e., $\mu_{1}(f) \neq 0$ ) and suppose that $\mu_{1}(f)$ has a simple zero $a$ in $G F\left(2^{r}\right)$. Then $f$ has one and only one zero $\alpha$ such that $\mu_{1}(\alpha)=a$.

The Galois rings $\mathcal{R}$ of characteristic 4 is defined to be the ring $(\mathbb{Z} / 4 \mathbb{Z})[\theta]$. Many papers have been studied concerning Gauss sums

[^0]over finite fields (see [1]). In this paper, we define and study Gauss sums over Galois rings $\mathcal{R}$. In particular, we give the absolute value of Gauss sum over $\mathcal{R}$.

## 2. Characters on the Galois rings $\mathcal{R}$

It is well-known in [cf. 3] that
(GR1) $\mathcal{R}$ is a finitely generated free $\mathbb{Z} / 4 \mathbb{Z}$-module and $|\mathcal{R}|=4^{r}$.
(GR2) $\mathcal{R}$ is a finite local commutative ring with the unique maximal ideal $M=2 \mathcal{R}$ and the residue field $K=\mathcal{R} / M \cong G F\left(2^{r}\right)$.
(GR3) $\operatorname{Gal}(\mathcal{R} /(\mathbb{Z} / 4 \mathbb{Z})) \cong \operatorname{Gal}\left(G F\left(2^{r}\right) / G F(2)\right)$ and the Frobenius automorphism $\sigma$ of $\mathcal{R}$ given by $\theta \mapsto \theta^{2}$ is a generator of $\operatorname{Gal}(\mathcal{R} /(\mathbb{Z} / 4 \mathbb{Z}))$.
(GR4) Let $\mathcal{R}^{*}$ and $K^{*}$ denote the unit group of $\mathcal{R}$ and $K$, respectively. Then

$$
\begin{equation*}
\mathcal{R}^{*} \cong K^{*} \times(1+M) \quad(\text { direct product of groups }) \tag{2.1}
\end{equation*}
$$

where $K^{*}$ is a cyclic group of order $2^{r}-1$ and $1+M$ is a group of order $2^{r}$ such that $1+M$ is a direct product of $r$ cyclic groups each of order 2 .

From (2.1), $\mathcal{R}^{*}$ contains a cyclic subgroup $\mathcal{T}_{r}^{*}$ of order $2^{r}-1$. Let $\theta$ be a generator of $\mathcal{T}_{r}^{*}$ (such $\theta$ is called a primitive element of $\mathcal{R}$ ) and

$$
\mathcal{T}_{r}=\mathcal{T}_{r}^{*} \cup\{0\}=\left\{\theta^{i} \mid 0 \leq i \leq 2^{r}-2\right\} \cup\{0\},
$$

which is called the Teichmüller set for $K(=\mathcal{R} / M)$ in $\mathcal{R}$. Then $\mathcal{T}_{r}$ is isomorphic to $G F\left(2^{r}\right)$ under the homomorphism obtained by reduction modulo 2. It can be shown that every element $s \in \mathcal{R}$ has the 2 -adic expansion $s=\alpha+2 \beta \quad\left(\alpha, \beta \in \mathcal{T}_{r}\right)$. Thus $M=2 \mathcal{T}_{r}, M^{2}=0$ and every element of $\mathcal{R}^{*}$ has a unique representation in the form

$$
\begin{equation*}
\alpha(1+2 \beta)\left(\alpha \in \mathcal{T}_{r}^{*}, \beta \in \mathcal{T}_{r}\right) . \tag{2.2}
\end{equation*}
$$

Also, from (GR3) the Frobenius automorphism $\sigma$ on $\mathcal{R}$ is given by $\sigma(s)=\sigma(\alpha+2 \beta)=\alpha^{2}+2 \beta^{2}$. In analogy with finite fields, the trace function $\operatorname{Tr}: \mathcal{R} \rightarrow \mathbb{Z} / 4 \mathbb{Z}$ is defined by

$$
\begin{equation*}
\operatorname{Tr}(s)=\sum_{\tau \in \operatorname{Gal}(\mathcal{R} /(\mathbb{Z} / 4 \mathbb{Z}))} \tau(s)=\sum_{i=0}^{r-1} \sigma^{i}(s) . \tag{2.3}
\end{equation*}
$$

Also, the additive characters $\lambda_{t}(t \in \mathcal{R})$ on $\mathcal{R}$ are defined by

$$
\begin{equation*}
\lambda_{t}(s)=\sqrt{-1}^{\operatorname{Tr}(t s)} \quad \text { for } \quad s \in \mathcal{R} \tag{2.4}
\end{equation*}
$$

We see that $\lambda_{0}$ is the trivial character on $\mathcal{R}, \lambda_{t}(s)=\lambda_{1}(t s)$ and $\overline{\lambda_{t}(s)}=$ $\lambda_{t}(-s)$. Also, if $t_{1} \neq t_{2}\left(t_{1}, t_{2} \in \mathcal{R}\right)$, then $\lambda_{t_{1}} \neq \lambda_{t_{2}}$.

Since $1+M$ has the structure of a multiplicative group of order $2^{r}$, $1+M$ is isomorphic to the additive group of $G F\left(2^{r}\right)$ via the map

$$
\begin{equation*}
1+2 \beta \mapsto y \quad\left(\beta \in \mathcal{T}_{r} \text { with } \beta \equiv y(\bmod M), y \in G F\left(2^{r}\right)\right) \tag{2.5}
\end{equation*}
$$

Hence there is a one-to-one correspondence between the set of all multiplicative characters on $1+M$ and the set of all additive characters on $G F\left(2^{r}\right)$. Thus, each multiplicative character $\chi$ on $\mathcal{R}^{*}$ can be written as

$$
\begin{equation*}
\chi(s)=\eta(\alpha) \psi_{x}(y) \tag{2.6}
\end{equation*}
$$

for all $s=\alpha(1+2 \beta)\left(\alpha \in \mathcal{T}_{r}^{*}, \beta \in \mathcal{T}_{r}\right.$ with $\beta \equiv y(\bmod M), y \in$ $G F\left(2^{r}\right)$ ), where $\eta$ is a character on $\mathcal{T}_{r}^{*}$ and $\psi_{x}$ is an additive character on $G F\left(2^{r}\right)^{+}\left(x \in G F\left(2^{r}\right)\right)$ which is given by

$$
\begin{equation*}
\psi_{x}(y)=(-1)^{\operatorname{tr}(x y)} \quad \text { for } \quad y \in G F\left(2^{r}\right) \tag{2.7}
\end{equation*}
$$

where $\operatorname{tr}(x)$ is the trace of $x$ from $G F\left(2^{r}\right)$ to $G F(2)$ given by $\operatorname{tr}(x)=$ $\sum_{j=0}^{r-1} x^{2^{j}}$.

## 3. Gauss sums over Galois rings $\mathcal{R}$ and its absolute value

Let $\mathbb{C}[\mathcal{R}]$ denote the space of all $\mathbb{C}$-valued functions on $\mathcal{R}$. Then the set of characteristic functions $\left\{\delta_{t} \mid t \in \mathcal{R}\right\}$, where

$$
\delta_{t}(s)=\left\{\begin{array}{lll}
1 & \text { if } & s=t \\
0 & \text { if } & s \neq t
\end{array}\right.
$$

is a basis of $\mathbb{C}[\mathcal{R}]$. Hence $\mathbb{C}[\mathcal{R}]$ is a $4^{r}$-dimensional $\mathbb{C}$-vector space. We define an inner product on $\mathbb{C}[\mathcal{R}]$ by

$$
\langle f, g\rangle=\frac{1}{|\mathcal{R}|} \sum_{s \in \mathcal{R}} f(s) \overline{g(s)}=\frac{1}{4^{r}} \sum_{s \in \mathcal{R}} f(s) \overline{g(s)} .
$$

Then $\left\{\delta_{t} \mid t \in \mathcal{R}\right\}$ is an orthonormal basis of $\mathbb{C}[\mathcal{R}]$ and the set of all additive characters on $\mathcal{R}^{+}$is an orthonormal basis of $\mathbb{C}[\mathcal{R}]$ by the character orthogonality condition [2, Theorem 4.4]

$$
\sum_{s \in \mathcal{R}} \lambda_{t}(s)=\left\{\begin{array}{lll}
4^{r} & \text { if } & \lambda_{t} \text { is trivial }  \tag{3.1}\\
0 & \text { if } & \lambda_{t} \text { is nontrivial. }
\end{array}\right.
$$

It is convenient to extend the domain of definition of a character $\chi$ from $\mathcal{R}^{*}$ to $\mathcal{R}$ by setting

$$
\chi(M)=\left\{\begin{array}{lll}
1 & \text { if } & \chi \text { is trivial }  \tag{3.2}\\
0 & \text { if } & \chi \text { is nontrivial. }
\end{array}\right.
$$

With above definition we have

$$
\sum_{s \in \mathcal{R}} \chi(s)= \begin{cases}4^{r} & \text { if } \quad \chi \text { is trivial }  \tag{3.3}\\ 0 & \text { if } \quad \chi \text { is nontrivial },\end{cases}
$$

and $\chi \in \mathbb{C}[\mathcal{R}]$. Thus

$$
\begin{equation*}
\chi=\sum_{\lambda_{t}}\left\langle\chi, \overline{\lambda_{t}}\right\rangle \overline{\lambda_{t}}=\sum_{t \in \mathcal{R}}\left\langle\chi, \overline{\lambda_{t}}\right\rangle \overline{\lambda_{t}}=\frac{1}{4^{r}} \sum_{t \in \mathcal{R}} g\left(\chi, \lambda_{t}\right) \overline{\lambda_{t}} \tag{3.4}
\end{equation*}
$$

where

$$
\begin{equation*}
g\left(\chi, \lambda_{t}\right)=\sum_{s \in \mathcal{R}} \chi(s) \lambda_{t}(s) . \tag{3.5}
\end{equation*}
$$

We call each $g\left(\chi, \lambda_{t}\right)$ the Gauss sum over $\mathcal{R}$. From (3.2), (3.3) and (3.5) we have

$$
g\left(\chi, \lambda_{t}\right)=\left\{\begin{array}{lll}
4^{r} & \text { if } & \chi \text { and } \lambda_{t} \text { are both trivial } \\
0 & \text { if } & \chi \text { is nontrivial and } \lambda_{t} \text { is trivial. }
\end{array}\right.
$$

Also, we have the following theorem.
Theorem 3.1. Let $\chi$ be a nontrivial character on $\mathcal{R}^{*}$. Then (a) If $t \in \mathcal{R}^{*}$, then $g\left(\chi, \lambda_{t}\right)=\chi\left(t^{-1}\right) g\left(\chi, \lambda_{1}\right)$. In particular, $\overline{g\left(\chi, \lambda_{1}\right)}=$ $\chi(-1) g\left(\bar{\chi}, \lambda_{1}\right)$. (b) If $t \in M$ and $\chi$ is nontrivial on $1+M$, then $g\left(\chi, \lambda_{t}\right)=0$.

Proof. For (a). If $t \in \mathcal{R}^{*}$, then

$$
\begin{aligned}
g\left(\chi, \lambda_{t}\right) & =\sum_{s \in \mathcal{R}^{*}} \chi(s) \lambda_{1}(t s)=\chi\left(t^{-1}\right) \sum_{s \in \mathcal{R}^{*}} \chi(t s) \lambda_{1}(t s) \quad(\text { setting } u=t s) \\
& =\chi\left(t^{-1}\right) \sum_{u \in \mathcal{R}^{*}} \chi(u) \lambda_{1}(u)=\chi\left(t^{-1}\right) g\left(\chi, \lambda_{1}\right) .
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
\overline{g\left(\chi, \lambda_{1}\right)} & =\sum_{s \in \mathcal{R}^{*}} \chi\left(s^{-1}\right) \lambda_{1}(-s)=\chi(-1) \sum_{s \in \mathcal{R}^{*}} \chi\left((-s)^{-1}\right) \lambda_{1}(-s) \\
& =\chi(-1) \sum_{s \in \mathcal{R}^{*}} \chi\left(s^{-1}\right) \lambda_{1}(s)=\chi(-1) g\left(\bar{\chi}, \lambda_{1}\right) .
\end{aligned}
$$

To prove (b), let $t \in M$. If $\chi$ is nontrivial on $1+M$, then by (2.6) we have $\chi=\eta \cdot \psi_{x}$, where $\eta$ is a character on $\mathcal{T}_{r}^{*}$ and $\psi_{x}$ is a nontrivial character on $G F\left(2^{r}\right)^{+}$. Thus

$$
\begin{aligned}
& g\left(\chi, \lambda_{t}\right)=\sum_{s \in \mathcal{R}^{*}} \chi(s) \lambda_{t}(s) \\
& \left(s=\alpha(1+2 \beta), \alpha \in \mathcal{T}_{r}^{*}, \beta \in \mathcal{T}_{r} \text { with } \beta \equiv y(\bmod M), y \in G F\left(2^{r}\right)\right) \\
& \quad=\sum_{\alpha \in \mathcal{T}_{r}^{*}} \sum_{y \in G F\left(2^{r}\right)} \eta(\alpha) \psi_{x}(y) \lambda_{t}(\alpha(1+2 y)) \\
& \quad=\sum_{\alpha \in \mathcal{T}_{r}^{*}} \sum_{y \in G F\left(2^{r}\right)} \eta(\alpha)(-1)^{\operatorname{tr}(x y)} \sqrt{-1}^{\operatorname{Tr}(t \alpha(1+2 y))}
\end{aligned}
$$

(by (2.4) and (2.7))

$$
=\sum_{\alpha \in \mathcal{T}_{r}^{*}} \eta(\alpha) \sqrt{-1}^{\operatorname{Tr}(t \alpha)} \sum_{y \in G F\left(2^{r}\right)}(-1)^{\operatorname{tr}(x y)} \sqrt{-1}^{\operatorname{Tr}(2 t \alpha y)}
$$

Since $t \alpha \in M$, i.e., $t \alpha \equiv 0(\bmod M)$, we get

$$
2 t \alpha y \equiv 0(\bmod 4)
$$

Hence

$$
g\left(\chi, \lambda_{t}\right)=\sum_{\alpha \in \mathcal{T}_{r}^{*}} \eta(\alpha) \lambda_{t}(\alpha) \sum_{y \in G F\left(2^{r}\right)} \psi_{x}(y)=0
$$

since $\sum_{y \in G F\left(2^{r}\right)} \psi_{x}(y)=0$ for a nontrivial character $\psi_{x}$.

Corollary 3.2. Let $\chi$ be a nontrivial character on $\mathcal{R}^{*}$. If $\chi$ is nontrivial on $1+M$, then

$$
\chi=4^{-r} g\left(\chi, \lambda_{1}\right) \sum_{t \in \mathcal{R}^{*}} \chi\left(t^{-1}\right) \overline{\lambda_{t}} .
$$

Proof.

$$
\begin{aligned}
\chi & =4^{-r} \sum_{t \in \mathcal{R}} g\left(\chi, \lambda_{t}\right) \overline{\lambda_{t}} \quad(\text { see }(3.4)) \\
& =4^{-r} \sum_{t \in \mathcal{R}^{*}} g\left(\chi, \lambda_{t}\right) \overline{\lambda_{t}}+4^{-r} \sum_{t \in M} g\left(\chi, \lambda_{t}\right) \overline{\lambda_{t}} \\
& =4^{-r} g\left(\chi, \lambda_{1}\right) \sum_{t \in \mathcal{R}^{*}} \chi\left(t^{-1}\right) \overline{\lambda_{t}} \quad \text { (by Theorem 3.1 (a) and (b)). }
\end{aligned}
$$

Theorem 3.3. Let $\chi$ be a nontrivial character on $\mathcal{R}^{*}$. If $\chi$ is nontrivial on $1+M$, then

$$
\begin{gather*}
g\left(\chi, \lambda_{1}\right) g\left(\bar{\chi}, \lambda_{1}\right)=4^{r} \chi(-1) \quad \text { and }  \tag{3.6}\\
\left|g\left(\chi, \lambda_{t}\right)\right|= \begin{cases}2^{r} & \text { if } t \in \mathcal{R}^{*} \\
0 & \text { if } t \in M\end{cases} \tag{3.7}
\end{gather*}
$$

Proof. For (3.6). Theorem 3.1 (a) and (b) imply

$$
\begin{aligned}
\sum_{t \in \mathcal{R}} g\left(\chi, \lambda_{t}\right) g\left(\bar{\chi}, \lambda_{t}\right) & =\sum_{t \in \mathcal{R}^{*}} g\left(\chi, \lambda_{t}\right) g\left(\bar{\chi}, \lambda_{t}\right)+\sum_{t \in M} g\left(\chi, \lambda_{t}\right) g\left(\bar{\chi}, \lambda_{t}\right) \\
& =g\left(\chi, \lambda_{1}\right) g\left(\bar{\chi}, \lambda_{1}\right) \sum_{t \in \mathcal{R}^{*}} 1,
\end{aligned}
$$

and so

$$
\begin{equation*}
\sum_{t \in \mathcal{R}} g\left(\chi, \lambda_{t}\right) g\left(\bar{\chi}, \lambda_{t}\right)=\left(4^{r}-2^{r}\right) g\left(\chi, \lambda_{1}\right) g\left(\bar{\chi}, \lambda_{1}\right) . \tag{3.8}
\end{equation*}
$$

On the other hand, (3.5) yields that

$$
\begin{aligned}
& \sum_{t \in \mathcal{R}} g\left(\chi, \lambda_{t}\right) g\left(\bar{\chi}, \lambda_{t}\right)=\sum_{t \in \mathcal{R}} \sum_{s \in \mathcal{R}^{*}} \sum_{u \in \mathcal{R}^{*}} \chi\left(s u^{-1}\right) \lambda_{s+u}(t) \\
& =\chi(-1) \sum_{t \in \mathcal{R}} \sum_{u \in \mathcal{R}^{*}} 1+\sum_{\substack{u \in \mathcal{R}^{*}}} \sum_{\substack{s \in \mathcal{R}^{*} \\
s+u \neq 0}} \chi\left(s u^{-1}\right) \sum_{t \in \mathcal{R}} \lambda_{s+u}(t) .
\end{aligned}
$$

Since $\sum_{t \in \mathcal{R}} \lambda_{s+u}(t)=0$ for $s, u \in \mathcal{R}^{*}$ with $s+u \neq 0$, we have

$$
\begin{equation*}
\sum_{t \in \mathcal{R}} g\left(\chi, \lambda_{t}\right) g\left(\bar{\chi}, \lambda_{t}\right)=\left(4^{r}-2^{r}\right) 4^{r} \chi(-1) \tag{3.9}
\end{equation*}
$$

By comparing (3.8) and (3.9) we have (3.6). Next, for (3.7). If $t \in M$, it follows from Theorem 3.1 (b). Let $t \in \mathcal{R}^{*}$. Then

$$
\begin{aligned}
\left|g\left(\chi, \lambda_{t}\right)\right|^{2} & =g\left(\chi, \lambda_{t}\right) \overline{g\left(\chi, \lambda_{t}\right)}=g\left(\chi, \lambda_{1}\right) \overline{g\left(\chi, \lambda_{1}\right)}(\text { by Theorem } 3.1 \text { (a)) } \\
& =\chi(-1) g\left(\chi, \lambda_{1}\right) g\left(\bar{\chi}, \lambda_{1}\right) \quad \text { (by Theorem } 3.1 \text { (a)) } \\
& =4^{r} \quad(\text { by }(3.6)) .
\end{aligned}
$$

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